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APPROXIMATIONS FOR p-ADIC TRANSITIVE ISOMETRIES

JOANNA FURNO

Abstract. For a fixed prime p, we give periodic approximations of p-adic transitive isometries to prove results on spectrum and entropy with respect to Haar measure. For translation by a p-adic rational number, these periodic approximations converge in the strong topology if and only if the rational number is an integer. Finally, we describe labeling algorithms for digraph representations of translations by p-adic rational numbers. These algorithms illuminate the p-adic expansion of rational numbers in the p-adic integers and connections to number theory.

1. Introduction

The p-adic numbers began as a tool in number theory and have spread to other fields, including dynamical systems and ergodic theory. For a few examples of some of the first appearances of p-adic numbers in dynamical systems and ergodic theory, see [15], [11], [6], [14], and [19]. A transformation is transitive modulo p^n if iterates of the transformation cycle through all balls of radius p^{-n}. A transformation is a *p-adic transitive isometry* if it is an isometry that is transitive modulo p^n for all $n \in \mathbb{N}$. V. Anashin studied p-adic transitive isometries in [1] and [2]. Quotient-preserving maps on profinite groups, a generalization of p-adic transitive isometries, are studied in [13], [3], and [4]. Translation by a unit in $\mathbb{Z}_p$ is an example of a p-adic transitive isometry. The ergodic properties of translations are studied in [2], [5], [8], and [9].

In section 2, we give approximations of p-adic transitive isometries and show that they are cyclic approximations by periodic transformations.

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A. B. Katok and A. M. Stepin study transformations admitting approximations by periodic transformations in [12]. Generalizations of Katok and Stepin's results include papers by T. Schwartzbauer [17] and [16] and by Richard J. Turek [18]. Using the results of Katok and Stepin in [12], we give results in Theorem 2.6 for spectrum and entropy with respect to Haar measure.

A sequence of transformations S_n on a measure space $(X, \mathcal{A}, \mu)$ *converges in the strong topology* on the set of transformations to a transformation T on $(X, \mathcal{A}, \mu)$ if

$$\mu \{x \in X : S_n(x) \neq T(x)\} \to 0 \text{ as } n \to \infty.$$

In section 3, we show by example that our approximations do not always converge in the strong topology. Theorem 3.2 states that the approximations converge to translation by $a \in \mathbb{Z}_p$ in the strong topology if and only if $a \in \mathbb{Z}_p^\times$ is an integer in $\mathbb{Z}$.

Hansheng Diao and Cesar E. Silva defined digraph representations of locally 1-Lipschitz p-adic rational functions in [7]. In section 4, we discuss the digraph representations for translations by rational numbers in the p-adic integers. Labeling Algorithm 4.1 and Labeling Algorithm 4.3 give methods for labeling the vertices of the digraph representation of translation by a rational number, using the digraph representation of translation by 1. Labeling Algorithm 4.3 gives a connection between the dynamics and number theory, since it serves the same purpose as the Euclidean algorithm. This connection is given explicitly in Theorem 4.5.

The results in section 3 and section 4 are part of the author's Ph.D. dissertation [10], completed under Jane Hawkins at the University of North Carolina at Chapel Hill.

1.1. Definitions.

For a fixed prime p, the set of p-adic integers is the set of formal power series in p,

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \mathbb{Z} \text{ and } 0 \leq a_i \leq p-1 \right\}.$$

By listing the coefficients of the series $\sum a_i p^i$ as a sequence (a_i), the set $\mathbb{Z}_p$ can be identified with the one-sided product space on p symbols. However, $\mathbb{Z}_p$ has an additional ring structure, where addition and multiplication are defined componentwise with carries. The set of units in $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^\times = \left\{ a \in \mathbb{Z}_p : a^{-1} \in \mathbb{Z}_p \right\}.$$

Then $a \in \mathbb{Z}_p^\times$ if and only if $|a|_p = 1$ (where $|a|_p$ denotes the p-adic absolute value, as defined below), which holds if and only if the first coordinate is nonzero. Moreover, the p-adic integers $\mathbb{Z}_p$ contain a copy of the integers $\mathbb{Z}$, which appear as series that end in repeating 0's for nonnegative integers or repeating $p-1$'s for negative integers. More generally, the elements of $\mathbb{Z}_p$ that end in repeating coefficients can be identified as rational numbers.

We define an order on $\mathbb{Z}_p$ by

$$\operatorname{ord}_p\left(\sum_{i=0}^{\infty} a_i p^i\right) = \min\{i : a_i \neq 0\}.$$

This order induces the p-adic absolute value on $\mathbb{Z}_p$ by

$$|a|_p = \begin{cases} 0 & \text{if } a = 0, \\ p^{-\operatorname{ord}_p(a)} & \text{if } a \neq 0. \end{cases}$$

The p-adic absolute value induces a metric, for which positive distances are always powers of p. The metric induces a totally disconnected topology, so all balls are both open and closed. The topology has a basis consisting of balls

$$B_{p^{-n}}(a) = \left\{x \in \mathbb{Z}_p : |x-a|_p \leq p^{-n}\right\},$$

where n is a nonnegative integer and $a \in \mathbb{Z}_p$. For a locally-compact abelian group, Haar measure is the translation-invariant measure that is unique up to scalar multiplication. For normalized Haar measure γ on the Borel sets $\mathcal{B}$ of $\mathbb{Z}_p$, the measure of a ball is equal to its radius. If $a \in \mathbb{Z}_p$, let a_i be the integers such that $0 \leq a_i \leq p-1$ and $a = \sum a_i p^i$. For $a, b \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, we say that $a \equiv b \mod p^n$ if $a_i = b_i$ for $0 \leq i \leq n-1$. Then $a \equiv b \mod p^n$ if and only if $|a-b|_p \leq p^{-n}$. For $n \in \mathbb{N}$, we say that $\sum_{i=n}^{\infty} a_i p^i$ is the *n-tail* of $a \in \mathbb{Z}_p$.

A transformation $T : \mathbb{Z}_p \to \mathbb{Z}_p$ is an isometry if $|T(x)-T(y)|_p = |x-y|_p$ for all $x, y \in \mathbb{Z}_p$. Let $\mathbb{F}_{p^n} = \left\{0, 1, \ldots, p^{n-1}\right\}$ be the finite field with p^n elements. If T is an isometry, then we define the transformation $T \mod p^n$ on $\mathbb{F}_{p^n}$ by $(T \mod p^n)(x) = T(x) \mod p^n = \sum_{i=0}^{n-1}(T(x))_i p^i$. If $x \equiv y \mod p^n$, then $|x-y|_p \leq p^{-n}$. If T is an isometry, $|x-y|_{\leq} p^{-n}$ implies that $|T(x)-T(y)|_p \leq p^{-n}$, so $T(x) \equiv T(y) \mod p^n$. Hence, $T \mod p^n$ is well defined. The function T is *transitive modulo p^n* if $T \mod p^n$ is transitive, so T permutes balls of radius p^{-n} in a single cycle.

2. Ergodic Properties of Transitive Isometries

In this section, we recall the classical definition by Katok and Stepin [12] of a measure-preserving transformation that admits an approximation by periodic transformations with speed $f(n)$. For a p-adic transitive isometry and any speed $f(n)$, we give a cyclic approximation by periodic transformations. After reviewing some definitions of ergodic properties, we use the approximations and results of Katok and Stepin [12] to give the spectral and ergodic properties that hold for p-adic transitive isometries with respect to Haar measure.

Let T be a transformation from a measure space $(X, \mathcal{B}, \mu)$ to itself. The transformation T is *measurable* with respect to μ if $T^{-1}A \in \mathcal{B}$ for all $A \in \mathcal{B}$. A measurable transformation T is *measure-preserving* with respect to μ if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

Let $(X, \mathcal{B}, \mu)$ be a measure space. Then $\xi = \{A_1, A_2, \ldots, A_n\}$ is a *partition* of X if $A_i \cap A_j = \emptyset$ for $i \neq j$ and $X = \bigcup A_i$, up to sets of measure zero. For a sequence of partitions ξ_n, we write $\xi_n \to \epsilon$ *as* $n \to \infty$ if for all $A \in \mathcal{B}$ and for each $n \in \mathbb{N}$ there exists $A(\xi_n)$, a union of elements in ξ_n, such that $\lim_{n\to\infty} \mu(A \Delta A(\xi_n)) = 0$, where Δ is the symmetric difference.

Let T be an invertible, measure-preserving transformation on a measure space $(X, \mathcal{B}, \mu)$. Let $f(n)$ be a sequence of positive real numbers. Then T admits a *cyclic approximation by periodic transformations (cyclic a.p.t.) with speed* $f(n)$ if there exist partitions

$$\xi_n = \{C_i(n) : i = 1, 2, \ldots, q(n)\}$$

and a sequence of transformations S_n such that

(1) $\xi_n \to \epsilon$ as $n \to \infty$,
(2) $S_n(C_i(n)) = C_{i+1}(n)$ for $1 \leq i < q(n)$ and $S_n(C_{q(n)}(n)) = C_0(n)$, and
(1) $\displaystyle\sum_{i=1}^{q(n)} \mu(TC_i(n) \Delta S_n C_i(n)) < f(q(n))$.

In Theorem 2.1, we define approximations S_n for a p-adic transitive isometry $T : \mathbb{Z}_p \to \mathbb{Z}_p$. For $n \in \mathbb{N}$, the approximation $S_n(x)$ agrees with $T(x)$ for the first n coefficients and fixes the n-tail of x.

Theorem 2.1. *Let $T : (\mathbb{Z}_p, \mathcal{B}, \gamma) \to (\mathbb{Z}_p, \mathcal{B}, \gamma)$ be a p-adic transitive isometry on $\mathbb{Z}_p$ with Haar measure. Let $f(n)$ be a sequence of positive real numbers. For $n \in \mathbb{N}$, define an approximation $S_n : (\mathbb{Z}_p, \mathcal{B}, \gamma) \to (\mathbb{Z}_p, \mathcal{B}, \gamma)$ by*

$$(S_n(x))_i = \begin{cases} (T(x))_i & \text{if } 0 \leq i < n \\ x_i & \text{if } i \geq n. \end{cases} \tag{2.1}$$

Then the sequence of transformations S_n is a cyclic a.p.t. with speed $f(n)$.

Proof. Let $f(n)$ be a sequence of positive real numbers. Let $T : \mathbb{Z}_p \to \mathbb{Z}_p$ be a p-adic transitive isometry. As announced by Anashin in [1] and proved in [2], if T is a p-adic transitive isometry, then T preserves Haar measure. Let $C_i(n) = B_{p^{-n}}(T^i(0))$ for $1 \leq i \leq p^n$. Since T is transitive mod p^n, $\xi_n = \{C_i(n) : i = 1, 2, \dots, q(n)\}$ is a partition of $\mathbb{Z}_p$ and $q(n) = p^n$. Define periodic S_n by (2.1). Since iterates of T travel through balls of radius p^{-n} in a cycle, we have $S_n(C_i(n)) = C_{i+1}(n)$ for $1 \le i < q(n)$ and $S_n(C_{q(n)}(n)) = C_0(n)$. Since S_n and T agree for the first n coefficients, we have $TC_i(n) = S_nC_i(n)$, so

$$\sum_{i=1}^{p^n} \mu(TC_i(n)\Delta S_nC_i(n)) = 0 < f(p^n).$$

Therefore, T admits an approximation by a cyclic a.p.t. with speed $f(n)$. $\square$

Example 2.2. Translation by $a \in \mathbb{Z}_p$ is defined by

$$\begin{aligned} T_a : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto x + a. \end{aligned}$$

If $a \in \mathbb{Z}_p^\times$, then T_a is an example of a p-adic transitive isometry. For $x, y \in \mathbb{Z}_p$,

$$|T_a(x) - T_a(y)|_p = |x + a - y - a|_p = |x - y|_p,$$

so T_a is an isometry. Since $a \in \mathbb{Z}_p^\times$, the translation T_a is minimal (see [5, Theorem 6.1]). Since the translation T_a is a minimal isometry, it is transitive mod p^n for all $n \in \mathbb{N}$ (see [7, Theorem 3.2]).

In Proposition 2.3 and Proposition 2.4, we give further properties of the transformations S_n. A sequence of transformations $S_n(x)$ on a metric space (X, d) converges to $T(x)$ *uniformly in* x if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(S_n(x), T(x)) < \epsilon$ for all $n \ge N$ and for all $x \in X$.

Proposition 2.3. *Let T be a transformation on $\mathbb{Z}_p$. For each $n \in \mathbb{N}$, define S_n by (2.1). Then the sequence $S_n(x)$ on $(\mathbb{Z}_p, |\cdot|_p)$ converges uniformly in x to $T(x)$.*

Proof. Let $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $p^{-N} < \epsilon$. By the definition of S_n, we have $(S_n(x))_i = (T(x))_i$ for $0 \le i < n$ and for all $x \in \mathbb{Z}_p$. Thus, if $n \ge N$, then

$$|S_n(x) - T(x)|_p \le p^{-n} < \epsilon$$

for all $x \in \mathbb{Z}_p$. Therefore, the sequence of periodic transformations $S_n(x)$ converges uniformly in x to the transformation $T(x)$. $\square$

Proposition 2.4. *Let T be an isometry on $\mathbb{Z}_p$. Let $n \in \mathbb{N}$ and define S_n by (2.1). Then S_n is an isometry on $\mathbb{Z}_p$.*

Proof. Let $x, y \in \mathbb{Z}_p$. Let k be the nonnegative integer such that $|x-y|_p = p^{-k}$. Then $x_i = y_i$ for $0 \leq i < k$ and $x_k \neq y_k$. Since T is an isometry, $(Tx)_i = (Ty)_i$ for $0 \leq i < k$ and $(Tx)_k \neq (Ty)_k$. We break the proof into two cases.

For the first case, suppose that $k < n$. Then $(S_n x)_i = (Tx)_i = (Ty)_i = (S_n y)_i$ for $0 \leq i < k$. Since $(Tx)_k \neq (Ty)_k$ and $k < n$, it follows that $(S_n x)_k \neq (S_n y)_k$. Thus, $|S_n(x) - S_n(y)|_p = p^{-k}$.

For the second case, suppose that $k \geq n$. Then $(S_n x)_i = (Tx)_i = (Ty)_i = (S_n y)_i$ for $0 \leq i < n$ and $(S_n x)_i = x_i = y_i = (S_n y)_i$ for $n \leq i < k$. Since $x_k \neq y_k$ and $k \geq n$, it follows that $(S_n x)_k \neq (S_n y)_k$. Thus, $|S_n(x) - S_n(y)|_p = p^{-k}$.

In either case, $|S_n(x) - S_n(y)|_p = |x - y|_p$, so S_n is an isometry on $\mathbb{Z}_p$. $\square$

To discuss the spectral and ergodic properties of transitive isometries on $\mathbb{Z}_p$, we need some further definitions. For a transformation T, define a unitary operator U_T on $L^2(X)$ by $U_T f(x) = f(Tx)$. Then $f \in L^2(X)$ is an *eigenfunction* with *eigenvalue* λ if $U_T f = \lambda f$. If all of the eigenvalues of U_T are simple eigenvalues, then T has *simple spectrum.* If the eigenfunctions of U_T span $L^2(X)$, then T has *discrete spectrum.*

A measure-preserving transformation T is *ergodic* with respect to μ if for all $A \in \mathcal{B}$ with $T^{-1}A = A$ either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$. Suppose T is a measure-preserving transformation on a finite measure space $(X, \mathcal{B}, \mu)$. The *entropy of a partition* $\xi = \{A_1, A_2, \ldots, A_k\}$ is

$$H(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log_2(\mu(A_i)),$$

where $0 \log_2(0)$ is defined to be 0. The refinement $\vee_{i=0}^{n-1} T^{-i}\xi$ is the partition consisting of the sets $\cap_{i=0}^{n-1} T^{-i} A_{j_i}$, where $0 \leq j_i \leq k$ for all $0 \leq i \leq n-1$. Then the entropy of T with respect to ξ is

$$h(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$$

and the *entropy* of T is the supremum over finite partitions,

$$h(T) = \sup_{\xi} h(T, \xi).$$

In [12, Theorem 3.1, Theorem 3.5 and Corollary 4.2], Katok and Stepin prove the following about the ergodic properties of invertible, measure-preserving transformations admitting a cyclic approximation by periodic transformations.

Theorem 2.5. *If an invertible, measure-preserving transformation T admits a cyclic a.p.t.*

(1) *with speed θ/n for $\theta < 1/2$, then T has simple spectrum.*
(2) *with speed $f(n)$ for any speed $f(n)$, then T has a discrete spectrum and all its eigenvalues are roots of unity.*
(3) *with speed $o(1/\log_2 n)$, then T has zero entropy.*

Next, we apply Theorem 2.5 to give spectral and ergodic properties of p-adic transitive isometries in Theorem 2.6.

Theorem 2.6. *Let $T : (\mathbb{Z}_p, \mathcal{B}, \gamma) \to (\mathbb{Z}_p, \mathcal{B}, \gamma)$ be a p-adic transitive isometry on $\mathbb{Z}_p$ with Haar measure. With respect to Haar measure, T has simple spectrum, has discrete spectrum where all eigenvalues are roots of unity, and has zero entropy.*

Proof. Let T be a p-adic transitive isometry on $\mathbb{Z}_p$. By Theorem 2.1, T admits a cyclic a.p.t. with speed $f(n)$, for any speed $f(n)$. Since T is an invertible transformation that preserves Haar measure, the properties in Theorem 2.6 follow from the results for the corresponding properties in Theorem 2.5. □

3. Strong Convergence Properties

If $a \in \mathbb{Z}_p^\times$, then T_a is a p-adic transitive isometry by Example 2.2, so T_a admits a cyclic a.p.t. by Theorem 2.1. For the translation T_a and $n \in \mathbb{N}$, let S_n be the approximation defined by (2.1). Then Proposition 2.3 states that $S_n(x)$ converges uniformly in x to T_a. However, the sequence S_n does not always converge to T in the strong topology. Theorem 3.2 states that the sequence S_n converges to T_a with respect to the strong topology if and only if a is an integer. Before proving Theorem 3.2, we prove Lemma 3.1 to describe which T_a fix the n-tail of an element $x \in \mathbb{Z}_p$.

Lemma 3.1. *If there exist $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$ such that $(T_a(x))_i = x_i$ for all $i \geq n$, then $a \in \mathbb{Z} \subset \mathbb{Z}_p$.*

Proof. Suppose there exists $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$ such that $(x + a)_i = x_i$ for all $i \geq n$. This equality can occur in one of two ways.

First, suppose that

$$\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} a_i p^i < p^n.$$

In this case, addition of x and a does not result in a carry to the nth coordinate. Then $(x + a)_n = x_n + a_n$, which is equal to x_n if and only if $a_n = 0$. If $a_n = 0$, then $x_n + a_n < p$, so there is no carry to the $n + 1$th coordinate. As an induction hypothesis, suppose that $x_i + a_i < p$ for some

$i \geq n$. Then $(x+a)_{i+1} = x_{i+1} + a_{i+1}$, which is equal to x_{i+1} if and only if $a_{i+1} = 0$. Moreover, if $a_{i+1} = 0$, then $x_{i+1} + a_{i+1} < p$. It follows by induction that $a_i = 0$ for all $i \geq n$, so a is a nonnegative integer.

Second, suppose that

$$\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} a_i p^i \geq p^n.$$

In this case, addition of x and a does result in a carry to the nth coordinate. Then $(x+a)_n = x_n + a_n + 1$, which is equal to $x_n \mod p$ if and only if $a_n = p-1$. If $a_n = p-1$, then $x_n + a_n + 1 \geq p$, so there is a carry to the $n+1$th coordinate. As an induction hypothesis, suppose that $x_i + a_i \geq p$ for some $i \geq n$. Then $(x+a)_{i+1} = x_{i+1} + a_{i+1} + 1$, which is equal to x_{i+1} modulo p if and only if $a_{i+1} = p-1$. Moreover, if $a_{i+1} = p-1$, then $x_{i+1} + a_{i+1} + 1 \geq p$. It follows by induction that $a_i = p-1$ for all $i \geq n$, so a is a negative integer.

In either case, a is an integer, considered as an element of $\mathbb{Z}_p$. □

In Theorem 3.2, we consider probability measures on $\mathbb{Z}_p$ beyond Haar measure. Since $\mathbb{Z}_p$ can be identified with the one-sided product space on p symbols, we can consider $\mathbb{Z}_p$ with respect to i.i.d. Bernoulli measures, which are invariant under Bernoulli shifts. Let $\vec{q} = (q_0, q_1, \ldots, q_{p-1})$ be a probability vector with $q_i > 0$ for $0 \leq i < p$. Then the i.i.d. Bernoulli measure μ defined by $\vec{q}$ gives balls of radius p^{-n} measure by

$$\mu\left(B_{p^{-n}}(\sum a_i p^i)\right) = \prod_{i=0}^{n-1} q_{a_i}.$$

Standard constructions extend this definition to the Borel σ-algebra $\mathcal{B}$. Haar measure is the i.i.d. Bernoulli measure defined by the probability vector with all weights equal to $1/p$.

The convergence behavior of the sequence S_n in the strong topology distinguishes between integer and non-integer elements of $\mathbb{Z}_p$. For this reason, we break the proof into cases based on whether $a \in \mathbb{Z}_p$ is a positive integer, a negative integer, or not an integer.

Theorem 3.2. *Let $a \in \mathbb{Z}_p^\times$ and let μ be an i.i.d. Bernoulli measure. For $n \in \mathbb{N}$, define S_n by (2.1). Then S_n converges in the strong topology to T_a if and only if $a \in \mathbb{Z} \subset \mathbb{Z}_p$.*

Proof. Let μ be an i.i.d. Bernoulli measure defined by a probability vector $(q_0, \ldots, q_{p-1})$. Let $Q = \max_{0 \leq i < p} q_i$ be the maximal weight.

First, suppose that $a \in \mathbb{Z}_p^\times$ is a positive integer. Let $n \in \mathbb{N}$ such that $n > \log_p(a)$. Since $0 < a < p^n$, we have $a_i = 0$ for all $i \geq n$. Let $x \in \cup_{k=0}^{p^n-a-1} B_{p^{-n}}(k)$. Then $x \equiv k \mod p^n$ for some $0 \leq k < p^n - a$, so

$\sum_{i=0}^{n-1} x_i p^i + a < p^n$. Thus, adding a to x does not result in a carry to the nth coordinate. For all $i \geq n$, this implies that

$$\begin{aligned}(x+a)_i &= x_i + a_i \\ &= x_i,\end{aligned}$$

so $T_a(x) = S_n(x)$. Similarly, if $x \in \cup_{k=p^n-a}^{p^n-1} B_{p^{-n}}(k)$, then adding a to x does result in a carry to the nth coordinate. Since $(x+a)_n = x_n + 1$, we have $T_a(x) \neq S_n(x)$. Then

$$\begin{aligned}d_\mu(T_a, S_n) &= \mu\left(\{x \in \mathbb{Z}_p : T_a(x) \neq S_n(x)\}\right) \\ &= \mu\left(\bigcup_{k=p^n-a}^{p^n-1} B_{p^{-n}}(k)\right) \leq aQ^n.\end{aligned}$$

Since $Q < 1$, the sequence of real numbers aQ^n converges to 0 as n goes to infinity. Therefore, S_n converges to T_a in the strong topology.

Next, suppose that $a \in \mathbb{Z}_p^\times$ is a negative integer. Let $n \in \mathbb{N}$ such that $n > \log_p(|a|)$. Since $-p^n < a < 0$, we have $a_i = p-1$ for all $i \geq n$. Moreover, $\sum_{i=0}^{n-1} a_i p^i = p^n + a > 0$. Let $x \in \cup_{|a|}^{p^n-1} B_{p^{-n}}(k)$. Then $x \equiv k \mod p^n$ for some $|a| \leq k < p^n$, so $\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} a_i p^i = \sum_{i=0}^{n-1} x_i p^i + p^n + a \geq p^n$. Thus, adding a to x does result in a carry to the nth coefficient. For all $i \geq n$, this implies that

$$\begin{aligned}(x+a)_i &= 1 + x_i + a_i \\ &= x_i + p \equiv x_i \mod p,\end{aligned}$$

so $T_a(x) = S_n(x)$. Similarly, if $x \in \cup_{k=0}^{|a|-1} B_{p^{-n}}(k)$, then adding a to x does not result in a carry to the nth coordinate. Since $(x+a)_n = x_n + p - 1$, we have $T_a(x) \neq S_n(x)$. Then

$$d_\mu(T_a, S_n) = \mu\left(\bigcup_{k=0}^{|a|-1} B_{p^{-n}}(k)\right) \leq |a|Q^n.$$

Since $Q < 1$, the sequence of real numbers $|a|Q^n$ converges to 0 as n goes to infinity. Therefore, S_n converges to T_a in the strong topology.

Finally, suppose that $a \in \mathbb{Z}_p^\times$ is not an integer. Let $n \in \mathbb{N}$. Then Lemma 3.1 implies that $T_a(x) \neq S_n(x)$ for all $x \in \mathbb{Z}_p$. If μ is an i.i.d. product measure, then

$$d_\mu(T_a, S_n) = \mu(\mathbb{Z}_p) = 1.$$

Since this equality holds for all $n \in \mathbb{N}$, the sequence S_n does not converge to T_a in the strong topology. □

4. Translations and the Euclidean Algorithm

Let T be an isometry on $\mathbb{Z}_p$ and let $n \in \mathbb{N}$. We recall the definition of the digraph $G(T, p^n)$ from Diao and Silva [7]. Let the vertex set be $V(G) = \{0, 1, \dots, p^n - 1\}$. Then the edge set $E(G)$ contains the directed edge (i, j) if and only if $T(B_{p^{-n}}(i)) = B_{p^{-n}}(j)$. This digraph representation of an isometry illustrates the action of an isometry on balls of radius p^{-n}.

Let $a \in \mathbb{Z}_p^\times$ and $n \in \mathbb{N}$. For the translation T_a, define the approximation S_n by (2.1). By Example 2.2 and Proposition 2.4, T_a and S_n are isometries on $\mathbb{Z}_p$. Moreover, $T_a(x) \equiv S_n(x) \mod p^n$ for all $x \in \mathbb{Z}_p$. Hence, iterates of T_a and S_n cycle through balls of radius p^{-n} in the same order and $G(T_a, p^n) = G(S_n, p^n)$. As examples, Figure 1 and Figure 2 give the digraph representations $G(T_1, 3^1)$ and $G(T_1, 3^2)$, respectively.

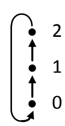

Figure 1. The digraph $G(T_1, 3^1)$.

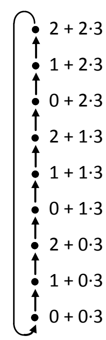

Figure 2. The digraph $G(T_1, 3^2)$.

The vertices of $G(T_1, p^n)$ are easy to label because the labels are the integers from 0 to $p^n - 1$ in counting order. In Labeling Algorithm 4.1 and Labeling Algorithm 4.3, we use $G(T_1, p^n)$ to label $G(T_a, p^n)$, where $a \in \mathbb{Z}_p$ is a positive integer or rational number. These algorithms reveal the p-adic expansions of these numbers and connections to number theory.

We can tell whether a p-adic integer is an integer or rational number based on its expansion. If $k \in \mathbb{N}$, then k is expressed as an element of $\mathbb{Z}_p$ by a series that ends in repeating 0's. Similarly, the series representation of $-k$ in $\mathbb{Z}_p$ ends in repeating $p - 1$'s. If $k \in \mathbb{Z}$ is not divisible by p, then $1/k$ has a series representation in $\mathbb{Z}_p$. The coefficients of this series can be found using the Euclidean algorithm.

Let a and b be two natural numbers such that $\gcd(a, b) = d$. It is a basic result of number theory that there exist integers x and y such that $xa + yb = d$. The Euclidean algorithm is a method to find these integers x and y. Let $n \in \mathbb{N}$. If k is not divisible by p, then $\gcd(k, p^n) = 1$. Thus, the Euclidean algorithm finds integers x_n and y_n such that $x_n k + y_n p^n = 1$. This linear combination implies that $x_n k \equiv 1 \mod p^n$, so $1/k \equiv x_n \mod p^n$. If $1/k$ has the series representation $\sum_{i=0}^{\infty} a_i p^i$ in $\mathbb{Z}_p$, then $x_n \equiv \sum_{i=0}^{n-1} a_i p^i \mod p^n$. In this manner, the Euclidean algorithm can be used to find the first n coefficients of the series expansion of $1/k$.

For $a = k$ and $b = p^n$ such that $\gcd(k, p^n) = 1$, Labeling Algorithm 4.3 also yields integers x and y that satisfy $xk + yp^n = 1$, as given in Theorem 4.5. Hence, Labeling Algorithm 4.3 serves the same purpose as the Euclidean algorithm in finding an inverse of k in $\mathbb{Z}_p$.

Let k be a positive integer that is not divisible by p. Labeling Algorithm 4.1 uses the fact that $T_k = T_1^k$ to label the digraph $G(T_k, p^n)$ from the labels on the digraph $G(T_1, p^n)$.

Labeling Algorithm 4.1. Let $k \in \mathbb{N}$ such that p does not divide k. The digraph $G(T_k, p^n)$ has p^n vertices in a cycle with labels as follows:

(1) Label any vertex as 0 in $G(T_k, p^n)$. In $G(T_1, p^n)$, begin at the vertex labeled 0.

(2) Travel along k edges in $G(T_1, p^n)$ and 1 edge in $G(T_k, p^n)$. Give the final vertex in $G(T_k, p^n)$ the same label as the final vertex in $G(T_1, p^n)$.

(3) Repeat step (2) until all vertices in $G(T_k, p^n)$ are labeled.

Example 4.2. Let $p = 3$, $n = 2$, and $k = 5$ in Labeling Algorithm 4.1. The digraph $G(T_5, 3^2)$ has 9 vertices. We pick one to label 0. Then the consecutive vertices in $G(T_5, 3^2)$ are given the label from every fifth vertex in $G(T_1, 3^2)$, starting from the vertex labeled 0. Figure 3 and Figure 4 show the digraph $G(T_1, 3^2)$ and the first two iterations of step (2) in Labeling Algorithm 4.1, with the traveled edges in black and the other edges in grey. Figure 5 shows the final result—the digraph $G(T_5, 3^2)$ with all of the vertices labeled.

Let k be a positive integer that is not divisible by p. Labeling Algorithm 4.3 uses the fact that $T_1 = T_{1/k}^k$ to label the digraph $G(T_{1/k}, p^n)$ from the labels on the digraph $G(T_1, p^n)$.

Labeling Algorithm 4.3. Let $k \in \mathbb{N}$ such that p does not divide k. The digraph $G(T_{1/k}, p^n)$ has p^n vertices in a cycle with labels from $\mathbb{F}_{p^n}$ as follows:

(1) Label any vertex as 0 in $G(T_{1/k}, p^n)$. In $G(T_1, p^n)$, begin at the vertex labeled 0.

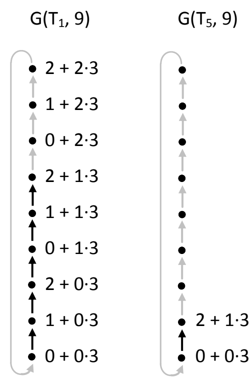

FIGURE 3. The first iteration of step (2) in Labeling Algorithm for $p = 3$, $n = 2$, and $k = 5$.

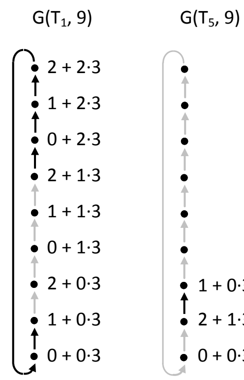

FIGURE 4. The second iteration of step (2) in Labeling Algorithm 4.1 for $p = 3$, $n = 2$, and $k = 5$.

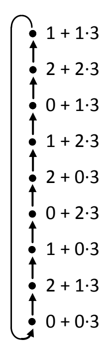

FIGURE 5. The digraph $G(T_5, 3^2)$.

(2) Travel along 1 edge in $G(T_1, p^n)$ and k edges in $G(T_{1/k}, p^n)$. Give the final vertex in $G(T_{1/k}, p^n)$ the same label as the final vertex in $G(T_1, p^n)$.
(3) Repeat step (2) until all vertices in $G(T_{1/k}, p^n)$ are labeled.

Example 4.4. Let $p = 3$, $n = 2$, and $k = 4$ in Labeling Algorithm 4.3. The digraph $G(T_{1/4}, 3^2)$ has 9 vertices. We pick one to label 0. Then every fourth vertex in $G(T_{1/4}, 3^2)$ is given the label from consecutive vertices in $G(T_1, 3^2)$, starting from the vertex labeled 0. Figure 6 shows the digraph $G(T_1, 3^2)$ and the third iteration of step (2) in Labeling Algorithm 4.3, with the traveled edges in black and the other edges in grey. Figure 7 shows the digraph $G(T_{1/4}, 3^2)$ with all of the vertices labeled.

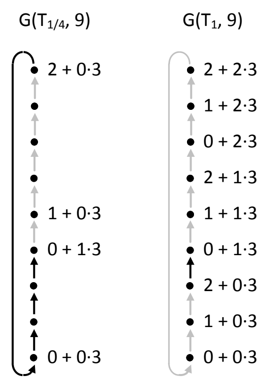

FIGURE 6. The third iteration of step (2) in Labeling Algorithm 4.3 for $p = 3$, $n = 2$, and $1/k = 1/4$.

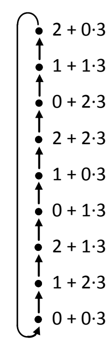

FIGURE 7. The digraph $G(T_{1/4}, 3^2)$.

Next, Theorem 4.5 gives a connection between p-adic dynamics and number theory through Labeling Algorithm 4.3, which serves the same purpose as the Euclidean algorithm.

Theorem 4.5. *Let $k \in \mathbb{N}$ such that p does not divide k. Fix $n \in \mathbb{N}$ and consider $G(T_{1/k}, p^n)$. Let $x_n \in \{0, 1, \ldots, p^n - 1\}$ be the integer such that $(0, x_n) \in E(G)$. Let y_n be the number of times that we cycle completely through $G(T_{1/k}, p^n)$ in Labeling Algorithm 4.3 before we label the vertex x_n. Then x_n and y_n are nonnegative integers that satisfy $x_n k = 1 + y_n p^n$.*

Proof. We give a combinatorial proof of Theorem 4.5 by counting edges in two different ways. Consider the digraph representation for $T_{1/k}$ on balls of radius p^{-n}. In Labeling Algorithm 4.3, every kth vertex in $G(T_{1/k}, p^n)$ is labeled with consecutive integers. Hence, we label the first vertex after 0 when we reach a multiple of k that is congruent to 1 modulo p^n. There are two ways to count how many edges in $G(T_{1/k}, p^n)$ are traveled to reach this vertex.

First, let $x_n \in \mathbb{F}_{p^n}$ be the integer such that $(0, x_n) \in E(G)$. In other words, x_n is the label of the vertex after 0 in $G(T_{1/k}, p^n)$. Since we are assigning labels with consecutive integers, x_n is also the number of times we have traveled k edges to reach the first vertex after 0. Thus, we have traveled through $x_n k$ edges to reach the first vertex after 0.

Second, let y_n count the number of times that we cycle completely through $G(T_{1/k}, p^n)$ before we label the vertex x_n. Thus, we have traveled though $1 + y_n p^n$ edges to reach the first vertex after 0. Therefore, x_n and y_n are integers such that $x_n k = 1 + y_n p^n$. □

Example 4.6. Using the Labeling Algorithm 4.3 to label the vertices of $G(T_{1/4}, 3^2)$, we label a vertex 0 and then label every fourth vertex with consecutive integers. The first vertex after 0 is eventually labeled with a $7 = 1 + 2 \cdot 3$. By the time that we reach the first vertex after 0, we have cycled 3 times through the 3^2 edges of $G(T_{1/4}, 3^2)$. Since $7 \cdot 4 = 1 + 3 \cdot 3^2$, we conclude that $7 = 1 + 2 \cdot 3 \equiv 1/4 \mod 3^2$.

Labeling Algorithm 4.1 and Labeling Algorithm 4.3 can be generalized to label the vertices of $G(T_a, p^n)$ for any rational number $a \in \mathbb{Z}_p^\times$. Suppose j/k is a rational number in reduced form, with $j \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then $j/k \in \mathbb{Z}_p^\times$ if and only if both j and k are not divisible by p. Since $T_{j/k}^k = T_1^j$, step (2) is modified so that we travel along k edges in $G(T_{j/k}, p^n)$ and j edges in $G(T_1, p^n)$. If j is negative, then we travel the edges of $G(T_1, p^n)$ in the opposite direction; that is, we travel the edges of $G(T_{-1}, p^n)$. Again, the label of the vertex after 0 in $G(T_{j/k}, p^n)$ is the expansion of j/k in $\mathbb{Z}_p$ modulo p^n.

Theorem 4.7. *Let $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that j, k, and p are pairwise relatively prime. Fix $n \in \mathbb{N}$ and consider $G(T_{j/k}, p^n)$. Let $x_n \in \{0, 1, \ldots, p^n - 1\}$ be the integer such that $(0, x_n) \in E(G)$. Then x_n is a nonnegative integer such that $x_n k \equiv j \mod p^n$. That is, the p-adic expansion of j/k agrees with the p-adic expansion of x_n in the first n coefficients.*

Example 4.8. For an example with a negative rational number, consider $G(T_{-4/3}, 5^1)$. Since $T_{-4/3}^3 = T_{-1}^4$ on $\mathbb{Z}_5$, traveling 3 edges in $G(T_{-4/3}, 5^1)$ corresponds to traveling 4 edges in $G(T_{-1}, 5^1)$. Moreover, $G(T_{-1}, 5^1)$ has the same edges as $G(T_1, 5^1)$, but in the opposite direction. Figure 8 shows the first step in labeling $G(T_{-4/3}, 5^1)$ and Figure 9 gives the completed graph. The vertex after 0 in Figure 9 is labeled 2. Hence, 2 is a nonnegative integer such that $3 \cdot 2 \equiv -4 \mod 5$. In other words, 2 is the first coefficient in the 5-adic expansion of $-4/3$.

To conclude, we discuss generalization of the results to $\mathbb{Z}_g$. For a fixed composite number $g \in \mathbb{N}$, the g-adic integers have definition similar to

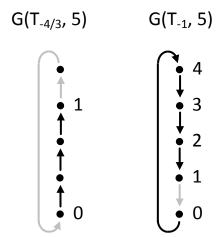

FIGURE 8. The first step in labeling $G(T_{-4/3}, 5)$.

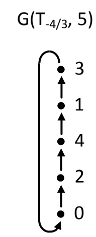

FIGURE 9. The digraph $G(T_{-4/3}, 5)$.

that of the p-adic integers. The results and proofs in this paper can be generalized to the g-adic integers by modifying notation and changing "p does not divide k" to "g and k are relatively prime."

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REFERENCES

[1] V. S. Anashin, *Uniformly distributed sequences of p-adic integers*, Discrete Math. Appl. **12** (2002), no. 6, 527–590.

[2] ———, *Ergodic transformations in the space of p-adic integers* in p-Adic Mathematical Physics: 2nd International Conference. Ed. Andrei Yu. Khrennikov, Zoran Rakic, and Igor V. Volovich. AIP Conference Proceedings, Vol. 826. Melville, NY: Amer. Inst. Phys., 2006. 3–24.

[3] ———, *Noncommutative algebraic dynamics: Ergodic theory for profinite groups*, Proc. Steklov Inst. Math. **265** (2009), no. 1, 30–58.

[4] Vladimir Anashin and Andrei Khrennikov, *Applied Algebraic Dynamics*. de Gruyter Expositions in Mathematics, 49. Berlin: Walter de Gruyter & Co., 2009.

[5] John Bryk and Cesar E. Silva, *Measurable dynamics of simple p-adic polynomials*, Amer. Math. Monthly **112** (2005), no. 3, 212–232.

[6] Zaqueu Coelho and William Parry, *Ergodicity of p-adic multiplications and the distribution of Fibonacci numbers* in Topology, Ergodic Theory, Real Algebraic Geometry. Ed. V. Turaev and A. Vershik. Amer. Math. Soc. Transl. Ser. 2, 202. Providence, RI: Amer. Math. Soc., 2001. 51–70.

[7] Hansheng Diao and Cesar E. Silva, *Digraph representations of rational functions over the p-adic numbers*, p-Adic Numbers Ultrametric Anal. Appl. **3** (2011), no. 1, 23–38.

[8] Ai-Hua Fan, Ming-Tian Li, Jia-Yan Yao, and Dan Zhou, *p-adic affine dynamical systems and applications*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 129–134.

[9] ———, *Strict ergodicity of affine p-adic dynamical systems on* $\mathbb{Z}_p$, Adv. Math. **214** (2007), no. 2, 666–700.

[10] Joanna Furno, *Ergodic Theory of p-adic Transformations.* Dissertation. University of North Carolina at Chapel Hill, 2013. Ann Arbor, MI: ProQuest LLC, 2013.

[11] M. Herman and J.-C. Yoccoz, *Generalizations of some theorems of small divisors to non-Archimedean fields*, in Geometric Dynamics (Rio de Janeiro, 1981). Ed. J. Palis, Jr. Lecture Notes in Mathematics, 1007. Berlin: Springer, 1983. 408–447.

[12] A. B. Katok and A. M. Stepin, *Approximation of ergodic dynamical systems by periodic transformations* (Russian), Dokl. Akad. Nauk SSSR **171** (1966), 1268–1271.

[13] James Kingsbery, Alex Levin, Anatoly Preygel, and Cesar E. Silva, *Measurable dynamics of maps on profinite groups*, Indag. Math. (N.S.) **18** (2007), no. 4, 561–581.

[14] D. A. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergodic Theory Dynam. Systems **8** (1988), no. 3, 411–419.

[15] Reinhard Oselies and Heiner Zieschang, *Ergodische Eigenschaften der Automorphismen p-adischer Zahlen*, Arch. Math. (Basel) **26** (1975), 144–153.

[16] T. Schwartzbauer, *Approximation of measure-preserving transformations*, Math. Systems Theory **6** (1972/73), 312–323.

[17] ———, *Methods of approximation in ergodic theory* in Entropy and Ergodic Theory. Selecta Statistica Canadiana, Vol. 2. Toronto, Ont.: Univ. Press Canada, 1975. 101–113.

[18] Richard J. Turek, *Approximation and ergodic theorems*, Adv. in Math. **51** (1984), no. 3, 258–270.

[19] Christopher F. Woodcock and Nigel P. Smart, *p-adic chaos and random number generation*, Experiment. Math. **7** (1998), no. 4, 333–342.

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