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ON EVENTUAL COLORING NUMBERS

YUKI IKEGAMI, HISAO KATO, AND AKIHIDE UEDA

Abstract. In [6], for each natural number p we defined eventual colorings within p of homeomorphisms which are generalizations of colorings of fixed-point free homeomorphisms, and we investigated the eventual coloring number $C(f, p)$ of a fixed-point free homeomorphism $f : X \to X$ with zero-dimensional set of periodic points. In [6], we constructed two indices $\varphi_n(k)$ and $\tau_n(k)$ for evaluating the eventual coloring number $C(f, p)$. The purpose of this paper is to construct a new index $\psi_n(k)$ which is more appropriate than the indices $\varphi_n(k)$ and $\tau_n(k)$.

1. Introduction

In this paper, we assume that all spaces are separable metric spaces and all maps are continuous functions. Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, ...\}$. For a separable metric space X, $\dim X$ denotes the covering dimension of X. For each map $f : X \to X$, let $P(f)$ be the set of all periodic points of f, i.e.,

$$P(f) = \{x \in X |\ f^j(x) = x \text{ for some } j \in \mathbb{N}\}.$$

For a subset K of X, $\mathrm{cl}(K)$, $\mathrm{int}(K)$ and $\mathrm{bd}(K)$ denote the closure, interior and the boundary of K in X, respectively. Let $\mathcal{C}$ be a family of subsets of X. For each $x \in X$, $\mathrm{ord}_x(\mathcal{C})$ denotes the number of elements of $\mathcal{C}$ which contain x, i.e.,

$$\mathrm{ord}_x(\mathcal{C}) = |\{C \in \mathcal{C} \mid x \in C\}|.$$

By a *swelling* of a family $\{A_s\}_{s\in S}$ of subsets of a space X, we mean any family $\{B_s\}_{s\in S}$ of subsets of X such that $A_s \subset B_s$ $(s \in S)$ and for every

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finite set of indices $s_1, s_2, ..., s_m \in S$,

$$\bigcap_{i=1}^{m} A_{s_i} \neq \phi \text{ if and only if } \bigcap_{i=1}^{m} B_{s_i} \neq \phi.$$

Conversely, for any cover $\{B_s\}_{s\in S}$ of X, a cover $\{A_s\}_{s\in S}$ of X is a *shrinking* of $\{B_s\}_{s\in S}$ if $A_s \subset B_s$ $(s \in S)$. A finite cover $\mathcal{C}$ of X is a *closed partition* of X provided that each element C of $\mathcal{C}$ is closed in X, $\mathrm{int}(C) \neq \emptyset$ and $C \cap C' = \mathrm{bd}(C) \cap \mathrm{bd}(C')$ for $C, C' \in \mathcal{C}$ with $C \neq C'$. Let $\mathcal{B}$ be a collection of subsets of a space X with $\dim X = n < \infty$. The collection $\mathcal{B}$ is *in general position in* X provided that if $\mathcal{S} \subset \mathcal{B}$ with $|\mathcal{S}| \leq n+1$, then $\dim(\bigcap\{S|\ S \in \mathcal{S}\}) \leq n - |\mathcal{S}|$. We need the following lemma of general position (see [6]).

Lemma 1.1. ([6, Lemma 2.2]) *Suppose that $f : X \to X$ is a fixed-point free homeomorphism of a separable metric space X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Let $\mathcal{C} = \{C_i|\ 1 \leq i \leq m\}$ $(m \in \mathbb{N})$ be an open cover of X and let $\mathcal{B} = \{B_i|\ 1 \leq i \leq m\}$ be a closed shrinking of $\mathcal{C}$. Then for any $k \in \mathbb{N}$ there is an open shrinking $\mathcal{C}' = \{C'_i|\ 1 \leq i \leq m\}$ of $\mathcal{C}$ such that*
(0) *$B_i \subset C'_i$,*
(1) *$\{f^j(\mathrm{bd}(C'))|\ C' \in \mathcal{C}', -k \leq j \leq k\}$ is in general position,*
(2) *$\mathrm{bd}(C') \cap P(f) = \emptyset$ for each $C' \in \mathcal{C}'$.*

Let $f : X \to X$ be a fixed-point free map of a separable metric space X, i.e., $f(x) \neq x$ for each $x \in X$. A subset C of X is called a *color* (see [11]) of f if $f(C) \cap C = \emptyset$. Note that $f(C) \cap C = \emptyset$ if and only if $C \cap f^{-1}(C) = \emptyset$. We say that a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is the *coloring number* of f (see [11]). The following is an important theorem of coloring numbers.

Theorem 1.2. ([1, Aarts, Fokkink and Vermeer]) *If $f : X \to X$ is a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$, then $C(f) \leq n + 3$.*

Let $f : X \to X$ be a fixed-point free map of a separable metric space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within* p of f ([6]) if $\bigcap_{i=0}^{p} f^{-i}(C) = \emptyset$. Note that C is a color of f if and only if C is eventually colored within 1.

Proposition 1.3. ([6, Proposition 1.4]) *Let $f : X \to X$ be a fixed-point free map of a separable metric space X and $p \in \mathbb{N}$. Then a subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e., for each $x \in C$, $f^i(x) \notin C$ with some $i \leq p$.*

In [6], we defined the eventual coloring number $C(f,p)$ of f as follows. A cover $\mathcal{C}$ of X is called an *eventual coloring* of f within p if each element $C \in \mathcal{C}$ is eventually colored of f within p. The minimal cardinality $C(f,p)$ of all closed (or open) eventual colorings of f within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. The coloring number $C(f)$ has been investigated by many mathematicians (e.g., see [1-5], [7] and [9-11]). In [6], we constructed two indices $\varphi_n(k)$ and $\tau_n(k)$ to evaluate the eventual coloring number $C(f,p)$. In this paper, we will construct a new index $\psi_n(k)$ which is more appropriate than the indices $\varphi_n(k)$ and $\tau_n(k)$.

2. The index $\psi_n(k)$

In [6], we constructed two indices $\varphi_n(k)$ and $\tau_n(k)$. For each $n \in \mathbb{N} \cup \{0\}$ and $k = 0, 1, 2, ..., n+1$, we defined the index $\varphi_n(k)$ as follows. Put $\varphi_n(0) = 1$ $(k = 0)$. For each $k = 1, 2, ..., n+1$, by induction on k we defined the index $\varphi_n(k)$ by

$$\varphi_n(k) = 2\varphi_n(k-1) + [\frac{n}{n+2-k}] \cdot (\varphi_n(k-1) + 1),$$

where $[x] = \max\{m \in \mathbb{N} \cup \{0\} | \ m \leq x\}$ for $x \in [0, \infty)$. Also, for each $n \in \mathbb{N} \cup \{0\}$ and $k = 0, 1, 2, ..., n+1$, we defined the index $\tau_n(k)$ by

$$\tau_n(k) = k(2n+1) + 1.$$

In [6], we proved the following theorem.

Theorem 2.1. ([6, Theorems 2.3 and 2.6]) *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then*

$$C(f, \min\{\varphi_n(k), \tau_n(k)\}) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n+1$.

In general, we have the following problem of eventual coloring numbers ([6, Problem 2.5]).

Problem 2.2. *For each $n \in \mathbb{N} \cup \{0\}$ and $1 \leq k \leq n+1$, determine the minimum $m(n,k)$ of natural numbers p satisfying the following conditions; if $f : X \to X$ is any fixed-point free homeomorphism of a separable metric space X such that $\dim X = n$ and $\dim P(f) \leq 0$, then $C(f,p) \leq n+3-k$.*

Now, we will construct a new index $\psi_n(k)$. Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n+1$. Put $R(n,k) = n - (n+2-k)[\frac{n}{n+2-k}]$. Note that n is divided by $(n+2-k)$ with the remainder $R(n,k)$. First, we put

$\psi_n(0) = 1$ $(k = 0)$. Next we consider the following two cases R(<) and R(=):

$$\text{R}(<)\ R(n,k) < n+1-k.$$

$$\text{R}(=)\ R(n,k) = n+1-k.$$

For each $1 \le k \le n+1$, we define the index $\psi_n(k)$ by

$$\psi_n(k) = \begin{cases} k(2[\frac{n}{n+2-k}]-1)+2 & (\text{if } R(n,k) < n+1-k), \\ k(2[\frac{n}{n+2-k}]+1)+1 & (\text{if } R(n,k) = n+1-k). \end{cases}$$

For actual values of the indices $\varphi_n(k), \tau_n(k)$ and $\psi_n(k)$, see the tables below. Then we have the following proposition.

Proposition 2.3. *For the indices $\varphi_n(k), \tau_n(k)$ and $\psi_n(k)$, the following inequalities hold.*

(1) *For each $n \in \mathbb{N} \cup \{0\}$ and $0 \le k \le n+1$,*

$$\psi_n(k) \le \min\{\varphi_n(k), \tau_n(k)\}.$$

(2) *For each $n \ge 2$ and $2 \le k \le n$,*

$$\psi_n(k) < \min\{\varphi_n(k), \tau_n(k)\}.$$

(3) *Moreover, for each $n \in \mathbb{N} \cup \{0\}$,*

$$\psi_n(0) = \varphi_n(0) = \tau_n(0) = 1,\ \psi_n(1) = \varphi_n(1) = 2,\ \psi_n(n+1) = \tau_n(n+1).$$

Proof. Let $n \in \mathbb{N} \cup \{0\}$. First, we will show $\psi_n(k) \le \varphi_n(k)$. If $k = 0$, then by definitions, $\psi_n(0) = \varphi_n(0) = 1$ $(n \ge 0)$. If $k = 1$, then $R(n,k) = R(n,1) = n$ $(= n+1-k)$, and hence the case R(=) holds. Then $\psi_n(1) = (2[\frac{n}{n+1}]+1)+1 = 2$. Hence $\varphi_n(1) = 2 = \psi_n(1)$ for each $n \ge 0$. Note that for $k \ge 1$,

$$k(2[\frac{n}{n+2-k}]-1)+2 < k(2[\frac{n}{n+2-k}]+1)+1$$

and hence in both of the cases R(<) and R(=),

$$\psi_n(k) \le k(2[\frac{n}{n+2-k}]+1)+1\ (k \ge 1).$$

It suffices to prove that

$$(*)_k\ k(2[\frac{n}{n+2-k}]+1)+1 \le \varphi_n(k)\ (k \ge 1).$$

We proceed by induction on $k = 1, ..., n+1$. If $k = 1$, then the both sides are 2 by the above argument. Let $2 \le k \le n+1$. We assume that $(*)_{k-1}$ holds, i.e.,

$$(k-1)(2[\frac{n}{n+2-(k-1)}]+1)+1 \le \varphi_n(k-1).$$

For simplicity, put $a = [\frac{n}{n+2-(k-1)}]$ and $b = [\frac{n}{n+2-k}]$. By the definition of $\varphi_n(k)$ and the above assumption, we have the following inequality $(**)_k$:

$$\begin{aligned}
\varphi_n(k) &= 2\varphi_n(k-1) + [\frac{n}{n+2-k}] \cdot (\varphi_n(k-1)+1) \\
&\geq 2\{(k-1)(2a+1)+1\} + b\{(k-1)(2a+1)+2\} \\
&= k(2ab+1) + k + (k+1)b + 2a(2k-b-2) \\
&= k(2[\frac{n}{n+2-(k-1)}][\frac{n}{n+2-k}]+1) + k + (k+1)[\frac{n}{n+2-k}] \\
&\quad +2[\frac{n}{n+2-(k-1)}](2k-[\frac{n}{n+2-k}]-2).
\end{aligned}$$

If $k=2$, then $[\frac{n}{n+2-(k-1)}] = [\frac{n}{n+2-(2-1)}] = 0$. By the inequality $(**)_2$,

$$\varphi_n(2) \geq 2+2+(2+1)[\frac{n}{n+2-2}] = 7 = 2(2[\frac{n}{n+2-2}]+1)+1 \geq \psi_n(2)\ (n \geq 1).$$

Moreover, we see that $\varphi_n(2) = 7$ $(n \geq 1)$ and $\psi_n(2) = 4$ $(n \geq 2)$ because that the condition R(<) $R(n,2) = 0 < n+1-2$ $(n \geq 2)$ is satisfied. Hence $\varphi_n(2) > \psi_n(2)$ $(n \geq 2)$.

Now, let us consider the case $k \geq 3$. Note that $n \geq k-1 \geq 2$ and $[\frac{n}{n+2-(k-1)}] \geq 1$. Also, we see that $k > [\frac{n}{n+2-k}]$ for each $3 \leq k \leq n+1$. In fact, suppose, on the contrary, that $k \leq [\frac{n}{n+2-k}]$. Then

$$\begin{aligned}
&n \geq k(n+2-k) = nk+2k-k^2 \\
\Rightarrow\ &k^2 \geq n(k-1)+2k.
\end{aligned}$$

Since $k \leq n+1$,

$$k^2 \geq n(k-1)+2k \geq (k-1)^2+2k = k^2+1.$$

This is a contradiction. By $(**)_k$,

$$\begin{aligned}
\varphi_n(k) &\geq k(2[\frac{n}{n+2-(k-1)}][\frac{n}{n+2-k}]+1) + k \\
&\qquad + (k+1)[\frac{n}{n+2-k}] + 2[\frac{n}{n+2-(k-1)}](2k-[\frac{n}{n+2-k}]-2) \\
&\geq k(2[\frac{n}{n+2-k}]+1) + k \\
&\qquad + (k+1)[\frac{n}{n+2-k}] + 2\{(k-[\frac{n}{n+2-k}]) + (k-2)\} \\
&> k(2[\frac{n}{n+2-k}]+1)+1\ (\geq \psi_n(k))\ (3 \leq k \leq n+1).
\end{aligned}$$

Consequently we see that $\psi_n(k) \leq \varphi_n(k)$.

Next, we will show that $\psi_n(k) \leq \tau_n(k)$. If $k=0$, then $\psi_n(0) = \tau_n(0) = 1$. For $k \geq 1$, it suffices to show that

$$k(2[\frac{n}{n+2-k}]+1)+1 \leq \tau_n(k).$$

If $1 \le k \le n$, then $[\frac{n}{n+2-k}] < n$. Hence

$$\psi_n(k) \le k(2[\frac{n}{n+2-k}]+1)+1 < k(2n+1)+1 = \tau_n(k)$$

for $1 \le k \le n$. If $k = n+1$ $(n \ge 0)$, then $[\frac{n}{n+2-k}] = n$. Since the case R(=) $R(n,k) = 0 = n+1-k$ holds, we see that

$$\begin{aligned}\psi_n(n+1) &= (n+1)(2[\frac{n}{n+2-(n+1)}]+1)+1 = (n+1)(2[n]+1)+1 \\ &= (n+1)(2n+1)+1 = \tau_n(n+1)\end{aligned}$$

for $n \ge 0$. Finally we can conclude that all conditions in Proposition 2.3 are satisfied. $\square$

Tables of three indices $\varphi_n(k), \tau_n(k)$ and $\psi_n(k)$

$\varphi_n(k)$

n \ k	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	7	30	-	-	-
3	1	2	7	22	113	-	-
4	1	2	7	22	90	544	-
5	1	2	7	22	69	278	1951

$\tau_n(k)$

n \ k	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	4	7	-	-	-	-
2	1	6	11	16	-	-	-
3	1	8	15	22	29	-	-
4	1	10	19	28	37	46	-
5	1	12	23	34	45	56	67

$\psi_n(k)$

n \ k	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	4	16	-	-	-
3	1	2	4	10	29	-	-
4	1	2	4	5	14	46	-
5	1	2	4	5	13	26	67

3. Main theorem

In this section, we prove the following theorem which is the main result of this paper. The proof is a modification of [6, Theorems 2.3 and 2.6], but we need more precise arguments.

Theorem 3.1. *Let $f : X \to X$ be a fixed-point free homeomorphism of a separable metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, \cdots, n + 1$.

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n + 1$. First, we will define the following index $\psi_{n,k}(l)$ for each $l = 0, 1, 2, ..., k$. If $l = 0$, we put $\psi_{n,k}(0) = 1$. Next, if $1 \leq l \leq k$, then we define the index $\psi_{n,k}(l)$ by

$$\psi_{n,k}(l) = \begin{cases} l(2[\frac{n}{n+2-k}] - 1) + 2 & \text{(if R(<) } R(n,k) < n + 1 - k \text{ holds)}, \\ l(2[\frac{n}{n+2-k}] + 1) + 1 & \text{(if R(=) } R(n,k) = n + 1 - k \text{ holds)}. \end{cases}$$

Note that $\psi_{n,k}(k) = \psi_n(k)$.

We will show that for each $0 \leq l \leq k$, there is an open cover $\mathcal{C}_l = \{C_{l,i} | \ 1 \leq i \leq n + 3 - l\}$ of X satisfying following conditions:
$(0)_l$ $\mathrm{cl}(C_{l,i})$ is eventually colored within $\psi_{n,k}(l)$ for $1 \leq i \leq n + 3 - k$,
$(1)_l$ $\mathrm{cl}(C_{l,i})$ is colored for $n + 3 - k < i \leq n + 3 - l$.

We proceed by induction on l. By Theorem 1.2, there is an open coloring $\mathcal{C}_0 = \{C_{0,i} | \ 1 \leq i \leq n+3\}$ of f. Since $\psi_{n,k}(0) = 1$, we may assume that $\mathcal{C}_0$ satisfies the conditions $(0)_0$ and $(1)_0$. Now we assume that $l \geq 1$ and there is an open cover $\mathcal{C}_{l-1} = \{C_{l-1,i} | \ 1 \leq i \leq n + 3 - (l - 1)\}$ $(l \geq 1)$ of X satisfying the conditions $(0)_{l-1}$ and $(1)_{l-1}$. By Lemma 1.1, we may assume that

$$\{f^j(\mathrm{bd}(C)) | \ C \in \mathcal{C}_{l-1}, -1 \leq j \leq 2[\frac{n}{n+2-k}] + 1\}$$

is in general position. Let $K_i = \mathrm{cl}(C_{l-1,i})$ for $1 \leq i \leq n+3-(l-1)$. We put

$$L_1 = K_1,\ L_i = \mathrm{cl}(K_i \backslash (K_1 \cup K_2 \cup \cdots \cup K_{i-1}))\ (i \geq 2).$$

Then the collection $\mathcal{L} = \{L_i |\ 1 \leq i \leq n+3-(l-1)\}$ is a closed partition of X. Note that

$$\mathrm{bd}(L_{i_1}) \cap \mathrm{bd}(L_{i_2}) \cap \cdots \cap \mathrm{bd}(L_{i_m}) \subset \mathrm{bd}(C_{i_1}) \cap \mathrm{bd}(C_{i_2}) \cap \cdots \cap \mathrm{bd}(C_{i_{m-1}})$$

for $1 \leq i_1 < i_2 < \cdots < i_m \leq n+3-(l-1)$. Then we see that for $x \in X$,

$$(\Delta)\ \mathrm{ord}_x(\mathcal{L}) - 1 \leq \mathrm{ord}_x\{\mathrm{bd}(C) |\ C \in \mathcal{C}_{l-1}\}.$$

Now, we will repaint $L_{n+3-(l-1)}$ by use of eventual colors L_i ($1 \leq i \leq n+3-k$). To this end, let $x \in L_{n+3-(l-1)}$. For the point x, we put

$$J_{n+3-(l-1)}(x) = \{j |\ 1 \leq j \leq 2[\frac{n}{n+2-k}] + 1 \text{ and } f^j(x) \notin L_{n+3-(l-1)}\}.$$

Since $L_{n+3-(l-1)}$ is a color of f,

$$|J_{n+3-(l-1)}(x)| \geq [\frac{n}{n+2-k}] + 1.$$

For each $j \in J_{n+3-(l-1)}(x)$, we put

$$I(j) = \{i \in \{1, 2, \cdots, n+3-k\} |\ f^j(x) \in L_i\}.$$

We will show that there is $j \in J_{n+3-(l-1)}(x)$ such that $|I(j)| < n+3-k$. Suppose, on the contrary, that $|I(j)| = n+3-k$ for all $j \in J_{n+3-(l-1)}(x)$. Then

$$f^j(x) \in \bigcap_{i=1}^{n+3-k} L_i \subset \bigcap_{i=1}^{n+2-k} \mathrm{bd}(C_i).$$

Since $\{f^j(\mathrm{bd}(C) |\ C \in \mathcal{C}_{l-1}, 0 \leq j \leq 2[\frac{n}{n+2-k}] + 1\}$ is in general position, we see that

$$([\frac{n}{n+2-k}] + 1)(n+2-k) \leq n.$$

However, we have

$$([\frac{n}{n+2-k}] + 1)(n+2-k) \geq n+1.$$

This is a contradiction. Hence there is $j \in J_{n+3-(l-1)}(x)$ such that $|I(j)| < n+3-k$.

We put

$$j(x) = \min\{j \in J_{n+3-(l-1)}(x) |\ |I(j)| < n+3-k\}.$$

Now, we have to consider two cases R($<$) $R(n,k) < n+1-k$ and R($=$) $R(n,k) = n+1-k$.

Case R(<): $R(n,k) < n+1-k$.

Note that in this case R(<), $k \geq l \geq 1$ and $k \geq 2$. Let $x \in L_{n+3-(l-1)}$. First, we will choose an open neighborhood $U(x)$ of x in X as follows. To this end, we will consider the following two cases $(\star)$ and $(\star\star)$:

$$(\star)\ j(x) < 2[\frac{n}{n+2-k}]$$

$$(\star\star)\ j(x) \geq 2[\frac{n}{n+2-k}]$$

Note that the two cases $(\star)$ and $(\star\star)$ depend on the point $x \in L_{n+3-(l-1)}$.

First, we consider the case $(\star)$ $j(x) < 2[\frac{n}{n+2-k}]$. In this case, we choose $L_{i(x)}$ $(1 \leq i(x) \leq n+3-k)$ such that $f^{j(x)}(x) \notin L_{i(x)}$. Then we can choose an open neighborhood $U(x)$ of x in X such that

$$(\star)\ (f^{j(x)}(\mathrm{cl}(U(x))) \cap (L_{n+3-(l-1)} \cup L_{i(x)}) = \emptyset.$$

Next, we consider the case $(\star\star)$ $j(x) \geq 2[\frac{n}{n+2-k}]$. To choose $U(x)$, we will show that the following conditions (1) and (2) hold.

(1) $|\{j \in J_{n+3-(l-1)}(x) |\ j < j(x)\}| = [\frac{n}{n+2-k}]$,
(2) There is $i(x)$ such that $1 \leq i(x) \leq n+3-k$, $f^{j(x)}(x) \notin L_{i(x)}$ and $f^{-1}(x) \notin L_{i(x)}$.

We will show that (1) holds. Since $\{f^j(\mathrm{bd}(C) |\ C \in \mathcal{C}_{l-1}, 0 \leq j \leq 2[\frac{n}{n+2-k}]+1\}$ is in general position,

$$|\{j \in J_{n+3-(l-1)}(x) |\ j < j(x)\}|(n+2-k) \leq n.$$

Hence $|\{j \in J_{n+3-(l-1)}(x) |\ j < j(x)\}| \leq [\frac{n}{n+2-k}]$. Suppose, on the contrary, that

$$|\{j \in J_{n+3-(l-1)}(x) |\ j < j(x)\}| \leq [\frac{n}{n+2-k}] - 1.$$

Since $L_{n+3-(l-1)}$ is a color of f and $x \in L_{n+3-(l-1)}$, by the above inequality we see that

$$j(x) \leq 2([\frac{n}{n+2-k}] - 1) + 1 = 2[\frac{n}{n+2-k}] - 1.$$

This is a contradiction to the case $(\star\star)$. Hence (1) is true.

We will show that (2) holds. Suppose, on the contrary, that (2) is not true. We assume that for each $1 \leq i \leq n+3-k$, $f^{j(x)}(x) \in L_i$ or $f^{-1}(x) \in L_i$. Hence

$$n+3-k \leq \mathrm{ord}_{f^{j(x)}(x)}(\mathcal{L}) + \mathrm{ord}_{f^{-1}(x)}(\mathcal{L}).$$

Note that $\{f^j(\mathrm{bd}(C))|\ C \in \mathcal{C}_{l-1}, -1 \leq j \leq 2[\frac{n}{n+2-k}]+1\}$ is in general position. Then

$$\mathrm{ord}_{f^{j(x)}(x)}\partial\mathcal{C}_{l-1} + \mathrm{ord}_{f^{-1}(x)}\partial\mathcal{C}_{l-1} + \Sigma_{j\in A}\mathrm{ord}_{f^j(x)}\partial\mathcal{C}_{l-1} \leq n,$$

where $\partial\mathcal{C}_{l-1} = \{\mathrm{bd}(C)|\ C \in \mathcal{C}_{l-1}\}$ and $A = \{j \in J_{n+3-(l-1)}(x)|\ j < j(x)\}$. Also, by (1),

$$\Sigma_{j\in A}\mathrm{ord}_{f^j(x)}\partial\mathcal{C}_{l-1} \geq (n+2-k)[\frac{n}{n+2-k}].$$

Then by the inequality (Δ),

$$(\mathrm{ord}_{f^{j(x)}(x)}(\mathcal{L})-1)+(\mathrm{ord}_{f^{-1}(x)}(\mathcal{L})-1) \leq \mathrm{ord}_{f^{j(x)}(x)}\partial\mathcal{C}_{l-1}+\mathrm{ord}_{f^{-1}(x)}\partial\mathcal{C}_{l-1}$$

$$\leq n-(n+2-k)[\frac{n}{n+2-k}] = R(n,k) < n+1-k.$$

Hence

$$n+3-k \leq \mathrm{ord}_{f^{j(x)}(x)}(\mathcal{L}) + \mathrm{ord}_{f^{-1}(x)}(\mathcal{L}) < n+3-k.$$

This is a contradiction. Therefore (2) is true.

In the case ($\star\star$), by (2) we can choose an open neighborhood $U(x)$ of x in X such that

($\star\star$) $f^{j(x)}(\mathrm{cl}(U(x)))\cap(L_{n+3-(l-1)}\cup L_{i(x)}) = \emptyset$ and $f^{-1}(\mathrm{cl}(U(x)))\cap L_{i(x)} = \emptyset$.

Consequently, in both of the cases ($\star$) and ($\star\star$), we have a desired open neighborhood $U(x)$ of each x ($\in L_{n+3-(l-1)}$) in X.

Now we consider the following family

$$\mathcal{U} = \{U(x) : x \in L_{n+3-(l-1)}\}.$$

Take a locally finite closed refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\bigcup\mathcal{W} = L_{n+3-(l-1)}$. Note that for each $W \in \mathcal{W}$, we can choose $x \in L_{n+3-(l-1)}$ such that $W \subset U(x)$. Then we define a function $\lambda : \mathcal{W} \to \{1, 2, ..., n+3-k\}$ by $\lambda(W) = i(x)$. For each $1 \leq i \leq n+3-k$, we put

$$E_i = \bigcup\{W \in \mathcal{W}|\ \lambda(W) = i\}$$

and

$$F_i = L_i \cup E_i\ (1 \leq i \leq n+3-k).$$

Since $\mathcal{W}$ is locally finite, E_i and F_i are closed in X for each i.

For each $1 \leq i \leq n+3-k$ and $z \in L_i$, put

$$p_i(z) = \min\{s|\ 1 \leq s \leq \psi_{n,k}(l-1), f^s(z) \notin L_i\}.$$

Also, for each $1 \leq i \leq n+3-k$ and $y \in E_i$, put

$$q_i(y) = \min\{s|\ 1 \leq s \leq 2[\frac{n}{n+2-k}]+1, f^s(y) \notin E_i \cup L_i = F_i\}.$$

Now we will show the following Claim.

Claim. If $y \in E_i$ with $f^{-1}(y) \in L_i$ $(1 \le i \le n+3-k)$, then $q_i(y) \le 2[\frac{n}{n+2-k}] - 1$.

Suppose, on the contrary, that $q_i(y) \ge 2[\frac{n}{n+2-k}]$. By the construction of E_i, there is $x \in L_{n+3-(l-1)}$ such that $y \in U(x) \in \mathcal{U}$ and $i(x) = i$. Since

$$f^{j(x)}(\mathrm{cl}(U(x))) \cap (L_{n+3-(l-1)} \cup L_i) = \emptyset$$

in both of the cases $(\star)$ and $(\star\star)$, we see that $j(x) \ge q_i(y) \ge 2[\frac{n}{n+2-k}]$. Hence the point x satisfies the case $(\star\star)$. By the choice of $U(x)$ in the case $(\star\star)$, we see that $f^{-1}(\mathrm{cl}(U(x))) \cap L_i = \emptyset$, and hence $f^{-1}(y) \notin L_i$. This is a contradiction. Thus $q_i(y) \le 2[\frac{n}{n+2-k}] - 1$.

Now, we will show that F_i is eventually colored within $\psi_{n,k}(l)(= l(2[\frac{n}{n+2-k}] - 1) + 2)$ $(k \ge 2, k \ge l \ge 1)$. Let $w \in F_i(= L_i \cup E_i)$. If $w \in E_i$,

$$q_i(w) \le 2[\frac{n}{n+2-k}] + 1 = (2[\frac{n}{n+2-k}] - 1) + 2 = \psi_{n,k}(1) \le \psi_{n,k}(l).$$

Hence $f^{q_i(w)}(w) \notin F_i$. If $w \in L_i$, then

$$p_i(w) \le \psi_{n,k}(l-1) \le \psi_{n,k}(l).$$

If $f^{p_i(w)}(w) \notin E_i$, then $f^{p_i(w)}(w) \notin F_i$. If $f^{p_i(w)}(w) \in E_i$, by $f^{p_i(w)-1}(w) \in L_i$ and the above claim,

$$q_i(f^{p_i(w)}(w)) \le 2[\frac{n}{n+2-k}] - 1.$$

Then

$$\begin{aligned} p_i(w) + q_i(f^{p_i(w)}(w)) &\le \psi_{n,k}(l-1) + (2[\frac{n}{n+2-k}] - 1) \\ &\le l(2[\frac{n}{n+2-k}] - 1) + 2 = \psi_{n,k}(l) \ (k \ge 2, k \ge l \ge 1). \end{aligned}$$

Then $f^{(p_i(w)+q_i(f^{p_i(w)}(w)))}(w) \notin F_i$. Therefore, F_i is eventually colored within $\psi_{n,k}(l)(= l(2[\frac{n}{n+2-k}] - 1) + 2)$ $(l \ge 1)$.

If we choose a small open swelling of the closed cover

$$\{F_i |\ 1 \le i \le n+3-k\} \cup \{L_j |\ n+4-k \le j \le n+3-l\}$$

of X, we obtain a desired open cover $\mathcal{C}_l = \{C_{l,i} : 1 \le i \le n+3-l\}$ of X satisfying the conditions $(0)_l$ and $(1)_l$.

Next, we consider the remaining case R(=) $R(n,k) = n+1-k$.

Case R(=): $R(n,k) = n+1-k$.

Let $x \in L_{n+3-(l-1)}$. Recall the definition of $j(x)$. In the case R(=), we choose $L_{i(x)}$ $(1 \le i(x) \le n+3-k)$ such that $f^{j(x)}(x) \notin L_{i(x)}$ and we choose an open neighborhood $U(x)$ of x in X such that $f^{j(x)}(\mathrm{cl}(U(x)) \cap (L_{n+3-(l-1)} \cup L_{i(x)}) = \emptyset$. Consider the collection $\mathcal{U} = \{U(x) | \ x \in L_{n+3-(l-1)}\}$ and take a locally finite closed refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\bigcup \mathcal{W} = L_{n+3-(l-1)}$. For each $W \in \mathcal{W}$, we can choose $U(x)$ such that $W \subset U(x)$. Also, we define a function $\lambda : \mathcal{W} \to \{1, 2, ..., n+3-k\}$ by $\lambda(W) = i(x)$. For each $1 \le i \le n+3-k$, put

$$E_i = \bigcup\{W \in \mathcal{W} | \ j(W) = i\}, \ F_i = L_i \cup E_i \ (1 \le i \le n+3-k).$$

We will show that F_i is eventually colored within $\psi_{n,k}(l) (= l(2[\frac{n}{n+2-k}]+1)+1)$ $(l \ge 1)$. For each $1 \le i \le n+3-k$ and $z \in L_i$, put

$$p_i(z) = \min\{s | \ 1 \le s \le \psi_{n,k}(l-1), f^s(z) \notin L_i\}.$$

Also, for each $1 \le i \le n+3-k$ and $y \in E_i$, put

$$q_i(y) = \min\{s | \ 1 \le s \le 2[\frac{n}{n+2-k}]+1, f^s(y) \notin E_i \cup L_i = F_i\}.$$

Let $w \in F_i (= L_i \cup E_i)$. If $w \in E_i$, then

$$q_i(w) \le 2[\frac{n}{n+2-k}]+1 \le \psi_{n,k}(l).$$

Then $f^{q_i(w)}(w) \notin F_i$. If $w \in L_i$, then

$$p_i(w) \le \psi_{n,k}(l-1) = (l-1)(2[\frac{n}{n+2-k}]+1)+1 \le \psi_{n,k}(l).$$

If $f^{p_i(w)}(w) \notin E_i$, then $f^{p_i(w)}(w) \notin F_i$. If $f^{p_i(w)}(w) \in E_i$,

$$q_i(f^{p_i(w)}(w)) \le 2[\frac{n}{n+2-k}]+1.$$

Hence

$$\begin{aligned} p_i(w) + q_i(f^{p_i(w)}(w)) &\le (l-1)(2[\frac{n}{n+2-k}]+1)+1+(2[\frac{n}{n+2-k}]+1) \\ &= l(2[\frac{n}{n+2-k}]+1)+1 = \psi_{n,k}(l). \end{aligned}$$

Then $f^{(p_i(w)+q_i(f^{p_i(w)}(w)))}(w) \notin F_i$. Therefore, F_i is eventually colored within $\psi_{n,k}(l)$. Similarly, we obtain a desired open cover $\mathcal{C}_l = \{C_{l,i} | \ 1 \le i \le n+3-l\}$ of X satisfying the conditions $(0)_l$ and $(1)_l$.

Consequently, in both of the cases R(<) $R(n,k) < n+1-k$ and R(=) $R(n,k) = n+1-k$, we have a desired open cover $\mathcal{C}_l = \{C_{l,i} | \ 1 \le i \le n+3-l\}$ of X satisfying the conditions $(0)_l$ and $(1)_l$. This implies that $C(f, \psi_{n,k}(l)) \le n+3-l$. If we take $l = k$, then $C(f, \psi_n(k)) \le n+3-k$. □

Corollary 3.2. *If $f : X \to X$ is any homeomorphism of a separable metric space X with* $\dim X = n < \infty$*, then*

$$C(f|Y, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, \cdots, n+1$, where $Y = X - P(f)$ and $f|Y : Y \to Y$ is the restriction of f.

For the case that f is a map of a compact metric space, by Theorem 3.1 and [6, Theorem 3.1] we have the following result.

Corollary 3.3. *Let $f : X \to X$ be a fixed-point free map of a compact metric space X with* $\dim X = n < \infty$*. If* $\dim P(f) \leq 0$*, then*

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n+1$.

Remark. The constructions of the three indices $\psi_n(k)$, $\varphi_n(k)$ and $\tau_n(k)$ are similar. We repaint one domain stage by stage. The different points are the numbers of eventual colors which are used for repainting one domain. For $\varphi_n(k)$, we use eventual colors as many as possible for repainting one domain. For $\tau_n(k)$, we use only two eventual colors. For $\psi_n(k)$, we use $(n+3-k)$ eventual colors.

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