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by

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CLOSED SUBSETS OF EUCLIDEAN SPACES CONTAINED IN PSEUDO-ARCS

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Abstract. In this paper we prove that if K is a compact subset of the Euclidean space $\mathbb{R}^k$ ($k \geq 3$) with the property that every nondegenerate component of K is a pseudo-arc, then there exists a pseudo-arc P with $K \subset P \subset \mathbb{R}^k$.

1. Introduction

J. R. Kline and R. L. Moore proved [7] that, in the plane, a compact set M is a subset of an arc if and only if every component of M is either a one-point set or an arc α such that no point of α, except its end points, is a limit point of $M - \alpha$. In his dissertation, published in [3], H. Cook studied the corresponding problem for the pseudo-arc and proved that if K is a compact plane set, then there exists a pseudo-arc P with $K \subset P \subset \mathbb{R}^2$ if and only if each of the nondegenerate components of K is a pseudo-arc. H. Cook has conjectured that this result is also true for $\mathbb{R}^k$ if $k \geq 3$. This conjecture was stated in the paper by David P. Bellamy in [1].

In this paper we prove Cook's conjecture by showing that, if $k \geq 3$ and K is a compact subset of the Euclidean space $\mathbb{R}^k$, then there exists a pseudo-arc P such that $K \subset P \subset \mathbb{R}^k$ if and only if each nondegenerate component of K is a pseudo-arc.

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Key words and phrases. Chainability, Crooked Chain, Euclidean Space, Indecomposability, Pseudo-arc.

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2. Chains

The proof of the main theorem of this paper depends on making a careful and technical surgery with chains. In this section we develop the tools to cut and paste the appropiate chains.

A *continuum* is a compact connected nondegenerate metric space. A continuum is *indecomposable* if it is not the union of two of its proper subcontinua. Given a metric space X, a *chain* in X is a collection of open sets $\mathcal{C} = \{C_1, \ldots, C_n\}$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$, $C_1 \nsubseteq \mathrm{cl}_X(C_2)$ and $C_n \nsubseteq \mathrm{cl}_X(C_{n-1})$. Each of the elements C_i is a *link* of $\mathcal{C}$. If $p \in C_1$ and $q \in C_n$, then $\mathcal{C}$ is said to be a chain from p to q. The chain $\mathcal{C}$ is *taut* if it satisfies the additional condition that $\mathrm{cl}_X(C_i) \cap \mathrm{cl}_X(C_j) \neq \emptyset$ if and only if $|i - j| \leq 1$. The *mesh* of $\mathcal{C}$ is the maximuum of the diameters of the sets C_i. Given a subset A of X, we say that the chain $\mathcal{C}$ *covers* A provided that $A \subset C_1 \cup \ldots \cup C_n$, and $\mathcal{C}$ *properly covers* A if $(A \cap C_1) -\ \mathrm{cl}_X(C_2) \neq \emptyset \neq (A \cap C_n) -\ \mathrm{cl}_X(C_{n-1})$. Given $\varepsilon > 0$, an *ε-chain* is a chain $\mathcal{C}$ such that $\mathrm{mesh}(\mathcal{C}) < \varepsilon$.

The continuum X is *chainable* provided that for each $\varepsilon > 0$ there exists a taut ε-chain whose links cover X. Given a family $\mathcal{A}$ of subsets of X, $\cup\mathcal{A}$ denotes the union of the elements of $\mathcal{A}$.

Recall that a *pseudo-arc* is a hereditarily indecomposable chainable continuum (up to homeomorphisms, there is only one pseudo-arc, see [2, Theorem 1].

The strategy for proving the main result in this paper is as follows.

1. We take a compact subset K of $\mathbb{R}^k$ ($k \geq 3$) such that every nondegenerate component of K is a pseudo-arc.

2. We construct a sequence $\{\mathcal{U}_r\}_{r=1}^{\infty}$ of chains in $\mathbb{R}^k$ such that for each $r \in \mathbb{N}$, $K \subset \cup\mathcal{U}_r$, $\mathcal{U}_{r+1}$ is "crooked enough" in $\mathcal{U}_r$ and $\lim \mathrm{mesh}(\mathcal{U}_r) = 0$. In this way, $P = \cap\{\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{U}_r) : r \in \mathbb{N}\}$ is the desired pseudo-arc. Of course, the difficult part is the construction of the chains $\mathcal{U}_r$.

3. The expression "crooked enough" means that $\mathcal{U}_{r+1}$ is crooked in the chain $72(\mathcal{U}_r)$, this is the chain constructed by taking the union of the first 72 links of $\mathcal{U}_r$, then the union of the next 72 links of $\mathcal{U}_r$ and so on. This helps to make the construction of $\mathcal{U}_{r+1}$ easier, but some technical details are needed to handle chains of the form $s(\mathcal{U}_r)$; the first section (Lemmas 2.1 to 2.8) are essentially devoted to this end.

4. The difference between the case proved in this paper ($k \geq 3$) and the theorem proved by H. Cook ($k = 2$) is that we have "more space" in which we can work. We work with rectangular boxes in $\mathbb{R}^k$ so we can dig channels in these boxes without disconnecting them.

This is the reason for which we need to construct our chains with links that are finite connected unions of boxes. The basic facts about boxes and street arcs are developed in 4.1 to 4.4.

5. The chains $\mathcal{U}_r$ are constructed inductively. Supposing that $\mathcal{U}_r$ has been constructed we proceed as follows. Given a component Q of K, since Q is either a one-point set or a pseudo-arc, we know that it is possible to construct an appropriate chain covering Q that is crooked in $\mathcal{U}_r$. Lemmas 3.1, 5.1 and 5.3, and Theorems 3.2 and 5.2 are useful to extend the chains covering the components of K to a finite union of pairwise disjoint chains covering K.

6. Finally, once we can cover K by a finite number of appropriate pairwise disjoint chains crooked in $\mathcal{U}_r$, we need to extend these chains to an appropiate single chain, this is made in Theorem 5.5.

The following lemma is easy to prove.

Lemma 2.1. *Let X be a continuum, A a subcontinuum of X, $\mathcal{U} = \{U_1, \ldots, U_n\}$ a taut chain that properly covers A and $\mathcal{V} = \{V_1, \ldots, V_n\}$ a sequence of open sets such that $A \subset \cup\mathcal{V}$ and $V_i \subset U_i$ for each $i \in \{1, \ldots, n\}$. Then $\mathcal{V}$ is a taut chain that properly covers A.*

Let $\mathcal{D} = \{D_1, \ldots, D_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$ be chains in a continuum. The chain $\mathcal{D}$ *refines* $\mathcal{C}$ if the closure of every link of $\mathcal{D}$ is contained in one link of $\mathcal{C}$. The chain $\mathcal{D}$ is *crooked* in the chain $\mathcal{C}$ if $\mathcal{D}$ refines $\mathcal{C}$ and, for any indices k, l, i and j with $D_k \subset C_i$, $D_l \subset C_j$ and $i + 2 < j$, there exist indices r and s, with either $k < r < s < l$ or $k > r > s > l$, such that $D_r \subset C_{j-1}$ and $D_s \subset C_{i+1}$. Given $s \in \mathbb{N}$, if $n = sk + r$ and $r \in \{0, \ldots, s-1\}$, the *s-chain* of $\mathcal{C}$ is the chain $s(\mathcal{C}) = \{C_1 \cup \ldots \cup C_s; C_{s+1} \cup \ldots \cup C_{2s}; \ldots; C_{(k-2)s+1} \cup \ldots \cup C_{(k-1)s}; C_{(k-1)s+1} \cup \ldots \cup C_{ks} \cup \ldots \cup C_{ks+r}\}$. Notice that each link of $s(\mathcal{C})$ is the union of at least s links of $\mathcal{C}$. We denote by $\mathcal{D} * \mathcal{C}$ the sequence $\{D_1, \ldots, D_m, C_1, \ldots, C_n\}$, the operation $*$ is extended to a finite family of chains in the natural way. Given $1 \leq i \leq j \leq n$, let $\mathcal{C}(i, j)$ denote the subchain $\{C_i, \ldots, C_j\}$ of $\mathcal{C}$.

The following lemma can be easily proved.

Lemma 2.2. *For every $s, t \in \mathbb{N}$ and each chain $\mathcal{C}$ in a continuum X, $t(s(\mathcal{C})) = ts(\mathcal{C})$.*

Lemma 2.3. *Suppose that X is a continuum and there exist $s \in \mathbb{N}$, points $p \neq q$ in X and a sequence $\{\mathcal{U}_k\}_{k=1}^{\infty}$ of taut chains in X such that for each $k \in \mathbb{N}$,*

(a) $\mathcal{U}_k$ is a chain from p to q,

(b) $\mathcal{U}_{k+1}$ is crooked in $s(\mathcal{U}_k)$ and

(c) mesh$(\mathcal{U}_k) < \frac{1}{k}$.

Let $P = \cap\{cl_X(\cup\mathcal{U}_k) : k \in \mathbb{N}\}$. Then P is a pseudo-arc.

Proof. Clearly, P is a chainable continuum. Using a standard argument (see for example [6, p. 39]) it is possible to show that P is hereditarily indecomposable. Thus, P is a pseudo-arc (see [2, Theorem 1]). □

Lemma 2.4. *Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ and $\mathcal{V} = \{V_1, \ldots, V_m\}$ be chains in a continuum X such that $\mathcal{V}$ refines $\mathcal{U}$. Let $1 \leq i < j \leq n$. Then*

(a) if there exist $1 \leq a < b \leq m$ such that $V_a \subset U_i$ and $V_b \subset U_j$ (or $V_b \subset U_i$ and $V_a \subset U_j$), then for each $k \in \{i, \ldots, j\}$, there exists $c \in \{a, \ldots, b\}$ such that $V_c \subset U_k$,

(b) if there exist $1 \leq a < b \leq m$ such that $V_a \cap U_i \neq \emptyset$ and $V_b \cap U_j \neq \emptyset$ (or $V_b \cap U_i \neq \emptyset$ and $V_a \cap U_j \neq \emptyset$), then for each $i < k < j$, there exists $c \in \{a, \ldots, b\}$ such that $V_c \subset U_k$,

(c) if $a \in \{1, \ldots, m\}$ and $V_a \subset U_i \cup \ldots \cup U_j$, then there exists $k \in \{i, \ldots, j\}$ such that $V_a \subset U_k$ and

(d) if $r, s \in \{1, \ldots, m\}$ are such that $r \leq s$, $V_r \subset U_1$ and $V_s \subset U_n$, then for every $1 \leq c \leq e \leq n$, there exist $r \leq a \leq b \leq s$ such that $V_a \subset U_c$, $V_b \subset U_e$ and $\cup\mathcal{V}(a, b) \subset \cup\mathcal{U}(c, e)$.

Proof. We prove (a). The proof of (b) is similar. Suppose that $V_a \subset U_i$ and $V_b \subset U_j$, the other case is similar. Suppose also that no V_c ($c \in \{a, \ldots, b\}$) is contained in U_k. Then $i < k < j$. Since $\mathcal{V}$ refines $\mathcal{U}$, $V_a \cup \ldots \cup V_b \subset (U_1 \cup \ldots \cup U_{k-1}) \cup (U_{k+1} \cup \ldots \cup U_n)$. Let $e = \max\{g \in \{a, \ldots, b\} : V_g \subset U_1 \cup \ldots \cup U_{k-1}\}$. Notice that $e < b$ and $V_{e+1} \subset U_{k+1} \cup \ldots \cup U_n$. So, $V_e \cap V_{e+1} = \emptyset$, a contradiction.

(c) and (d) are easy to prove. □

Lemmas 2.5 and 2.6 can be proved by applying Lemma 2.4.

Lemma 2.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be chains in a continuum X such that $\mathcal{V}$ is crooked in $\mathcal{U}$. Then $\mathcal{V}$ is crooked in $s(\mathcal{U})$ for each $s \in \mathbb{N}$.*

Lemma 2.6. *Let X be a continuum and let $\mathcal{U} = \{U_1, \ldots, U_n\}$, $\mathcal{V} = \{V_1, \ldots, V_m\}$ and $\mathcal{W} = \{W_1, \ldots, W_m\}$ be chains in X such that $\mathcal{V}$ is crooked in $\mathcal{U}$ and $W_i \subset V_i$ for each $i \in \{1, \ldots, m\}$. Then $\mathcal{W}$ is crooked in $s(\mathcal{U})$ for each $s \geq 2$.*

Lemma 2.7. *Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ and $\mathcal{V} = \{V_1, \ldots, V_m\}$ be chains in a continuum X such that $\mathcal{V}$ is crooked in $\mathcal{U}$. Let $a \in \{1, \ldots, m-1\}$. Let $W_1, \ldots, W_{a+1}$, $Y_{a-1}, \ldots, Y_1$ be open subsets of X such that the family $\mathcal{Z} = \{Y_1, \ldots, Y_{a-1}, W_1, \ldots, W_{a+1}, V_{a+2}, \ldots, V_m\}$ is a chain and satisfies:*

(a) for each $i \in \{1, \ldots, a+1\}$, $W_i \subset V_i$ and

(b) for each $i \in \{1, \ldots, a-1\}$, $Y_i \subset V_{a-i+1}$.

Then $\mathcal{Y}$ is crooked in $2(\mathcal{U})$.

Proof. Let $2(\mathcal{U}) = \{R_1, \ldots, R_k\}$. Let $Z_1 = Y_1, \ldots, Z_{a-1} = Y_{a-1}$, $Z_a = W_1, \ldots, Z_{2a} = W_{a+1}$, $Z_{2a+1} = V_{a+2}, \ldots, Z_{a+m-1} = V_m$. Let $i, j \in \{1, \ldots, k\}$ and $h, l \in \{1, \ldots, a+m-1\}$ be such that $i + 2 < j$, $Z_h \subset R_i$ and $Z_l \subset R_j$. Since $\mathcal{Z}$ refines $\mathcal{U}$, Lemma 2.4 (c) implies that there exist $i_0, j_0 \in \{1, \ldots, n\}$ such that $Z_h \subset U_{i_0} \subset R_i$ and $Z_l \subset U_{j_0} \subset R_j$. Notice that $i_0 + 4 < j_0$.

Let $u, v \in \{1, \ldots, m\}$ be defined as

$$u = \begin{cases} a - h + 1, & \text{if } 1 \leq h \leq a - 1, \\ h - a + 1, & \text{if } a \leq h, \end{cases} \quad \text{and}$$
$$v = \begin{cases} a - l + 1, & \text{if } 1 \leq l \leq a - 1, \\ l - a + 1, & \text{if } a \leq l. \end{cases}$$

Then $Z_h \subset V_u$ and $Z_l \subset V_v$. Let $i_1, j_1 \in \{1, \ldots, n\}$ be such that $V_u \subset U_{i_1}$ and $V_v \subset U_{j_1}$.

Since $U_{i_1} \cap U_{i_0} \neq \emptyset$ and $U_{j_1} \cap U_{j_0} \neq \emptyset$, we have $i_1 + 2 < j_1$.

Let $i', j' \in \{1, \ldots, k\}$ be such that $U_{i_1+1} \subset R_{i'}$ and $U_{j_1-1} \subset R_{j'}$. Since $U_{i_1} \cap U_{i_0} \neq \emptyset$ and $U_{i_0} \subset R_i$, we have $i_1 + 1 \in \{i_0, i_0 + 1, i_0 + 2\}$ and $i' \in \{i, i+1\}$. Similarly, $j' \in \{j-1, j\}$.

We analyze the case that $h < l$. The case $l < h$ is similar.

We claim that there exist $h < r_0 < s_0 < l$ such that $Z_{r_0} \subset U_{j_1-1}$ and $Z_{s_0} \subset U_{i_1+1}$. We analyze 6 cases.

Case 1. $1 \leq h < l \leq a - 1$.

In this case $V_{a-h+1} = V_u \subset U_{i_1}$ and $V_{a-l+1} = V_v \subset U_{j_1}$. Since $\mathcal{V}$ is crooked in $\mathcal{U}$, there exist $a-l+1 < r < s < a-h+1$ such that $V_r \subset U_{i_1+1}$ and $V_s \subset U_{j_1-1}$. Since $h < a-s+1 < a-r+1 < l$, $Z_{a-s+1} = Y_{a-s+1} \subset V_s$ and $Z_{a-r+1} = Y_{a-r+1} \subset V_r$, we define $r_0 = a - s + 1$ and $s_0 = a - r + 1$.

The following two cases are similar to Case 1 and the existence of r_0 and s_0 is a consequence of the crookedness of $\mathcal{V}$ in $\mathcal{U}$.

Case 2. $a \leq h < l \leq 2a$.

Case 3. $2a + 1 \leq h < l \leq a + m - 1$.

Case 4. $1 \leq h \leq a - 1$ and $a \leq l \leq 2a$.

In this case $u = a - h + 1$, $v = l - a + 1$, $Z_h = Y_h$, $1 \leq l - a + 1 \leq a + 1$, $V_{a-h+1} = V_u \subset U_{i_1}$, $W_{l-a+1} = Z_l$ and $V_{l-a+1} = V_v \subset U_{j_1}$. Since $i_1 + 2 < j_1$, $V_u \subset U_{i_1}$ and $V_v \subset U_{j_1}$, we have $u \neq v$. So, we only have to consider two subcases.

Subcase 4.1. $a - h + 1 < l - a + 1$.

Since $\mathcal{V}$ is crooked in $\mathcal{U}$, there exist $a - h + 1 < r < s < l - a + 1$ such that $V_r \subset U_{j_1-1}$ and $V_s \subset U_{i_1+1}$. Since $h < a < 2a - h < r + a - 1 < s + a - 1 < l \leq 2a$, $Z_{r+a-1} = W_r \subset V_r$ and $Z_{s+a-1} = W_s \subset V_s$, we can define $r_0 = r + a - 1$ and $s_0 = s + a - 1$.

Subcase 4.2. $l-a+1 < a-h+1$.

Since $\mathcal{V}$ is crooked in $\mathcal{U}$, there exist $a-h+1 > r > s > l-a+1$ such that $V_r \subset U_{j_1-1}$ and $V_s \subset U_{i_1+1}$. Since $1 \leq h = a-(a-h+1)+1 < a-r+1 < a-s+1 < a-(l-a+1)+1 = 2a-l \leq a \leq l$, $Z_{a-r+1} = Y_{a-r+1} \subset V_r$ and $Z_{a-s+1} = Y_{a-s+1} \subset V_s$, we can define $r_0 = a-r+1$ and $s_0 = a-s+1$.

Case 5. $a \leq h \leq 2a$ and $2a+1 \leq l \leq a+m-1$.

In this case $u = h-a+1$, $v = l-a+1$, $Z_h = W_{h-a+1}$, $Z_l = V_{l-a+1}$, $V_{h-a+1} \subset U_{i_1}$ and $V_{l-a+1} \subset U_{j_1}$. Since $\mathcal{V}$ is crooked in $\mathcal{U}$, there exist $h-a+1 < r < s < l-a+1$ such that $V_r \subset U_{j_1-1}$ and $V_s \subset U_{i_1+1}$. Notice that $h < r+a-1 < s+a-1 < l$, $Z_{r+a-1} \subset V_r$ (in both cases, when $Z_{r+a-1} = W_r$ and when $Z_{r+a-1} = V_r$) and $Z_{s+a-1} \subset V_s$. Thus, we can define $r_0 = r+a-1$ and $s_0 = s+a-1$.

Case 6. $1 \leq h \leq a-1$ and $2a+1 \leq l \leq a+m-1$.

In this case $Z_h = Y_h$, $u = a-h+1$, $Z_l = V_{l-a+1}$ and $V_{a-h+1} \subset U_{i_1}$. Since $h \leq a-1 < 2a-h < 2a+1 \leq l$ and $Z_{2a-h} = W_{a-h+1} \subset V_{a-h+1} = V_u$, we can apply Case 5 to Z_{2a-h} and Z_l to obtain r_0, s_0 such that $h < 2a-h < r_0 < s_0 < l$, $Z_{r_0} \subset U_{j_1-1}$ and $Z_{s_0} \subset U_{i_1+1}$.

This completes the proof of the existence of r_0 and s_0.

Since $Z_{r_0} \subset U_{j_1-1} \subset R_{j'}$, $Z_{s_0} \subset U_{i_1+1} \subset R_{i'}$, $i' \in \{i, i+1\}$ and $j' \in \{j-1, j\}$, applying Lemma 2.4 (a), we obtain that there exist $r_0 \leq r_1 < s_1 \leq s_0$ such that $Z_{r_1} \subset R_{j-1}$ and $Z_{s_1} \subset R_{i+1}$. Therefore, $\mathcal{Z}$ is crooked in $2(\mathcal{U})$. $\square$

Lemma 2.8. *Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ and $\mathcal{V} = \{V_1, \ldots, V_m\}$ be chains in a continuum X such that $\mathcal{V}$ is crooked in $\mathcal{U}$. Suppose that there exists a finite sequence $k_0, \ldots, k_{2r+1}$ such that $1 = k_0$, $k_{2r+1} = m$, $k_1 \leq k_2$, $k_3 \leq k_4$, $\ldots, k_{2r-1} \leq k_{2r}$, $k_0 + 2 < k_1$, $k_2 + 2 < k_3$, $\ldots$, $k_{2r} + 2 < k_{2r+1}$, there exist two subsets $\{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_r\}$ of $\{1, \ldots, n\}$, and a family $\{\mathcal{W}_1, \ldots, \mathcal{W}_r\}$ of chains crooked in $\mathcal{U}$, where for each $a \in \{1, \ldots, r\}$, $\mathcal{W}_a = \{W_1^{(a)}, \ldots, W_{s_a}^{(a)}\}$, moreover the following holds for each $a \in \{1, \ldots, r\}$:*

(a) $i_a \leq j_a$,

(b) $W_1^{(a)} \cap V_c \neq \emptyset$ if and only if $c = k_{2a-1} - 1$ and $W_{s_a}^{(a)} \cap V_c \neq \emptyset$ if and only if $c = k_{2a} + 1$,

(c) $W_1^{(a)} \cup V_{k_{2a-1}} \subset U_{i_a}$ and $W_{s_a}^{(a)} \cup V_{k_{2a}} \subset U_{j_a}$,

(d) $(\cup \mathcal{W}_a) \cup (V_{k_{2a-1}} \cup \ldots \cup V_{k_{2a}}) \subset U_{i_a} \cup \ldots \cup U_{j_a}$,

(e) $(W_2^{(a)} \cup \ldots \cup W_{s_a-1}^{(a)}) \cap (V_1 \cup \ldots \cup V_m) = \emptyset$ and

(f) $\cup \mathcal{W}_1, \ldots, \cup \mathcal{W}_r$ are pairwise disjoint.

Let

$$\mathcal{Y} = \mathcal{V}(1, k_1 - 1) * \mathcal{W}_1 * \mathcal{V}(k_2 + 1, k_3 - 1) * \mathcal{W}_2 * \ldots * \mathcal{V}(k_{2(r-1)} + 1, k_{2r-1} - 1) * \mathcal{W}_r * \mathcal{V}(k_{2r} + 1, k_{2r+1}).$$

Then $\mathcal{Y}$ is a chain that is crooked in $3(\mathcal{U})$.

Proof. It is easy to show that $\mathcal{Y}$ is a chain that refines $\mathcal{U}$ (and then it refines $3(\mathcal{U})$). Let $3(\mathcal{U}) = \{Z_1, \ldots, Z_f\}$, $\mathcal{Y} = \{Y_1, \ldots, Y_e\}$ and $\mathcal{Y}_0 = \mathcal{V}(1, k_1 - 1) \cup \mathcal{V}(k_2 + 1, k_3 - 1) \cup \ldots \cup \mathcal{V}(k_{2r} + 1, k_{2r+1})$.

In order to see that $\mathcal{Y}$ is crooked in $3(\mathcal{U})$, suppose that $i, j \in \{1, \ldots, f\}$ and $h, l \in \{1, \ldots, e\}$ are such that $i + 2 < j$, $Y_h \subset Z_i$ and $Y_l \subset Z_j$. We may assume that $h < l$. The other case is similar.

By Lemma 2.4 (c), there exist links U_b and U_c of $\mathcal{U}$ such that $Y_h \subset U_b \subset Z_i$ and $Y_l \subset U_c \subset Z_j$. In the case that $Y_h \in \mathcal{W}_a$ for some $a \in \{1, \ldots, r\}$, $Y_h \subset \cup\mathcal{W}_a \subset U_{i_a} \cup \ldots \cup U_{j_a}$. By Lemma 2.4 (c), b can be chosen in the set $\{i_a, \ldots, j_a\}$. Similarly, in the case that $Y_l \in \mathcal{W}_a$ for some $a \in \{1, \ldots, r\}$, c can be chosen in the set $\{i_a, \ldots, j_a\}$.

Since each Z_a contains at least three links of $\mathcal{U}$ and $2 < j - i$, we have that $7 \leq c - b$.

We choose $s, t \in \{1, \ldots, m\}$ in the following way.

In the case that $Y_h \in \mathcal{Y}_0$, there exists $s \in \{1, \ldots, m\}$ such that $Y_h = V_s$. In the case that $Y_h \notin \mathcal{Y}_0$, there exists $a(h) \in \{1, \ldots, r\}$ such that $Y_h \in \mathcal{W}_{a(h)}$, so $Y_h \subset U_b$ and $b \in \{i_{a(h)}, \ldots, j_{a(h)}\}$. By Lemma 2.4 (a), there exists $s \in \{k_{2a(h)-1}, \ldots, k_{2a(h)}\}$ such that $V_s \subset U_b$. In any case, $V_s \subset U_b$. Similarly, when $Y_l \in \mathcal{Y}_0$, there exists $t \in \{1, \ldots, m\}$ such that $Y_l = V_t$. In the case that $Y_l \notin \mathcal{Y}_0$, there exists $a(l) \in \{1, \ldots, r\}$ such that $Y_l \in \mathcal{W}_{a(l)}$, so there exists $t \in \{k_{2a(l)-1}, \ldots, k_{2a(l)}\}$ such that $V_t \subset U_c$ (and $c \in \{i_{a(l)}, \ldots, j_{a(l)}\}$). In any case, $V_t \subset U_c$.

Since $\mathcal{V}$ is crooked in $\mathcal{U}$, there exist $u, v \in \{1, \ldots, m\}$ such that $V_u \subset U_{c-1}$, $V_v \subset U_{b+1}$ and either $s < u < v < t$ or $t < v < u < s$. We suppose that $s < u < v < t$. The case that $t < v < u < s$ is similar. We analyze 2 cases.

Case 1. There is no $a \in \{1, \ldots, r\}$ such that $\mathcal{V}(k_{2a-1}, k_{2a})$ contains two elements of the set $\{V_s, V_u, V_v, V_t\}$.

Notice that the chain $\mathcal{V}$ can be divided in the following sequence of subchains:

$$\mathcal{V}(1, k_1 - 1),\ \mathcal{V}(k_1, k_2),\ \mathcal{V}(k_2 + 1, k_3 - 1),\ \mathcal{V}(k_3, k_4), \ldots,$$
$$\mathcal{V}(k_{2(r-1)} + 1, k_{2r-1} - 1),\ \mathcal{V}(k_{2r-1}, k_{2r}),\ \mathcal{V}(k_{2r} + 1, k_{2r+1}).$$

Our assumption in this case says that no two elements of $\{V_s, V_u, V_v, V_t\}$ belong to the same of the following subchains of $\mathcal{V}$:

$$\mathcal{V}(k_1, k_2),\ \mathcal{V}(k_3, k_4),\ \ldots\ ,\ \mathcal{V}(k_{2r-1}, k_{2r}).$$

For each $w \in \{u, v\}$, we will choose $x(w) \in \{1, \ldots, e\}$ according to the following. If $V_w \notin \mathcal{V}(k_1, k_2) \cup \mathcal{V}(k_3, k_4) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, by construction of $\mathcal{Y}$, there exists $x(w) \in \{1, \ldots, e\}$ such that $V_w = Y_{x(w)}$. If $V_w \in \mathcal{V}(k_{2a(w)-1}, k_{2a(w)})$, for some $a(w) \in \{1, \ldots, r\}$, by Lemma 2.4 (c), there exists $b(w) \in \{i_{a(w)}, \ldots, j_{a(w)}\}$ such that $V_w \subset U_{b(w)}$. By Lemma 2.4 (a), there exists $z(w) \in \{1, \ldots, s_{a(w)}\}$ such that $W_{z(w)}^{(a(w))} \subset U_{b(w)}$. Let $x(w) \in \{1, \ldots, e\}$ be such that $Y_{x(w)} = W_{z(w)}^{(a(w))}$.

We claim that $x(u) < x(v)$. We consider the possible cases.

If $V_u, V_v \notin \mathcal{V}(k_1, k_2) \cup \mathcal{V}(k_3, k_4) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, then $V_u = Y_{x(u)}$ and $V_v = Y_{x(v)}$. By definition of the operator $*$, the elements of $\mathcal{Y}$ that belong to $\mathcal{V}(1, k_1 - 1) \cup \mathcal{V}(k_2 + 1, k_3 - 1) \cup \ldots \cup \mathcal{V}(k_{2(r-1)} + 1, k_{2r-1} - 1) \cup \mathcal{V}(k_{2r} + 1, k_{2r+1})$, preserve their original order as in the chain $\mathcal{V}$. Since $u < v$, we conclude that $x(u) < x(v)$.

In the case that $V_u, V_v \in \mathcal{V}(k_1, k_2) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, we have $V_u \in \mathcal{V}(k_{2a(u)-1}, k_{2a(u)})$ and $V_v \in \mathcal{V}(k_{2a(v)-1}, k_{2a(v)})$. Notice that our assumption for this case implies that $V_v \notin \mathcal{V}(k_{2a(u)-1}, k_{2a(u)})$ and, since $u < v$, the chain $\mathcal{V}(k_{2a(u)-1}, k_{2a(u)})$ precedes the chain $\mathcal{V}(k_{2a(v)-1}, k_{2a(v)})$ (in $\mathcal{Y}$), so $a(u) < a(v)$. Thus, in $\mathcal{Y}$, the subchain $\mathcal{W}_{a(u)}$ precedes the subchain $\mathcal{W}_{a(v)}$. Hence, the link $Y_{x(u)} = W^{(a(u))}_{z(u)}$ precedes the link $Y_{x(v)} = W^{(a(v))}_{z(v)}$. Therefore, $x(u) < x(v)$.

The last two cases,

$V_u \in \mathcal{V}(k_1, k_2) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, $V_v \notin \mathcal{V}(k_1, k_2) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$ and

$V_u \notin \mathcal{V}(k_1, k_2) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, $V_v \in \mathcal{V}(k_1, k_2) \cup \ldots \cup \mathcal{V}(k_{2r-1}, k_{2r})$, can be treated in a similar way.

This ends the proof that $x(u) < x(v)$.

Using similar arguments we can prove that $h < x(u)$ and $x(v) < l$.

In the case that $Y_{x(u)} = W^{(a(u))}_{z(u)}$, we have $Y_{x(u)} \cup V_u \subset U_{b(u)}$. Since $V_u \subset U_{c-1}$, we conclude that $b(u) \in \{c-2, c-1, c\}$, so $Y_{x(u)}$ is contained in one of the links U_{c-2}, U_{c-1} or U_c. The other possibility is that $Y_{x(u)} = V_u \subset U_{c-1}$. Thus, in any case $Y_{x(u)}$ is contained in one of the links U_{c-2}, U_{c-1} or U_c. Since $Y_h \subset U_b$, $b + 7 \leq c$ and $\mathcal{Y}$ refines $\mathcal{U}$, by Lemma 2.4 (a), there exists $h < h_1 \leq x(u)$ such that $Y_{h_1} \subset U_{c-3} \subset Z_{j-1}$.

Similarly, there exists $x(v) \leq l_1 < l$ such that $Y_{l_1} \subset U_{b+3} \subset Z_{i+1}$.

Therefore, Y_h, Y_{h_1}, Y_{l_1} and Y_l satisfy the condition that defines crookedness.

Case 2. There exist $a \in \{1, \ldots, r\}$ and two elements of the set $\{V_s, V_u, V_v, V_t\}$ that belong to $\mathcal{V}(k_{2a-1}, k_{2a})$.

Let $w_1, w_2 \in \{1, \ldots, e\}$ be such that $Y_{w_1} = W^{(a)}_1$ and $Y_{w_2} = W^{(a)}_{s_a}$. Then $w_2 - w_1 = s_a - 1$ and $w_2 = w_1 + s_a - 1$.

Subcase 2.1. $Y_h, Y_l \in \mathcal{W}_a \subset \mathcal{Y}$.

By Lemma 2.5, $\mathcal{W}_a$ is crooked in $3(\mathcal{U})$. Thus, it is possible to find elements in $\mathcal{W}_a \subset \mathcal{Y}$ that satisfy the condition that defines crookedness of $\mathcal{Y}$ in $3(\mathcal{U})$.

Subcase 2.2. $Y_h \notin \mathcal{W}_a$.

In this subcase, we claim that $h < w_1$.

In this case (2), there exists $x \in \{u, v, t\}$ such that $V_x \in \mathcal{V}(k_{2a-1}, k_{2a})$. Thus, $k_{2a-1} \leq x \leq k_{2a}$.

If $Y_h \in \mathcal{Y}_0$, by definition $V_s = Y_h$. In the order of the chain $\mathcal{Y}$, Y_h is before or after the subchain $\mathcal{W}_a$. Since the chain $\mathcal{Y}$ does not have any of the elements of $\mathcal{V}(k_{2a-1}, k_{2a})$, we have $s < k_{2a-1}$ or $k_{2a} < s$. We are assuming that $s < u < v < t$. Hence, $s < k_{2a-1}$. Therefore, in the order of the chain $\mathcal{Y}$, Y_h is before the subchain $\mathcal{W}_a$. Since $Y_{w_1} = W_1^{(a)} \in \mathcal{W}_a$, we conclude that $h < w_1$.

Now, we analyze the case $Y_h \notin \mathcal{Y}_0$. By definition $Y_h \in \mathcal{W}_{a(h)}$ and $s \in \{k_{2a(h)-1}, \ldots, k_{2a(h)}\}$. Notice that $a(h) \neq a$. Since $k_{2a(h)-1} \leq s \leq k_{2a(h)}$, $k_{2a-1} \leq x \leq k_{2a}$ and $s < x$, we obtain that $a(h) < a$. Hence, in the chain $\mathcal{Y}$, the subchain $\mathcal{W}_{a(h)}$ is before the subchain $\mathcal{W}_a$. Thus, $h < w_1$.

This completes the proof of the inequality $h < w_1$.

By the hypothesis for Case 2, we conclude that either $\{V_u, V_v\} \subset \mathcal{V}(k_{2a-1}, k_{2a})$, $\{V_v, V_t\} \subset \mathcal{V}(k_{2a-1}, k_{2a})$ or $\{V_u, V_t\} \subset \mathcal{V}(k_{2a-1}, k_{2a})$. Since $u < v < t$, we have that either $\{V_u, V_v\} \subset \mathcal{V}(k_{2a-1}, k_{2a})$ or $\{V_v, V_t\} \subset \mathcal{V}(k_{2a-1}, k_{2a})$. Thus, $V_v \in \mathcal{V}(k_{2a-1}, k_{2a})$ and there exists $y \in \{u, t\}$ such that $V_y \in \mathcal{V}(k_{2a-1}, k_{2a})$.

By Lemma 2.4 (c), $V_v \subset U_z$ for some $z \in \{i(a), \ldots, j(a)\}$. Since $V_v \subset U_{b+1}$, we have $z \in \{b, b+1, b+2\}$. Thus, $i(a) \leq b+2$.

By Lemma 2.4 (c), $V_y \subset U_{z_1}$ for some $z_1 \in \{i(a), \ldots, j(a)\}$. Since either $V_y \subset U_{c-1}$ or $V_y \subset U_c$, we have $z_1 \in \{c-2, c-1, c, c+1\}$. Thus, $c-2 \leq j(a)$.

We divide Subcase 2.2. in two subsubcases.

Subsubcase 2.2.1. $V_t \in \mathcal{V}(k_{2a-1}, k_{2a})$.

In this subsubcase $t \in \{k_{2a-1}, \ldots, k_{2a}\}$. By the way we choose V_t, since $V_t \notin \mathcal{Y}_0$, we have $Y_l \neq V_t$ and $Y_l \notin \mathcal{Y}_0$. Then $Y_l \in \mathcal{W}_{a(l)}$, $t \in \{k_{2a(l)-1}, \ldots, k_{2a(l)}\}$, $V_t \subset U_c$ and $c \in \{i_{a(l)}, \ldots, j_{a(l)}\}$. Since $t \in \{k_{2a(l)-1}, \ldots, k_{2a(l)}\} \cap \{k_{2a-1}, \ldots, k_{2a}\}$, we have $a = a(l)$.

Let $c_1 \in \{1, \ldots, s_a\}$ be such that $Y_l = W_{c_1}^{(a)}$. Since $Y_{w_1} = W_1^{(a)}$, the definition of the operator $*$ implies that $w_1 \leq l$. So, $h < w_1 \leq l$. Since $i(a) \leq b+2$ and $7 \leq c-b$, we have $i(a)+5 \leq c$. Since $Y_{w_1} = W_1^{(a)} \subset U_{i(a)}$ and $W_{c_1}^{(a)} = Y_l \subset U_c$, the crookedness of $\mathcal{W}_a$ in $\mathcal{U}$, implies that there exist $c_2, c_3 \in \{1, \ldots, c_1\}$ such that $1 < c_2 < c_3 < c_1$, $W_{c_2}^{(a)} \subset U_{c-1}$ and $W_{c_3}^{(a)} \subset U_{i(a)+1}$. By Lemma 2.4 (a), there exist $c_4 \in \{1, \ldots, c_2\}$ and $c_5 \in \{c_3, \ldots, c_1\}$ such that $W_{c_4}^{(a)} \subset U_{c-3}$ and $W_{c_5}^{(a)} \subset U_{b+3}$. Notice that $W_{c_4}^{(a)} = Y_{w_1+c_4-1}$, $W_{c_3}^{(a)} = Y_{w_1+c_3-1}$, $h < w_1 \leq w_1+c_4-1 \leq w_1+c_2-1 < w_1+c_5-1 < w_1+c_1-1 = l$, $Y_h \subset U_b \subset Z_i$, $Y_{w_1+c_4-1} \subset U_{c-3} \subset Z_{j-1}$, $Y_{w_1+c_5-1} \subset U_{b+3} \subset Z_{i+1}$ and $Y_l \subset Z_j$. Thus, Y_h, $Y_{w_1+c_4-1}$, $Y_{w_1+c_5-1}$ and Y_l satisfy the condition that defines crookedness (of $\mathcal{Y}$ in $3(\mathcal{U})$).

Subsubcase 2.2.2. $V_t \notin \mathcal{V}(k_{2a-1}, k_{2a})$.

Proceeding as in the proof that $h < w_1$, it can be shown that $w_2 < l$. Since $i(a) \leq b+2 < c-2 \leq j(a)$, by Lemma 2.4 (a), there exist $c_2, c_3 \in \{1, \ldots, s_a\}$ such that $c_2 < c_3$, $W_{c_2}^{(a)} \subset U_{b+2}$ and $W_{c_3}^{(a)} \subset U_{c-2}$. Since $b+5 \leq c-2$, crookedness of $\mathcal{W}_a$ implies that there exist $c_4, c_5 \in \{c_2, \ldots, c_3\}$ such that $c_2 < c_4 < c_5 < c_3$, $W_{c_4}^{(a)} \subset U_{c-3}$ and $W_{c_5}^{(a)} \subset U_{b+3}$. Notice that $h < w_1 < w_1+c_4-1 < w_1+c_5-1 < w_1+s_a-1 = w_2 < l$, $Y_h \subset U_b \subset Z_i$, $Y_{w_1+c_4-1} = W_{c_4}^{(a)} \subset U_{c-3} \subset Z_{j-1}$, $Y_{w_1+c_5-1} = W_{c_5}^{(a)} \subset U_{b+3} \subset Z_{i+1}$ and $Y_l \subset Z_j$. Thus, Y_h, $Y_{w_1+c_4-1}$, $Y_{w_1+c_5-1}$ and Y_l satisfy the condition that defines crookedness (of $\mathcal{Y}$ in $3(\mathcal{U})$).

Subcase 2.3. $Y_l \notin \mathcal{W}_a$.

This subcase is similar to Subcase 2.2. $\square$

3. Covering Compact Sets

Given a continuum X, let 2^X and $C(X)$ be the respective hyperspaces of nonempty closed subsets and of subcontinua of X, endowed with the Hausdorff metric [5, Theorem 2.2]. We will use that 2^X and $C(X)$ are compact [5, Theorem 3.5 and Corollary 3.7]. Given a subset B of X, let $\langle B\rangle = \{A \in C(X) : A \subset B\}$. It is well known that if B is open in X, then $\langle B\rangle$ is open in $C(X)$.

Lemma 3.1. *Let X be a continuum. Let K be a closed subset of X, $U_1, \ldots, U_n$ open subsets of X and $C_1, \ldots, C_n$ pairwise distinct components of K such that $C_1 \subset U_1$, $\ldots$,$C_n \subset U_n$ and each component of K is contained in some U_i. Then there exist pairwise disjoint closed subsets $K_1, \ldots, K_n$ of X such that for each $i \in \{1, \ldots, n\}$, $C_i \subset K_i \subset U_i$ and $K = K_1 \cup \ldots \cup K_n$.*

Proof. We proceed by induction. If $n = 1$, define $K_1 = K$.

Suppose that the claim in the lemma is valid for some $n \geq 1$ and let K, $U_1, \ldots, U_{n+1}$ and $C_1, \ldots, C_{n+1}$ be as in the hypothesis.

Let $U = U_1 \cup \ldots \cup U_n$. Consider the sets $P = (K - U_{n+1}) \cup C_1 \cup \ldots \cup C_n$ and $Q = (K-U) \cup C_{n+1} \cup (\cup\{D : D$ is a component of K and D is not contained in any U_i with $i \in \{1, \ldots, n\}\})$. Notice that $P \cup Q \subset K$.

We see that Q is closed in X. Let $p \in K$ be such that $p = \lim p_m$, where for each $m \in \mathbb{N}$, $p_m \in D_m$, D_m is a component of K and D_m is not contained in any U_i with $i \in \{1, \ldots, n\}$. By the compactness of $C(X)$, we may assume that $\lim D_m = D_0$ for some $D_0 \in C(X)$. Notice that $D_0 \subset K$. Let D be the component of K such that $p \in D$. Since $p \in D_0$, we have $D_0 \subset D$. If there exists $i \in \{1, \ldots, n\}$ such that $D \subset U_i$, then $D_0 \subset U_i$ and there exists $m \in \mathbb{N}$ such that $D_m \subset U_i$, a contradiction. This shows that $D \subset Q$ and $p \in Q$. Therefore, Q is closed in X.

Suppose that there exists a component D of K such that $D \cap P \neq \emptyset$ and $D \cap Q \neq \emptyset$. By hypothesis, there exists $i \in \{1, \dots, n+1\}$ such that $D \subset U_i$. If $i = n+1$, since $D \cap P \neq \emptyset$, we have that $D = C_j$ for some $j \in \{1, \dots, n\}$. Thus, $D \subset U$. Since $D \cap Q \neq \emptyset$ and $D \subset U_j$, we conclude that $D = C_{n+1}$, a contradiction. In the case that $i \in \{1, \dots, n\}$, we have $D \subset U_i \subset U$. Since $D \cap Q \neq \emptyset$, we obtain that $D = C_{n+1} \subset U_{n+1}$. Since $D \cap P \neq \emptyset$, we conclude that $D = C_j$ for some $j \in \{1, \dots, n\}$, which is also a contradiction. We have shown that no component D of K intersects both sets P and Q. By [5, Theorem 12.9], there exist closed disjoint subsets K_0 and K_{n+1} of K such that $K = K_0 \cup K_{n+1}$, $P \subset K_0$ and $Q \subset K_{n+1}$.

Notice that K_0 is compact and $C_1, \dots, C_n$ are pairwise different components of K_0 such that $C_1 \subset U_1$, …, $C_n \subset U_n$. Given a component D of K_0, D is a component of K. Since $D \cap Q = \emptyset$, there exists $i \in \{1, \dots, n\}$ such that $D \subset U_i$.

We have shown that K_0, $U_1, \dots, U_n$ and $C_1, \dots, C_n$ satisfy the inductive hypothesis. Hence, there exist pairwise disjoint closed subsets $K_1, \dots, K_n$ of X such that for each $i \in \{1, \dots, n\}$, $C_i \subset K_i \subset U_i$ and $K_0 = K_1 \cup \dots \cup K_n$. It is easy to show that the sets $K_1, \dots, K_{n+1}$ satisfy the required properties. $\square$

Theorem 3.2. *Let X be a continuum and $\mathcal{B}_1$ a basis of open subsets of X such that $\mathcal{B}_1$ is closed under finite unions and under finite intersections. Suppose that K is a compact subspace of X such that each of its components is chainable. Then for each $\varepsilon > 0$, there exist $n \in \mathbb{N}$, components $C_1, \dots, C_n$ of K and taut ε-chains $\mathcal{U}_1, \dots, \mathcal{U}_n$ of X such that $cl_X(\cup\mathcal{U}_1), \dots, cl_X(\cup\mathcal{U}_n)$ are pairwise disjoint, $K \subset (\cup\mathcal{U}_1) \cup \dots \cup (\cup\mathcal{U}_n)$, $\mathcal{U}_1 \cup \dots \cup \mathcal{U}_n \subset \mathcal{B}_1$ and for each $i \in \{1, \dots, n\}$, $\mathcal{U}_i$ properly covers C_i.*

Proof. Given a component C of K, by the chainability of C it follows that there exists a family $\mathcal{D}_C = \{D_1^C, \dots, D_{n(C)}^C\}$ of closed subsets of X such that for each $i \in \{1, \dots, n(C)\}$, diameter$(D_i^C) < \varepsilon$, $D_i^C \cap D_j^C \neq \emptyset$ if and only if $|i - j| \leq 1$, $C = D_1^C \cup \dots \cup D_{n(C)}^C$, $D_1^C - D_2^C \neq \emptyset$ and $D_{n(C)}^C - D_{n(C)-1}^C \neq \emptyset$. By the normality of X, there exists a taut ε-chain $\mathcal{U}_C = \{U_1^C, \dots, U_{n(C)}^C\}$ of elements of $\mathcal{B}_1$ such that $D_i^C \subset U_i^C$ for each $i \in \{1, \dots, n(C)\}$, and in the case that $n(C) > 1$, we assume that $D_1^C - \mathrm{cl}_X(U_2^C) \neq \emptyset$ and $D_{n(C)}^C - \mathrm{cl}_X(U_{n(C)-1}^C) \neq \emptyset$.

Let $\mathcal{C} = \mathrm{cl}_{C(X)}(\{C : C \text{ is a component of } K\})$.

Since $\mathcal{C}_0 = \{D \in C(X) : D \subset K\}$ is a closed subset of $C(X)$, $\mathcal{C}_0$ contains $\mathcal{C}$. Since each $D \in \mathcal{C}_0$ is contained in some component C of K, we obtain that $D \subset C \subset \cup\mathcal{U}_C$. Then $\mathfrak{W} = \{\langle \cup\mathcal{U}_C \rangle : C \text{ is a component of } K\}$ is an open cover of $\mathcal{C}_0$ and then it is an open cover of the compact set $\mathcal{C}$.

Thus, there exist $n \in \mathbb{N}$ and $C_1, \ldots, C_n$ (pairwise distinct) components of K such that $\mathcal{C} \subset \langle \cup \mathcal{U}_{C_1} \rangle \cup \ldots \cup \langle \cup \mathcal{U}_{C_n} \rangle$.

Given a component C of K, C belongs to $\mathcal{C}$, so there exists $i \in \{1, \ldots, n\}$ such that $C \subset \cup \mathcal{U}_{C_i}$.

Thus, we can apply Lemma 3.1 to K, $\cup \mathcal{U}_{C_1}, \ldots, \cup \mathcal{U}_{C_n}$ and $C_1, \ldots, C_n$ and obtain that there exist pairwise disjoint closed subsets $K_1, \ldots, K_n$ of X such that, for each $i \in \{1, \ldots, n\}$, $C_i \subset K_i \subset \cup \mathcal{U}_{C_i}$ and $K = K_1 \cup \ldots \cup K_n$.

For each $i \in \{1, \ldots, n\}$, choose an element $V_i \in \mathcal{B}_1$ such that $K_i \subset V_i \subset \cup \mathcal{U}_{C_i}$. We also assume that $\mathrm{cl}_X(V_1), \ldots, \mathrm{cl}_X(V_n)$ are pairwise disjoint.

For each $i \in \{1, \ldots, n\}$, let $\mathcal{U}_i = \{V_i \cap U_1^{C_i}, \ldots, V_i \cap U_{n(C_i)}^{C_i}\}$. Given $j \in \{1, \ldots, n(C_i)-1\}$, $\emptyset \neq D_j^{C_i} \cap D_{j+1}^{C_i} \subset (V_i \cap U_j^{C_i}) \cap (V_i \cap U_{j+1}^{C_i})$. Thus, $\mathcal{U}_i$ is a taut ε-chain in X that properly covers C_i. Clearly, $\mathrm{cl}_X(\cup \mathcal{U}_1), \ldots, \mathrm{cl}_X(\cup \mathcal{U}_n)$ are pairwise disjoint and $K \subset (\cup \mathcal{U}_1) \cup \ldots \cup (\cup \mathcal{U}_n)$. □

4. Street Arcs and Boxes

Given a subset A of $\mathbb{R}^k$ and $\delta > 0$, let $N(A, \delta)$ be the union of δ-balls in $\mathbb{R}^k$ centered in points of A. A *street arc* is an arc α in $\mathbb{R}^k$ such that there exists a finite collection of arcs $\alpha_1, \ldots, \alpha_n$ such that $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ and each α_i is a convex segment parallel to one of the axis in $\mathbb{R}^k$, that is, α_i is of the form $\alpha_i = \{x_1\} \times \ldots \times \{x_{j-1}\} \times [a, b] \times \{x_{j+1}\} \times \ldots \times \{x_k\}$, where $j \in \{1, \ldots, k\}$, $a \leq b$ and $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k) \in \mathbb{R}^{k-1}$. We will use the basis for the topology in $\mathbb{R}^k$ that is defined as the collection of all finite unions of boxes, that is, we define $\mathcal{B}$ as the family of subsets of $\mathbb{R}^k$ that are finite unions of elements of $\mathcal{B}_0$, where $\mathcal{B}_0 = \{(a_1, b_1) \times \ldots \times (a_k, b_k) \subset \mathbb{R}^k : a_i < b_i \text{ for each } i \in \{1, \ldots, n\}\}$. The elements of $\mathcal{B}_0$ are *k-boxes.*

The following two lemmas are easy to prove.

Lemma 4.1. *The family $\mathcal{B}$ satisfies the following properties.*

(a) $\mathcal{B}$ is closed under finite intersections,

(b) if $U, V \in \mathcal{B}$, then $U-$ $cl_{\mathbb{R}^k}(V) \in \mathcal{B}$,

(c) the components of an element of $\mathcal{B}$ belong to $\mathcal{B}$.

Lemma 4.2. *Let $k \geq 3$. Let U be an open connected subset of $\mathbb{R}^k$ and let p and q be distinct points in U.*

(a) Suppose that $\alpha_1, \ldots, \alpha_n$ are pairwise disjoint subcontinua of $\mathbb{R}^k$ such that $\alpha_1 \cup \ldots \cup \alpha_n \subset U$ and each α_i is either a street arc or a one-point set, with end points a_i and b_i ($a_i = b_i$ when α_i is degenerate). Then there exists a street arc α in U, joining p and q, such that $\alpha_1 \cup \ldots \cup \alpha_n \subset \alpha$, and in the natural order of α, $p < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < q$.

(b) Suppose that α is a street arc with one end point q in U and let $p \in U - \alpha$. Then there exists a street arc $\beta \subset U$ such that β joins p to q and $\alpha \cap \beta = \{q\}$.

Lemma 4.3. *Let $k \geq 3$, $\mathcal{G} \subset \mathcal{B}$ be a finite set such that each $W \in \mathcal{G}$ is connected and let α be a street arc in $\mathbb{R}^k$. Then for each $\delta > 0$ there exists $V \in \mathcal{B}$ such that V is connected, $\alpha \subset V \subset N(\alpha, \delta)$ and for each $W \in \mathcal{G}$, $W-$ $cl_{\mathbb{R}^k}(V)$ is nonempty and connected.*

Proof. Let $\mathcal{D} = \{D_1, \ldots, D_n\} \subset \mathcal{B}_0$ be a family of boxes such that each element of $\mathcal{G}$ is union of elements of $\mathcal{D}$, where for each $i \in \{1, \ldots, n\}$, $D_i = (a_1^{(i)}, b_1^{(i)}) \times \ldots \times (a_k^{(i)}, b_k^{(i)})$. Let $\alpha = \alpha_1 \cup \ldots \cup \alpha_m$, where each α_i is a convex segment in $\mathbb{R}^k$, parallel to one of the canonical axis of $\mathbb{R}^k$.

For each $j \in \{1, \ldots, k\}$, let $\pi_j : \mathbb{R}^k \to \mathbb{R}$ be the projection on the j^{th}-coordinate. Then for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$, there exist $c(i,j) \leq e(i,j)$ such that $\pi_j(\alpha_i) = [c(i,j), e(i,j)]$, where $c(i,j) = e(i,j)$ for all but at most one j.

For each $l \in \{1, \ldots, k\}$, let $F_l = \{a_l^{(i)} : i \in \{1, \ldots, n\}\} \cup \{b_l^{(i)} : i \in \{1, \ldots, n\}\} \cup \{c(i,l) : i \in \{1, \ldots, m\}\} \cup \{e(i,l) : i \in \{1, \ldots, m\}\}$. Arrange the elements of F_l in an increasing sequence $f_1^{(l)} < \cdots < f_{r_l}^{(l)}$.

Thus, for each $i \in \{1, \ldots, n\}$ the family $\mathcal{E}_i = \{D_i \cap ([f_{u_1}^{(1)}, f_{u_1+1}^{(1)}] \times \ldots \times [f_{u_k}^{(k)}, f_{u_k+1}^{(k)}]) : u_v \in \{1, \ldots, r_v - 1\}$ for each $v \in \{1, \ldots, k\}\}$ is a subdivision of D_i in closed (in D_i) boxes, some of them having their complete boundary and some of them having a part (or nothing) of their boundary. Notice that α can touch only some portions of the edges of the boxes in $\mathcal{E}_i$. In particular, α does not touch the interior of any box in $\mathcal{E}_i$. Moreover, notice that the following property holds:

(a) if $p, q \in D_i$ and they do not belong to the same element of $\mathcal{E}_i$, then there exists a finite sequence $\{E_1, \ldots, E_w\}$ of elements in $\mathcal{E}_i$ such that $p \in E_1$, $q \in E_w$ and for each $j \in \{1, \ldots, w-1\}$, $E_j \cap E_{j+1}$ is a common $(k-1)$-dimensional face of E_j and E_{j+1}.

Let $\varepsilon = \frac{1}{4} \min(\{f_{u+1}^{(l)} - f_u^{(l)} : l \in \{1, \ldots, k\}$ and $u \in \{1, \ldots, r_l - 1\}\} \cup \{\delta\})$. Then no $(k-1)$-dimensional face of an element of $\mathcal{E}_1 \cup \ldots \cup \mathcal{E}_n$ can be covered by $N(\alpha, \varepsilon)$.

For each $j \in \{1, \ldots, m\}$, let $V_j = (\min(\pi_1(\alpha_j)) - \varepsilon, \max(\pi_1(\alpha_j)) + \varepsilon) \times \ldots \times (\min(\pi_k(\alpha_j)) - \varepsilon, \max(\pi_k(\alpha_j)) + \varepsilon)$. Then V_j is a 2ε-box covering α_j.

Let $V = V_1 \cup \ldots \cup V_k$.

Notice that $V \in \mathcal{B}$, V is connected and $\alpha \subset V \subset N(\alpha, \varepsilon)$.

Given $i \in \{1, \ldots, n\}$ and $E \in \mathcal{E}_i$, since $\mathrm{cl}_{\mathbb{R}^k}(V) \cap E$ is either the empty set or a union of 2ε-boxes covering some convex segments contained in the edges of E, we have that $E-$ $\mathrm{cl}_{\mathbb{R}^k}(V)$ is connected.

Given $i \in \{1, \ldots, n\}$, we show that D_i- $\mathrm{cl}_{\mathbb{R}^k}(V)$ is connected. Take two points $p, q \in D_i-$ $\mathrm{cl}_{\mathbb{R}^k}(V)$ such that they do not belong to the same element of $\mathcal{E}_i$, then there exists a finite sequence $\{E_1, \ldots, E_w\}$ of elements in $\mathcal{E}_i$ such that $p \in E_1$, $q \in E_w$ and $E_j \cap E_{j+1}$ is a common $(k-1)$-dimensional face of E_j and E_{j+1} for each $j \in \{1, \ldots, w-1\}$. By the choice of ε, for each $j \in \{1, \ldots, w-1\}$, $E_j \cap E_{j+1} - \mathrm{cl}_{\mathbb{R}^k}(V) \neq \emptyset$. Since $E_1 - \mathrm{cl}_{\mathbb{R}^k}(V), \ldots, E_w - \mathrm{cl}_{\mathbb{R}^k}(V)$ are connected, $(E_1 - \mathrm{cl}_{\mathbb{R}^k}(V)) \cup \ldots \cup (E_w - \mathrm{cl}_{\mathbb{R}^k}(V))$ is a connected subset of $D_i - \mathrm{cl}_{\mathbb{R}^k}(V)$ containing p and q.

Finally, we prove that if $W \in \mathcal{G}$, then $W - \mathrm{cl}_{\mathbb{R}^k}(V)$ is nonempty and connected. By definition of $\mathcal{D}$, there exists $i_W \in \{1, \ldots, n\}$ such that $D_{i_W} \subset W$. Then, there exists $E_W \in \mathcal{E}_{i_W}$ such that $E_W \subset D_{i_W}$. Hence, $E_W - \mathrm{cl}_{\mathbb{R}^k}(V)$ is a nonempty subset of W. Thus, $W - \mathrm{cl}_{\mathbb{R}^k}(V)$ is nonempty.

Take two elements $p, q \in W - \mathrm{cl}_{\mathbb{R}^k}(V)$. Since $\mathcal{D}$ is an open cover of W, there exist $s \in \mathbb{N}$ and $i_1, \ldots, i_s \in \{1, \ldots, n\}$ such that $p \in D_{i_1}$, $q \in D_{i_s}$ and $D_{i_j} \cap D_{i_{j+1}} \neq \emptyset$ for each $j \in \{1, \ldots, s-1\}$. Given $j \in \{1, \ldots, s-1\}$, by the choice of the sets F_l, there exists $E \in \mathcal{E}_{i_j} \cap \mathcal{E}_{i_{j+1}}$ such that $E \subset D_{i_j} \cap D_{i_{j+1}}$. Thus, $D_{i_j} \cap D_{i_{j+1}} - \mathrm{cl}_{\mathbb{R}^k}(V)$ is nonemtpy. Therefore, $(D_{i_1} - \mathrm{cl}_{\mathbb{R}^k}(V)) \cup \ldots \cup (D_{i_1} - \mathrm{cl}_{\mathbb{R}^k}(V)$ is a connected subset of $W - \mathrm{cl}_{\mathbb{R}^k}(V)$ containing p and q. $\square$

Applying induction to Lemma 4.3, we obtain the following.

Corollary 4.4. *Let $k \geq 3$, $\mathcal{D} \subset \mathcal{B}$ be a finite set such that each $W \in \mathcal{D}$ is connected and let $\alpha_1, \ldots, \alpha_n$ be pairwise disjoint street arcs in $\mathbb{R}^k$. Then for each $\delta > 0$ there exist $V_1, \ldots, V_n \in \mathcal{B}$ such that for each $i \in \{1, \ldots, n\}$, V_i is connected, $\alpha_i \subset V_i \subset N(\alpha_i, \delta)$ and for each $W \in \mathcal{D}$, $W - cl_{\mathbb{R}^k}(V_1 \cup \ldots \cup V_n)$ is nonempty and connected.*

5. Main Result

Given two nonempty compact disjoint subsets A and B of $\mathbb{R}^k$, let

$$\mathrm{dist}(A, B) = \min\{\|a - b\| : a \in A \text{ and } b \in B\}.$$

The first part of the following lemma can be proved by using [4, Theorem VI 4]. The second part follows from [4, Corollary to Theorem IV.4].

Lemma 5.1. *Suppose that K is a compact metric space such that each of its nondegenerate components is chainable. Then $\dim(K) = 1$. In particular, if $K \subset \mathbb{R}^k$ for some $k \geq 3$ and U is an open connected subset of $\mathbb{R}^k$, then $U - K$ is nonempty and connected.*

In the next theorem, we will see that the open links obtained in Theorem 3.2 can be constructed to be connected when we work in $\mathbb{R}^k$ $(k \geq 3)$.

Theorem 5.2. *Suppose that K is a compact subspace of $\mathbb{R}^k$, with $k \geq 3$, such that each nondegenerate component of K is chainable. Then for each $\varepsilon > 0$, there exist $n \in \mathbb{N}$, components $C_1, \ldots, C_n$ of K and taut ε-chains* *of connected subsets $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathbb{R}^k$ such that $cl_{\mathbb{R}^k}(\cup\mathcal{U}_1), \ldots, cl_{\mathbb{R}^k}(\cup\mathcal{U}_n)$ are pairwise disjoint, $K \subset (\cup\mathcal{U}_1) \cup \ldots \cup (\cup\mathcal{U}_n)$, $\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n \subset \mathcal{B}$ (where $\mathcal{B}$ is the collection of finite unions of boxes in $\mathbb{R}^k$, defined in the previous section) and for each $i \in \{1, \ldots, n\}$, $\mathcal{U}_i$ properly covers C_i.*

Proof. Let $\varepsilon > 0$ and let X_0 be a k-cell such that $N(K, \varepsilon) \subset \mathrm{int}_{\mathbb{R}^k}(X_0)$. Let d be the usual metric for $\mathbb{R}^k$. By Lemma 5.1, $\dim(K) \leq 1$. By Theorem 3.2, there exist $n \in \mathbb{N}$, components $C_1, \ldots, C_n$ of K and taut ε-chains of open subsets $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of X_0 such that $\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{V}_1), \ldots, \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{V}_n)$ are pairwise disjoint, $K \subset (\cup\mathcal{V}_1) \cup \ldots \cup (\cup\mathcal{V}_n)$, $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n \subset \mathcal{B}$ and for each $i \in \{1, \ldots, n\}$, $\mathcal{V}_i$ properly covers C_i.

We want to prove that we can choose the taut chains $\mathcal{V}_i$ in such a way that their links are connected.

Let $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n$. Given $V \in \mathcal{V}$, we have that V intersects K and V has diameter less than ε, so V is an open subset of X_0 contained in $\mathrm{int}_{\mathbb{R}^k}(X_0)$. Thus, V is open in $\mathbb{R}^k$.

Suppose that there exists $V \in \mathcal{V}$ such that V is not connected. Let R_1 and R_2 be components of V. Fix points $x_1 \in R_1 - K$ and $x_2 \in R_2 - K$. Let β be the convex segment in $\mathbb{R}^k$ joining x_1 and x_2. Since $d(x_1, x_2) < \varepsilon$ and diameter$(V) < \varepsilon$, we have diameter$(V \cup \beta) < \varepsilon$, so there exists $\delta > 0$ such that diameter$(V \cup N(\beta, \delta)) < \varepsilon$. Since $N(\beta, \delta)$ is homeomorphic to $\mathbb{R}^k$ and $\dim(K) \leq 1$, $N(\beta, \delta) \cap K$ does not separate $N(\beta, \delta)$, so there exists a street arc $\alpha \subset N(\beta, \delta) - K$, joining x_1 and x_2. Let $\delta_1 > 0$ be such that $N(\alpha, \delta_1) \subset N(\beta, \delta)$.

Let $\mathcal{D} = \{D \subset \mathbb{R}^k : D$ is a component of U for some $U \in \mathcal{V}\}$. By Lemma 4.1 (c), $\mathcal{D} \subset \mathcal{B}$. Since each element of $\mathcal{B}$ has a finite number of components, $\mathcal{D}$ is finite.

By Lemma 4.3, there exists $T_1 \in \mathcal{B}$ such that T_1 is connected, $\alpha \subset T_1 \subset \mathrm{cl}_{\mathbb{R}^k}(T_1) \subset N(\alpha, \delta_1) - K$ and for each $W \in \mathcal{D}$, $W -\mathrm{cl}_{\mathbb{R}^k}(T_1)$ is nonempty and connected, and there exists $T \in \mathcal{B}$ such that T is connected, $\alpha \subset T \subset \mathrm{cl}_{\mathbb{R}^k}(T) \subset T_1$ and for each $W \in \mathcal{D}$, $W -\mathrm{cl}_{\mathbb{R}^k}(T)$ is nonempty and connected.

We may assume that $V \in \mathcal{V}_1$.

We construct chains $\mathcal{V}'_1, \ldots, \mathcal{V}'_n$ in the following way. First we replace the chain $\mathcal{V}_1$ by the chain $\mathcal{V}'_1$, by changing the link V by $V \cup T$ and each link R in $\mathcal{V}_1 - \{V\}$ is replaced by $R -\mathrm{cl}_{\mathbb{R}^k}(T_1)$. For the rest of the chains $\mathcal{V}_i$ ($i \in \{2, \ldots, n\}$) we simply replace each link R by $\mathcal{V}_i$ by $R -\mathrm{cl}_{\mathbb{R}^k}(T_1)$.

It is easy to check that the chains $\mathcal{V}'_1, \ldots, \mathcal{V}'_n$ have the same properties as those we mentioned for $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in the first paragraph of the proof of this theorem. Moreover, if $\mathcal{D}' = \{D \subset \mathbb{R}^k : D$ is a component of U for some $U \in \mathcal{V}'_1 \cup \ldots \cup \mathcal{V}'_n\}$, then the number of elements of $\mathcal{D}'$ is equal to the number of elements of $\mathcal{D}$ minus one.

Thus, we have shown that if one element of $\mathcal{V}$ is not connected, we can construct another $\mathcal{V}' = \mathcal{V}'_1 \cup \ldots \cup \mathcal{V}'_n$ satisfying the same required properties but with the additional property that the total number of components of the elements of $\mathcal{V}$ is reduced by one. Hence, we can repeat this procedure until we get that all of the links in the chains are connected. $\square$

Lemma 5.3. *Let X be a continuum. Suppose that K is a compact subspace of X such that each of its nondegenerate components is a pseudo-arc. Let $\mathcal{U}$ be an open (in X) taut chain covering K. Then there exists $\delta > 0$ such that each δ-chain in X that properly covers some component of K is crooked in $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$. Let $\delta_0 > 0$ be such that each subset of X with diameter less than δ_0 that intersects K is contained in some U_i. If $0 < \delta < \delta_0$ and $\mathcal{V}$ is a δ-chain that properly covers a component of K, then $\mathcal{V}$ refines $\mathcal{U}$. If $n \leq 3$, by definition $\mathcal{V}$ is crooked in $\mathcal{U}$. Thus, we can suppose that $n \geq 4$.

Suppose that the lemma does not hold. Then for each $m \in \mathbb{N}$, there exists a $(\frac{1}{m})$-chain $\mathcal{V}_m = \{V_1^{(m)}, \ldots, V_{k_m}^{(m)}\}$ in X that properly covers a component C_m of K and $\mathcal{V}_m$ is not crooked in $\mathcal{U}$.

Let $M \in \mathbb{N}$ be such that $\frac{1}{M} < \delta_0$. Given $m \geq M$, by the choice of δ_0, $\mathcal{V}_m$ refines $\mathcal{U}$. Since $\mathcal{V}_m$ is not crooked in $\mathcal{U}$, there exist $i_m, j_m \in \{1, \ldots, n\}$ and $r_m, s_m \in \{1, \ldots, k_m\}$ such that $i_m + 2 < j_m$, $V_{r_m}^{(m)} \subset U_{i_m}$, $V_{s_m}^{(m)} \subset U_{j_m}$, $r_m < s_m$ (if $r_m > s_m$, we can rename the elements of $\mathcal{V}_m$) and if $t_m = \min\{t \in \{r_m, \ldots, s_m\} : V_t^{(m)} \subset U_{j_m - 1}\}$ (see Lemma 2.4 (a)), then $r_m < t_m$ and for each $t \in \{t_m + 1, \ldots, s_m\}$, $V_t^{(m)} \not\subset U_{i_m+1}$.

Taking a subsequence if necessary, we may assume that all the indexes i_m coincide and the same happens for the indexes j_m, that is, we may assume that there exist $i_0, j_0 \in \{1, \ldots, n\}$ such that $i_m = i_0$ and $j_m = j_0$ for every $m \geq M$.

For each $m \geq M$, let $A_m = \mathrm{cl}_X(V_{r_m}^{(m)} \cup \ldots \cup V_{t_m-1}^{(m)})$ and $B_m = \mathrm{cl}_X(V_{t_m}^{(m)} \cup \ldots \cup V_{s_m}^{(m)})$. Taking a subsequence, if necessary, we may assume that $\lim A_m = A$ and $\lim B_m = B$, in 2^X, for some nonempty closed subsets A and B of X. Since $\mathcal{V}_m$ is a $(\frac{1}{m})$-chain for each $m \in \mathbb{N}$, we obtain that A and B are connected. Given $m \geq M$, since $A_m \subset N(C_m, \frac{1}{m}) \subset N(K, \frac{1}{m})$, we obtain that $A \subset K$. Similarly, $B \subset K$. Since $A_m \cap B_m \neq \emptyset$, we have $A \cap B \neq \emptyset$.

Given $i \in \{r_m, \ldots, t_m - 1\}$, $V_i^{(m)} \subset U_j$ for some $j \in \{1, \ldots, n\} - \{j_0 - 1\}$. Since $V_{r_m} \subset U_{i_0}$, Lemma 2.4 (a) implies that $V_{r_m}^{(m)} \cup \ldots \cup V_{t_m-1}^{(m)} \subset U_1 \cup \ldots \cup U_{j_0-2}$. This implies that $A \subset \mathrm{cl}_X(U_1 \cup \ldots \cup U_{j_0-2})$.

Let $i \in \{t_m, \ldots, s_m\}$. We claim that $V_i^{(m)} \subset U_{i_0+2} \cup \ldots \cup U_n$. Suppose to the contrary that this claim does not hold. Let $j = \max\{t \in \{t_m, \ldots, s_m\} : V_t^{(m)} \subset U_{i_0+2} \cup \ldots \cup U_n\}$. Then $j < i$. Since $V_{j+1}^{(m)} \cap (U_{i_0+2} \cup \ldots \cup U_n) \neq \emptyset$ and $\mathcal{V}$ refines $\mathcal{U}$, we have $V_{j+1}^{(m)} \subset U_{i_0+1}$. This contradicts the choice of r_m and s_m and ends the proof of the claim. This claim implies that $B \subset \mathrm{cl}_X(U_{i_0+2} \cup \ldots \cup U_n)$.

Since $A_m \cap U_{i_0} \neq \emptyset$ for each $m \geq M$, we have $A \cap\ \mathrm{cl}_X(U_{i_0}) \neq \emptyset$, so $A \not\subseteq B$. Similarly, $B \cap\ \mathrm{cl}_X(U_{j_0}) \neq \emptyset$ and $B \not\subseteq A$. Therefore, A, B are subcontinua of K such that $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$. This contradicts the assumption that all nondegenerate components of K are pseudo-arcs and ends the proof of the lemma. $\square$

Lemma 5.4. *Let $\mathcal{U}$ be a taut chain of connected links in $\mathbb{R}^k$, with $k \geq 3$, such that $\mathcal{U}$ is a chain from point p to q. Suppose that $\mathcal{Z}$ is a chain from p to q such that the links of $\mathcal{Z}$ are connected and $\mathcal{Z}$ refines $\mathcal{U}$. Then for each $\varepsilon > 0$ there exists a taut chain $\mathcal{V} \subset \mathcal{B}$, from p to q, such that $\mathcal{V}$ refines $\mathcal{Z}$, the elements of $\mathcal{V}$ are connected and $\mathcal{V}$ is crooked in $3(\mathcal{U})$.*

Proof. Since $\cup\mathcal{Z}$ is an open connected subset of $\mathbb{R}^k$, there exists a k-cell $A \subset \cup\mathcal{Z}$ containing p and q in its interior. Then there exists a pseudo-arc $Q \subset \mathrm{int}_{\mathbb{R}^k}(A)$ such that $p, q \in Q$. Let $\delta > 0$ be as in Lemma 5.3 satisfying also that $\delta < \varepsilon$ and each subset of Q of diameter less that δ is contained in some link of $\mathcal{Z}$. Since Q is chainable and all of its pairs of points are end points, there exists a finite family $\mathcal{F} = \{F_1, \ldots, F_n\}$ of compact subsets of $\mathbb{R}^k$ such that $p \in F_1 - F_2$, $q \in F_n - F_{n-1}$, $F_i \cap F_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and $\mathrm{mesh}(\mathcal{F}) < \delta$. By the normality of $\mathbb{R}^k$, it is possible to construct a taut δ-chain $\mathcal{W} = \{W_1, \ldots, W_n\} \subset \mathcal{B}$ such that $p \in W_1-\ \mathrm{cl}_{\mathbb{R}^k}(W_2)$, $q \in W_n-\ \mathrm{cl}_{\mathbb{R}^k}(W_{n-1})$, $F_i \subset W_i$ for each $i \in \{1, \ldots, n\}$ and $\mathrm{mesh}(\mathcal{W}) < \delta$.

Let $\mathcal{D} = \{D \in \mathcal{B} : D$ is a component of W_i for some $i \in \{1, \ldots, n\}$ and $D \cap Q \neq \emptyset\}$. Then $\mathcal{D}$ is an open (in $\mathbb{R}^k$) cover of Q. By the connectedness of Q, there exists a chain $\mathcal{E} = \{E_1, \ldots, E_m\} \subset \mathcal{D}$ such that $p \in E_1$ and $q \in E_m$. Using the normality and the local connectedness of $\mathbb{R}^k$ it is possible to construct a taut chain $\mathcal{V} = \{V_1, \ldots, V_m\} \subset \mathcal{B}$ such that $p \in V_1-\ \mathrm{cl}_{\mathbb{R}^k}(V_2)$, $q \in V_m-\ \mathrm{cl}_{\mathbb{R}^k}(V_{m-1})$ and for each $i \in \{1, \ldots, m\}$, $V_i \subset \mathrm{cl}_{\mathbb{R}^k}(V_i) \subset E_i$, V_i is connected and $V_i \cap Q \neq \emptyset$. Then $\mathcal{V}$ refines $\mathcal{W}$.

By the choice of δ, $\mathcal{V}$ refines $\mathcal{Z}$.

Finally, we prove that $\mathcal{V}$ is crooked in $3(\mathcal{U})$. Let $\mathcal{U} = \{U_1, \ldots, U_c\}$ and $3(\mathcal{U}) = \{Y_1, \ldots, Y_e\}$. Let $i, j \in \{1, \ldots, e\}$ and $r, s \in \{1, \ldots, m\}$ be such that $i + 2 < j$, $V_r \subset Y_i$ and $V_s \subset Y_j$. We consider the case $r < s$. The other one $(s < r)$ is similar.

By Lemma 2.4 (c), there exist $a, b \in \{1, \ldots, c\}$, such that $V_r \subset U_a \subset Y_i$ and $V_s \subset U_b \subset Y_j$. By definition of $3(\mathcal{U})$, $a + 6 < b$.

Let $u, v \in \{1, \ldots, n\}$ be such that $E_r \subset W_u$ and $E_s \subset W_v$. By the choice of δ, $\mathcal{W}$ is crooked in $\mathcal{U}$, with $W_u \subset U_{a_1}$ and $W_v \subset U_{b_1}$ for some $a_1, b_1 \in \{1, \ldots, c\}$. Notice that $a_1 \in \{a-1, a, a+1\}$ and $b_1 \in \{b-1, b, b+1\}$. Thus, $a_1 + 4 < b_1$. Then there exist $x, y \in \{1, \ldots, n\}$ such that $W_x \subset U_{b_1-1}$, $W_y \subset U_{a_1+1}$ and either $u < x < y < v$ or $v < y < x < u$.

By Lemma 2.4 (a), there exist $r < r_1 < s_1 < s$ such that $V_{r_1} \subset W_x \subset U_{b_1-1} \subset Y_{j-1} \cup Y_j$ and $V_{s_1} \subset W_y \subset U_{a_1+1} \subset Y_i \cup Y_{i+1}$. Since $\mathcal{V}$ refines $3(\mathcal{U})$, there exist $r < r_2 \leq r_1$ and $s_1 \leq s_2 < s$ such that $V_{r_2} \subset U_{b-3} \subset Y_{j-1}$ and $V_{s_2} \subset U_{a+3} \subset Y_{j+1}$. Therefore, $\mathcal{V}$ is crooked in $3(\mathcal{U})$. □

Theorem 5.5. *Let K be a compact subspace of $\mathbb{R}^k$, with $k \geq 3$, such that each nondegenerate component of K is a pseudo-arc. Let $p, q \in \mathbb{R}^k - K$ and $\mathcal{U} = \{U_1, \ldots, U_n\} \subset \mathcal{B}$ be a taut chain such that $p \neq q$, $p \in U_1$, $q \in U_n$, each U_i is connected and $K \subset U_1 \cup \ldots \cup U_n$. Then for each $\varepsilon > 0$, there exists a taut ε-chain $\mathcal{Y} = \{Y_1, \ldots, Y_m\} \subset \mathcal{B}$ such that $p \in Y_1$, $q \in Y_m$, each Y_i is connected, $K \subset Y_1 \cup \ldots \cup Y_m$ and $\mathcal{Y}$ is crooked in $72(\mathcal{U})$.*

Proof. Let X be a k-cell such that $K \subset \mathrm{int}_{\mathbb{R}^k}(X)$. Take $\delta > 0$, as in Lemma 5.3 such that $\delta < \varepsilon$. We may assume that $N(K, 2\delta) \subset X$, $\{p, q\} \cap \mathrm{cl}_{\mathbb{R}^k}(N(K, 2\delta)) = \emptyset$ and for each $j \in \{1, \ldots, n\}$, $U_j - \mathrm{cl}_{\mathbb{R}^k}(N(K, \delta)) \neq \emptyset$. By Theorem 5.2, there exist $s \in \mathbb{N}$, components $C_1, \ldots, C_s$ of K and taut δ-chains $\mathcal{R}_1, \ldots, \mathcal{R}_s$ of (open) connected subsets of $\mathbb{R}^k$ such that $\mathrm{cl}_X(\cup\mathcal{R}_1), \ldots, \mathrm{cl}_X(\cup\mathcal{R}_s)$ are pairwise disjoint, $K \subset (\cup\mathcal{R}_1) \cup \ldots \cup (\cup\mathcal{R}_s)$, $\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_s \subset \mathcal{B}$ and for each $i \in \{1, \ldots, s\}$ $\mathcal{R}_i$ properly covers C_i. Then each $\mathcal{R}_i$ is crooked in $\mathcal{U}$ and $\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_1) \cup \ldots \cup \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_s) \subset N(K, 2\delta)$, so $\{p, q\} \cap \mathrm{cl}_{\mathbb{R}^k}((\cup\mathcal{R}_1) \cup \ldots \cup (\cup\mathcal{R}_s)) = \emptyset$.

Given $i \in \{1, \ldots, s\}$, let $\mathcal{R}_i = \{R_1^{(i)}, \ldots, R_{r_i}^{(i)}\}$. In the case that $r_i = 1$, $\mathcal{R}_i = \{R_1^{(i)}\}$ and we replace K by K_0 as follows. Take $j_0 \in \{1, \ldots, n\}$ such that $\mathrm{cl}_{\mathbb{R}^k}(R_1^{(i)}) \subset U_{j_0}$, take a point $z \in U_{j_0} - \mathrm{cl}_{\mathbb{R}^k}(R_1^{(i)})$ and an arc $\chi \subset U_{j_0}$ joining z to a point $z^* \in C_i$. Shortening χ if necessary, we may assume that χ does not intersect $\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_j)$ for any $j \neq i$. Let $R_2^{(i)} \in \mathcal{B}$ be such that $R_2^{(i)}$ is connected, $\chi \subset R_2^{(i)} \subset U_{j_0}$ and $R_1^{(i)} \not\subset \mathrm{cl}_{\mathbb{R}^k}(R_2^{(i)})$. Then we replace K by $K_0 = K \cup \chi$ and $\mathcal{R}_i$ by $\mathcal{R}'_i = \{R_1^{(i)}, R_2^{(i)}\}$. Thus, it is enough to find the chain $\mathcal{Y}$ using K_0 and $\mathcal{R}'_i$. Of course, K_0 has a component that is not a pseudo-arc, but we will not need this property anymore. Thus, we may assume that $r_i > 1$ for each $i \in \{1, \ldots, s\}$.

Note that there exist $u_i, v_i \in \{1, \ldots, n\}$ such that $u_i \leq v_i$, $\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_i) \subset U_{u_i} \cup \ldots \cup U_{v_i}$, no element of $\mathcal{R}_i$ is either contained in U_{u_i-1} or in U_{v_i+1}, and there exist a_i, $b_i \in \{1, \ldots, r_i\}$ such that $R_{a_i}^{(i)} \subset U_{u_i}$ and $R_{b_i}^{(i)} \subset U_{v_i}$.

We will apply Lemma 2.8, so we need that the first link of $\mathcal{R}_i$ be contained in U_{u_i}. That is, we need that $a_i = 1$. When $a_i > 1$, we will modify $\mathcal{R}_i$ according to the following procedure.

Suppose that $a_i > 1$.

For each $j \in \{1, \dots, a_i - 1\}$, choose a point $x_j \in R_j^{(i)} \cap R_{j+1}^{(i)} - K$ (see Lemma 5.1). By Lemma 5.1 and successive applications of Lemma 4.2 it is possible to construct street arcs $\alpha_2, \dots, \alpha_{a_i}$ such that $\alpha = \alpha_2 \cup \dots \cup \alpha_{a_i}$ is a street arc, α_{a_i} joins a point in $R_{a_i}^{(i)} - (\mathrm{cl}_{\mathbb{R}^k}(R_{a_i-1}^{(i)}) \cup K)$ to x_{a_i-1}, $\alpha_{a_i} \subset R_{a_i}^{(i)} - K$, and for each $j \in \{2, \dots, a_i - 1\}$, $\alpha_j \subset R_j^{(i)} - K$ and α_j joins x_{j-1} to x_j.

By Lemma 4.3, there exists $V \in \mathcal{B}$ such that V is connected, $\alpha \subset V$, $\mathrm{cl}_{\mathbb{R}^k}(V) \subset (R_2^{(i)} \cup \dots \cup R_{a_i}^{(i)}) - K$ and for each $W \in \mathcal{R}_i$, $W-\ \mathrm{cl}_{\mathbb{R}^k}(V)$ is nonempty and connected.

For each $j \in \{2, \dots, a_i\}$, let $S_{a_i+1-j} \in \mathcal{B}$ be such that S_{α_i+1-j} is connected and $\alpha_j \subset S_{a_i+1-j} \subset \mathrm{cl}_{\mathbb{R}^k}(S_{a_i+1-j}) \subset V \cap R_j^{(i)}$. Consider the sequence

$$\mathcal{C} = \{S_1, \dots ,S_{a_i-1}, R_1^{(i)}, R_2^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V), \dots, R_{a_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V), R_{a_i+1}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V), R_{a_i+2}^{(i)}, \dots ,R_{r_i}^{(i)}\}.$$

We claim that the following properties hold.

(a) $\mathcal{C}$ is a taut chain with connected links and $\cup\mathcal{C}$ is connected,

(b) $\mathcal{C}$ is crooked in $2(\mathcal{U})$,

(c) $S_1 \subset U_{u_i}$,

(d) $\mathcal{C} \subset \mathcal{B}$,

(e) $\cup\mathcal{C} \subset \cup\mathcal{R}_i \subset U_{u_i} \cup \dots \cup U_{v_i}$ and

(f) $K \cap (\cup\mathcal{R}_i) \subset \cup\mathcal{C}$.

Given $j \in \{1, \dots, r_i - 1\}$, if $(R_j^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V)) \cap (R_{j+1}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V)) = \emptyset$, then $C_i \cap (R_1^{(i)} \cup \dots \cup R_j^{(i)}) - \mathrm{cl}_{\mathbb{R}^k}(V)$ and $C_i \cap (R_{j+1}^{(i)} \cup \dots \cup R_{r_i}^{(i)}) - \mathrm{cl}_{\mathbb{R}^k}(V)$ is a separation of C_i, a contradiction. Thus, $(R_j^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V)) \cap (R_{j+1}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V)) \neq \emptyset$. Using this fact it is easy to show that $\mathcal{C}$ is a chain. Crookedness of $\mathcal{C}$ in $2(\mathcal{U})$ is an immediate consequence of Lemma 2.7. The rest of the properties are easy to check.

In the case that $a_i = 1$, the chain $\mathcal{R}_i$ has properties (a)-(f) (its first link is contained in U_{u_i}).

With a similar argument, considering the cases $b_i < w_i$ or $b_i = w_i$, it is possible to construct a chain $\mathcal{S}_i$ satisfying properties (a), (c)-(f) (for $\mathcal{C} = \mathcal{S}_i$) and the following properties.

(b') $\mathcal{S}_i$ is crooked in $4(\mathcal{U})$,

(g) the last link of $\mathcal{S}_i$ is contained in U_{v_i},

(h) since $r_i > 1$ and the length of $\mathcal{S}_i$ is greater than or equal to r_i, we have $s_i > 1$.

Let $\mathcal{S}_i = \{S_1^{(i)}, \ldots, S_{s_i}^{(i)}\}$.

For each $i \in \{1, \ldots, s\}$, since $\mathrm{int}_{\mathbb{R}^k}(K) = \emptyset$, we can choose points w_i, z_i and k-boxes A_i, B_i such that $w_i \in A_i \subset S_1^{(i)} - (K \cup \mathrm{cl}_{\mathbb{R}^k}(S_2^{(i)}))$ and $z_i \in B_i \subset S_{s_i}^{(i)} - (K \cup \mathrm{cl}_{\mathbb{R}^k}(S_{s_i-1}^{(i)}))$. Since the sets $\mathrm{cl}_X(\cup\mathcal{R}_1), \ldots, \mathrm{cl}_X(\cup\mathcal{R}_s)$ are pairwise disjoint, we obtain that the sets $A_1, \ldots, A_s, B_1, \ldots, B_s$ are pairwise disjoint.

Let $\varepsilon_1 > 0$ be such that $\varepsilon_1 < \varepsilon$, $N(\{w_i\}, 3\varepsilon_1) \subset A_i$ and $N(\{z_i\}, 3\varepsilon_1) \subset B_i$.

By (a) and Lemma 5.1 it is possible to choose a street arc β_i joining w_i to z_i such that $\beta_i \subset (\cup\mathcal{S}_i) - K$. Since $K \cup \beta_i$ is one-dimensional, $\cup\mathcal{S}_i - (K \cup \beta_i)$ is open and connected, so we can choose points w_i', z_i' and a street arc γ_i joining w_i' to z_i' such that $w_i' \in A_i$, $z_i' \in B_i$ and $\gamma_i \subset (\cup\mathcal{S}_i) - (K \cup \beta_i)$.

By Lemma 5.1, $(\cup\mathcal{U}) - (K \cup \gamma_1 \cup \ldots \cup \gamma_s)$ is open and connected, so by Lemma 4.2 (a), there exists a street arc $\beta \subset (\cup\mathcal{U}) - (K \cup \gamma_1 \cup \ldots \cup \gamma_s)$ such that β joins p to q, $\beta_1 \cup \ldots \cup \beta_s \subset \beta$ and, in the natural order of β, $p < w_1 < z_1 < w_2 < z_2 < \ldots < w_s < z_s < q$.

Let $\lambda > 0$ be such that $\lambda < \varepsilon_1$ and $\mathrm{cl}_{\mathbb{R}^k}(N(\beta, \lambda)) \subset (\cup\mathcal{U}) - (K \cup \gamma_1 \cup \ldots \cup \gamma_s)$. By Lemma 4.3, there exists $V_0 \in \mathcal{B}$ such that V_0 is connected, $\beta \subset V_0 \subset \mathrm{cl}_{\mathbb{R}^k}(V_0) \subset N(\beta, \lambda)$ and for each $W \in (\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_s) \cup \{A_1, \ldots, A_s\} \cup \{B_1, \ldots, B_s\}$, $W - \mathrm{cl}_{\mathbb{R}^k}(V_0)$ is nonempty and connected. Given $i \in \{1, \ldots, s\}$, $\mathrm{cl}_{\mathbb{R}^k}(V_0) \cap \gamma_i = \emptyset$, so $w_i' \in S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)$. Similarly, $z_i' \in S_{s_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)$.

Following the arc β construct a taut chain $\mathcal{Z}$ of connected elements of $\mathcal{B}$ such that $\mathcal{Z}$ goes from p to q, $\beta \subset \cup\mathcal{Z} \subset V_0$ and for each $i \in \{1, \ldots, s\}$, $\mathcal{Z}$ has a subchain $\mathcal{Z}_i = \{Z_1^{(i)}, \ldots, Z_{c_i}^{(i)}\}$ such that $\beta_i \subset \cup\mathcal{Z}_i \subset \cup\mathcal{S}_i \subset \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_i)$, $\mathcal{Z}_i$ goes from w_i to z_i, $\mathrm{mesh}(\mathcal{Z}_i) < \varepsilon_1$, $\mathcal{Z}$ refines $\mathcal{U}$ and $\mathcal{Z}$ preserves the order of the arc, in particular, in the order of $\mathcal{Z}$, $\mathcal{Z}_1$ appears first, $\mathcal{Z}_2$ after $\mathcal{Z}_1$, $\mathcal{Z}_3$ after $\mathcal{Z}_2$, etc. Then $Z_1^{(i)} \subset N(\{w_i\}, \varepsilon_1)$ and $Z_{c_i}^{(i)} \subset N(\{z_i\}, \varepsilon_1)$. Since $A_i \cap B_i = \emptyset$, we have $Z_1^{(i)} \cap Z_{c_i}^{(i)} = \emptyset$.

By Lemma 5.4, there exists a taut chain $\mathcal{V} = \{V_1, \ldots, V_m\} \subset \mathcal{B}$ of connected sets such that $\mathcal{V}$ refines $\mathcal{Z}$, $\mathcal{V}$ is crooked in $3(\mathcal{U})$, $\mathrm{mesh}(\mathcal{V}) < \min(\{\varepsilon_1\} \cup \{\frac{1}{6}\mathrm{dist}(\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_i), \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_j)) : i, j \in \{1, \ldots, s\} \text{ and } i \neq j\} \cup \{\mathrm{dist}(p, \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_i)) : i \in \{1, \ldots, s\}\} \cup \{\mathrm{dist}(q, \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_i)) : i \in \{1, \ldots, s\}\})$ and $\mathcal{V}$ goes from p to q.

Let ζ be a street arc joining p to q such that $\zeta \subset \cup\mathcal{V}$.

Since $\mathcal{Z}$ preserves the order of the arc β, given $i \in \{1, \ldots, s\}$, the link $Z_1^{(i)}$ precedes the link $Z_{c_i}^{(i)}$. By Lemma 2.4 (a), there exist $k_{2i-1}, k_{2i} \in \{1, \ldots, m\}$ such that $k_{2i-1} \leq k_{2i}$, $V_{k_{2i-1}} \subset Z_1^{(i)}$, $V_{k_{2i}} \subset Z_{c_i}^{(i)}$, $V_{k_{2i-1}} \cup \ldots \cup V_{k_{2i}} \subset \cup\mathcal{Z}_i$ and $1 \leq k_1 \leq k_2 \leq \ldots \leq k_{2s} \leq m$. Notice that $V_{k_{2i-1}-1} \subset N(V_{k_{2i-1}}\varepsilon_1) \subset N(Z_1^{(i)}, \varepsilon_1) \subset N(\{w_i\}, 2\varepsilon_1) \subset A_i$. Similarly, $V_{k_{2i}+1} \subset B_i$.

Since $\cup\mathcal{Z}_i \subset \cup\mathcal{S}_i \subset \cup\mathcal{R}_i$, we have $V_{k_{2i-1}} \cup \ldots \cup V_{k_{2i}} \subset U_{u_i} \cup \ldots \cup U_{v_i}$ and $(V_{k_{2i-1}} \cup \ldots \cup V_{k_{2i}}) \cap (V_{k_{2j-1}} \cup \ldots \cup V_{k_{2j}}) = \emptyset$, if $i \neq j$. Since $V_{k_{2i-1}} \subset Z_1^{(i)} \subset \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_i)$ and $V_{k_{2i}} \subset Z_{c_i}^{(i)} \subset \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{S}_i)$, by the choice of mesh$(\mathcal{V})$, if we define $k_0 = 1$ and $k_{2s+1} = m$, we have that $k_0 + 3 < k_1$, $k_2 + 3 < k_3$, $\ldots$, $k_{2s} + 3 < k_{2s+1}$.

Given $i \in \{1, \ldots, s\}$, choose points p_i, q_i and k-boxes P_i, Q_i such that $p_i \in P_i \subset \mathrm{cl}_{\mathbb{R}^k}(P_i) \subset V_{k_{2i-1}-1} - \mathrm{cl}_{\mathbb{R}^k}(\zeta \cup V_1 \cup \ldots \cup V_{k_{2i-1}-2} \cup V_{k_{2i-1}} \cup \ldots \cup V_m)$ and $q_i \in Q_i \subset \mathrm{cl}_{\mathbb{R}^k}(Q_i) \subset V_{k_{2i}+1} - \mathrm{cl}_{\mathbb{R}^k}(\zeta \cup V_1 \cup \ldots \cup V_{k_{2i}} \cup V_{k_{2i}+2} \cup \ldots \cup V_m)$. Then $p_i \in A_i$ and $q_i \in B_i$.

Choose street arcs η_i, λ_i such that $\eta_i \subset A_i - \zeta$, $\lambda_i \subset B_i - \zeta$, η_i joins p_i to w_i' and λ_i joins q_i to z_i'. Since $A_1, \ldots, A_s, B_1, \ldots, B_s$ are pairwise disjoint, then $\eta_1, \ldots, \eta_s, \lambda_1, \ldots, \lambda_s$ are pairwise disjoint.

By Corollary 4.4, there exist connected sets $L_i, M_i \in \mathcal{B}$ such that $\eta_i \subset L_i \subset \mathrm{cl}_{\mathbb{R}^k}(L_i) \subset A_i - \zeta$, $\lambda_i \subset M_i \subset \mathrm{cl}_{\mathbb{R}^k}(M_i) \subset B_i - \zeta$ and $W - \mathrm{cl}_{\mathbb{R}^k}(L_1 \cup \ldots \cup L_s \cap M_1 \cup \ldots \cup M_s)$ is nonempty and connected for each $W \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_s \cup \mathcal{V} \cup \{P_1, \ldots, P_s, Q_1, \ldots, Q_s\}$. Choose connected sets $L_i', M_i' \in \mathcal{B}$ such that $\eta_i \subset L_i' \subset \mathrm{cl}_{\mathbb{R}^k}(L_i') \subset L_i$ and $\lambda_i \subset M_i' \subset \mathrm{cl}_{\mathbb{R}^k}(M_i') \subset M_i$.

Set
$\mathcal{W}_i = \{(S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)) \cup L_i', (S_2^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)), \ldots, (S_{s_i-1}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)), (S_{s_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)) \cup M_i'\}$.

Since $w_i' \in L_i \cap S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)$, we obtain that $(S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)) \cup L_i$ is connected. Similarly, $(S_{s_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)) \cup M_i$ is connected. Hence, the elements of $\mathcal{W}_i$ are connected. Since $\mathrm{cl}_{\mathbb{R}^k}(V_0) \cap \gamma_i = \emptyset$ and $\gamma_i \subset \cup\mathcal{S}_i$ joins $w_i' \in S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(S_2^{(i)})$ to $z_i' \in S_{s_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(S_{s_i-1}^{(i)})$, it follows that $\{(S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)), (S_2^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)), \ldots, (S_{s_i-1}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)), (S_{s_i}^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0))\}$ is a taut chain. Since $L_i \subset S_1^{(i)}$ and $M_i \subset S_{s_i}^{(i)}$, by Lemma 2.6, $\mathcal{W}_i$ is a taut chain crooked in $2(4(\mathcal{U})) = 8(\mathcal{U})$.

Let $L = L_1 \cup \ldots \cup L_s$, $M = M_1 \cup \ldots \cup M_s$ and $J = \{k_1 - 1, \ldots, k_{2s-1} - 1\} \cup \{k_2 + 1, \ldots, k_{2s} + 1\}$. For each $j \in \{1, \ldots, m\}$, let

$$V_j' = \begin{cases} V_j - \mathrm{cl}_{\mathbb{R}^k}(L \cup M), & \text{if } j \notin J, \\ (V_j - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)) \cup P_i, & \text{if } j = k_{2i-1} - 1 \text{ for some } i \in \{1, \ldots, s\}, \\ (V_j - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)) \cup Q_i, & \text{if } j = k_{2i} + 1 \text{ for some } i \in \{1, \ldots, s\}. \end{cases}$$

Let $\mathcal{V}' = \{V_1', \ldots, V_m'\}$.

Since $\zeta \subset (V_1 - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)) \cup \ldots \cup (V_m - \mathrm{cl}_{\mathbb{R}^k}(L \cup M))$, the family $\mathcal{V}_0' = \{(V_1 - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)), \ldots, (V_m - \mathrm{cl}_{\mathbb{R}^k}(L \cup M))\}$ is a taut chain. By Lemma 2.6, $\mathcal{V}_0'$ is crooked in $6(\mathcal{U})$. Since for each $i \in \{1, \ldots, s\}$, $V_{k_{2i-1}-1}$ is the only element in $\mathcal{V}$ such that $\mathrm{cl}_{\mathbb{R}^k}(V_{k_{2i-1}-1}) \cap \mathrm{cl}_{\mathbb{R}^k}(P_i) \neq \emptyset$, and $V_{k_{2i}+1}$

is the only element in $\mathcal{V}$ such that $\mathrm{cl}_{\mathbb{R}^k}(V_{k_{2i}+1})\cap \mathrm{cl}_{\mathbb{R}^k}(Q_i) \neq \emptyset$. Lemma 2.6 implies that $\mathcal{V}'$ is a taut chain of connected sets that is crooked in $6(\mathcal{U})$.

Since $\emptyset \neq P_i - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)) \subset (V_{k_{2i-1}-1} - \mathrm{cl}_{\mathbb{R}^k}(L \cup M)) \cap P_i$, we have $V'_{k_{2i-1}-1}$ is connected. Similarly, $V'_{k_{2i}+1}$ is connected. Thus, the elements of $\mathcal{V}'$ are connected.

We will check that the conditions of Lemma 2.8 are satisfied for $\mathcal{V}'$.

Let $\mathcal{W}_i = \{W_1^{(i)}, \ldots, W_{s_i}^{(i)}\}$. Since $p_i \in \eta_i \cap P_i \subset W_1^{(i)} \cap V'_{k_{2i-1}-1}$, we have $W_1^{(i)} \cap V'_{k_{2i-1}-1} \neq \emptyset$.

Similarly, $W_{s_i}^{(i)} \cap V'_{k_{2i}+1} \neq \emptyset$.

Since $\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{V}) \subset \cup\mathcal{Z} \subset V_0$ and $P_i \cup Q_i \subset \cup\mathcal{V}$ for each $i \in \{1, \ldots, s\}$, we have that for each $i \in \{1, \ldots, s\}$, $(W_2^{(i)} \cup \ldots \cup W_{s_i-1}^{(i)}) \cap (\cup\mathcal{V}) = \emptyset$.

Moreover, if $\mathrm{cl}_{\mathbb{R}^k}(W_1^{(i)})\cap \mathrm{cl}_{\mathbb{R}^k}(V'_c) \neq \emptyset$, then $\mathrm{cl}_{\mathbb{R}^k}(L'_i)\cap \mathrm{cl}_{\mathbb{R}^k}(V'_c) \neq \emptyset$. This implies that either $c = k_{2e-1} - 1$ for some $e \in \{1, \ldots, s\}$ and $\mathrm{cl}_{\mathbb{R}^k}(L'_i) \cap P_e \neq \emptyset$ or $c = k_{2e} + 1$ for some $e \in \{1, \ldots, s\}$ and $L'_i \cap Q_e \neq \emptyset$. Notice that for each $e \in \{1, \ldots, s\}$, $P_e \subset V'_{k_{2e-1}-1} \subset A_e \subset \cup\mathcal{S}_e \subset \cup\mathcal{R}_e$ and $Q_e \subset V'_{k_{2e}+1} \subset B_e \subset \cup\mathcal{R}_e$.

In the case that there exists $e \in \{1, \ldots, s\}$ such that $c = k_{2e-1} - 1$ and $L_i \cap P_e \neq \emptyset$, since $L_i \subset \cup\mathcal{R}_i$, we have $i = e$ and $c = k_{2e-1} - 1$.

In the case that there exists $e \in \{1, \ldots, s\}$ such that $c = k_{2e} + 1$ and $L_i \cap Q_e \neq \emptyset$, we have $i = e$, $c = k_{2e} + 1$. Since $L_i \subset A_i \subset S_1^{(i)} - (K\cup \mathrm{cl}_{\mathbb{R}^k}(S_2^{(i)}))$ and $Q_i \subset B_i \subset S_{s_i}^{(i)} - (K\cup \mathrm{cl}_{\mathbb{R}^k}(S_{s_i-1}^{(i)}))$, we obtain that $L_i \cap Q_i = \emptyset$, a contradiction, so this case is impossible and we conclude that $c = k_{2i-1} - 1$.

This completes the proof that $\mathrm{cl}_{\mathbb{R}^k}(W_1^{(i)})\cap \mathrm{cl}_{\mathbb{R}^k}(V'_c) \neq \emptyset$ if and only if $c = k_{2i-1} - 1$, and we have proved that $W_1^{(i)} \cap V'_{k_{2i-1}-1} \neq \emptyset$. Similarly, $\mathrm{cl}_{\mathbb{R}^k}(W_{s_i}^{(i)})\cap \mathrm{cl}_{\mathbb{R}^k}(V'_c) \neq \emptyset$ if and only if $c = k_{2i} + 1$, and we have proved that $W_{s_i}^{(i)} \cap V'_{k_{2i}+1} \neq \emptyset$.

Notice that $W_1^{(i)} \cup V'_{k_{2i-1}} \subset (S_1^{(i)} - \mathrm{cl}_{\mathbb{R}^k}(V_0)) \cup L'_i \cup Z_1^{(i)} \subset U_{u_i} \cup A_i \subset U_{u_i}$. Similarly, $W_{s_i}^{(i)} \cup V'_{k_{2i}} \subset U_{v_i}$.

Notice also that for each $i \in \{1, \ldots, s\}$, $(\cup\mathcal{W}_i) \cup (V'_{k_{2i-1}} \cup \ldots \cup V'_{k_{2i}}) \subset (\cup\mathcal{S}_i) \cup (\cup\mathcal{Z}_i) \subset \cup\mathcal{S}_i \subset \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_i) \subset U_{u_i} \cup \ldots \cup U_{v_i}$.

Since for each $i \in \{1, \ldots, s\}$, $\cup\mathcal{W}_i \subset \cup\mathcal{S}_i \subset \mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{R}_i)$, we obtain $\cup\mathcal{W}_1, \ldots, \cup\mathcal{W}_r$ are pairwise disjoint.

Define $\mathcal{Y} = \mathcal{V}'(1, k_1 - 1) * \mathcal{W}_1 * \mathcal{V}'(k_2 + 1, k_3 - 1) * \mathcal{W}_2 * \ldots * \mathcal{V}'(k_{2(r-1)} + 1, k_{2r-1} - 1) * \mathcal{W}_r * \mathcal{V}'(k_{2r} + 1, k_{2r+1})$.

Since $\mathcal{V}'$ and each $\mathcal{W}_i$ is crooked in $24(\mathcal{U})$, by Lemma 2.8, $\mathcal{Y}$ is crooked in $72(\mathcal{U})$.

Clearly, $\mathcal{Y}$ is a taut ε-chain of elements of $\mathcal{B}$ and each link of $\mathcal{Y}$ is connected. Since $\mathrm{cl}_{\mathbb{R}^k}(L \cup M) \subset \mathrm{cl}_{\mathbb{R}^k}((\cup\mathcal{R}_1) \cup \ldots \cup (\cup\mathcal{R}_s))$, we have $p \in V_1'$, so p belongs to the first link of $\mathcal{Y}$. Similarly, q belongs to the last link of $\mathcal{Y}$.

Since $K \subset (\cup\mathcal{S}_1) \cup \ldots \cup (\cup\mathcal{S}_s)$ and $K \cap \mathrm{cl}_{\mathbb{R}^k}(V_0) = \emptyset$, we have $K \subset (\cup\mathcal{W}_1) \cup \ldots \cup (\cup\mathcal{W}_s) \subset \cup\mathcal{Y}$. □

Theorem 5.6. *Let $k \geq 3$. Suppose that K is a compact subspace of $\mathbb{R}^k$ such that each nondegenerate component of K is a pseudo-arc. Then there exists a pseudo-arc P in $\mathbb{R}^k$ such that $K \subset P$.*

Proof. Using Theorem 5.5 it is possible to define inductively a sequence $\{\mathcal{U}_r\}_{r=1}^{\infty}$ of taut chains in $\mathbb{R}^k$ such that for each $r \in \mathbb{N}$,

(a) $\mathcal{U}_r$ is a chain from p to q,
(b) $\mathcal{U}_{r+1}$ is crooked in $72(\mathcal{U}_r)$,
(c) $\mathrm{mesh}(\mathcal{U}_r) < \frac{1}{r}$, and
(d) $K \subset \cup\mathcal{U}_r$.

Then, by Lemma 2.3, the set $P = \cap\{\mathrm{cl}_{\mathbb{R}^k}(\cup\mathcal{U}_r) : r \in \mathbb{N}\}$ is a pseudo-arc containing K. □

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