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by

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OK-EXTENDIBLE FILTERS ON ω

ANDRÉS MILLÁN

Abstract. In this note we prove that every meager filter can be extended to an OK-point and that there are $2^{\mathfrak{c}}$-many nonmeager and null filters having OK-point extensions as well. These results generalize a construction by K. Kunen. Also, we notice that is consistent with ZFC that some measure zero filters cannot have OK-point extensions. Finally, we prove that despite of the fact that there exist $2^{\mathfrak{c}}$-many OK-points, its generic existence is independent of the axioms of ZFC.

1. Introduction

OK point ultrafilters were introduced by K. Kunen in [4] in order to prove that the remainder of the Stone-Čech compactification of ω is not homogeneous. Kunen constructed OK-points by using a system of infinite sets of ω with strong combinatorial properties. However, it was not clear for which kind of filters other than the cofinite filter that a similar construction could be performed. Also, it was shown in [4] that in ZFC, OK points are relatively abundant in the sense that there are $2^{\mathfrak{c}}$-many of them but, it was not obvious whether "small" filters could be extended to OK-points. The lack of interest about these issues could be attributed to the fact that papers [1], [2] and [6] had not yet been published and possibly those questions were not relevant at that time. This note can be considered as a first attempt to answer them.

Our notation and terminology is fairly standard. The cofinite filter will be denoted $\mathscr{F}_{\mathrm{cof}} = \{A \subseteq \omega\colon |\omega \setminus A| < \omega\}$. Letters $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ will always denote a filter containing $\mathscr{F}_{\mathrm{cof}}$. Letters $\mathscr{U}$ and $\mathscr{V}$ will denote

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Thanks to profesors K. Kunen and J. Roitman for sending me copies of their papers.

nonprincipal ultrafilters. The set of nonprincipal ultrafilters on ω will be denoted by ω^*. For any $\mathscr{F}$, let $\mathcal{I}_{\mathscr{F}} = \{A \subseteq X : X \setminus A \in \mathscr{F}\}$ be the dual ideal of $\mathscr{F}$ and $\mathcal{I}^+_{\mathscr{F}} = \mathcal{P}(X) \setminus \mathcal{I}_{\mathscr{F}}$. The filter generated by a family of sets $\mathcal{A}$ will be denoted as $\langle \mathcal{A} \rangle$. Given $\mathscr{F}$ we say that $\mathcal{B} \subseteq \mathscr{F}$ is a basis of $\mathscr{F}$ if for every $F \in \mathscr{F}$ there is a $B \in \mathcal{B}$ such that $B \subseteq F$. If such a $\mathcal{B}$ has size $< \mathfrak{c}$ we say that $\mathscr{F}$ is $< \mathfrak{c}$-generated. The character of an ultrafilter $\mathscr{U}$ is defined as $\chi(\mathscr{U}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathscr{U}$ and $\mathcal{B}$ is a basis$\}$. It is known that $\omega_1 \leq \chi(\mathscr{U}) \leq \mathfrak{c}$ for every $\mathscr{U} \in \omega^*$. If X is countably infinite 2^X will denote the set $\{f | f : X \to \{0,1\}\}$. This set can be topologized by taking the discrete topology on $\{0,1\}$ and then product topology on 2^X. Also, a probability measure can be defined on 2^X by taking the measure μ_0 on $\{0,1\}$ defined by $\mu_0(\{0\}) = \mu_0(\{1\}) = 1/2$ and then, the product measure. If $A \subseteq X$ then, χ_A denotes the characteristic function of A. A filter $\mathscr{F}$ is either meager or null provided the set $\hat{\mathscr{F}} = \{\chi_A \in 2^X : A \in \mathscr{F}\}$ is. Letters $\mathcal{M}$ y $\mathcal{N}$ denote respectively the meager and null ideals on 2^X. If $A \subseteq \omega \times \omega$ and $n < \omega$ then, $(A)_n = \{m < \omega : (n,m) \in A\}$. Given $\mathscr{F}$ and $\mathscr{G}$ the Fubini product $\mathscr{F} \otimes \mathscr{G}$ is the filter defined by

$$\mathscr{F} \otimes \mathscr{G} = \{A \subseteq \omega \times \omega : \{n < \omega : (A)_n \in \mathscr{G}\} \in \mathscr{F}\}.$$

Notice that $\mathscr{V} \otimes \mathscr{U}$ is always an ultrafilter.

Finally, $\mathfrak{d}$ and $\mathfrak{b}$ will denote respectively the minimum size of a dominating and unbounded family on ω^ω, $\mathsf{cov}(\mathcal{M})$ the minimum size of a family of meager sets covering 2^ω and, $\mathfrak{u} = \min\{\chi(\mathscr{U}) : \mathscr{U} \in \omega^*\}$.

2. OK-extendibility

Definition 2.1 (K. Kunen [4])**.** A nonprincipal ultrafilter $\mathscr{U}$ on ω is an OK-point if for every $\{L_n : n < \omega\} \subseteq \mathscr{U}$ there exists a sequence $\langle V_\alpha \in \mathscr{U} : \alpha < \mathfrak{c} \rangle$ such that for every $n \geq 1$ y $F \in [\mathfrak{c}]^n$

$$\bigcap_{\alpha \in F} V_\alpha \subseteq^* L_n.$$

If this is the case, we say that the sequence $\langle V_\alpha \in \mathscr{U} : \alpha < \mathfrak{c} \rangle$ is OK for $\{L_n : n < \omega\}$.

Notice that the terms of $\langle V_\alpha \in \mathscr{U} : \alpha < \mathfrak{c} \rangle$ are not necesarily different.

Proposition 2.2 (K. Kunen [4])**.** *Every P-point is an* OK*-point.*

Proof. If $\mathscr{U}$ is a P-point and $\{L_n : n < \omega\} \subseteq \mathscr{U}$ there exists a $U \in \mathscr{U}$ such that $U \subseteq^* L_n$ for every $n < \omega$. Define $\langle V_\alpha \in \mathscr{U} : \alpha < \mathfrak{c} \rangle$ by making $V_\alpha = U$ for every $\alpha < \mathfrak{c}$. □

Definition 2.3. A filter $\mathscr{F}$ is OK-extendible provided there exists an OK-point $\mathscr{U}$ such that $\mathscr{F} \subseteq \mathscr{U}$.

Definition 2.4 (K. Kunen [4])**.** An Independent Linked System with respect to $\mathscr{F}$ (ILS w.r.t. $\mathscr{F}$) is a system $\{A^{\beta}_{\alpha,n} \colon \alpha,\beta < \mathfrak{c}; n \geq 1\}$ of infinite subsets of ω satisfying the following conditions:

(a) $\forall \beta < \mathfrak{c}, n \geq 1$, $\sigma \in [\mathfrak{c}]^n$, $\tau \in [\mathfrak{c}]^{n+1}$; $\bigcap_{\alpha\in\sigma} A^{\beta}_{\alpha,n} \in \mathcal{I}^{+}_{\mathscr{F}}$ and $\bigcap_{\alpha\in\tau} A^{\beta}_{\alpha,n+1} \in [\omega]^{<\omega}$.

(b) $\forall \alpha, \beta < \mathfrak{c}, n \geq 1$ $A^{\beta}_{\alpha,n} \subseteq A^{\beta}_{\alpha,n+1}$.

(c) $\forall \tau \in [\mathfrak{c}]^{<\omega}$, $\beta \in \tau$, $n_\beta \geq 1$, $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$, $\bigcap_{\beta\in\tau}\bigcap_{\alpha\in\sigma_\beta} A^{\beta}_{\alpha,n_\beta} \in \mathcal{I}^{+}_{\mathscr{F}}$.

Definition 2.5. We call a filter $\mathscr{F}$, OK-friendly provided that there is an ILS w.r.t. to $\mathscr{F}$.

Theorem 2.6 (K. Kunen [4])**.** *The filter $\mathscr{F}_{\mathrm{cof}}$ is* OK*-friendly.*

Proof. (P. Simon [4]) Let $\mathcal{P}(\omega) = \{X_\alpha \colon \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\omega)$ and $S = \{(k,f) \colon k < \omega \ \&\ f \in \mathcal{P}(\mathcal{P}(k))^{\mathcal{P}(k)}\}$. If we put

$$A^{\beta}_{\alpha,n} = \{(k,f) \in S \colon X_\alpha \cap k \in f(X_\beta \cap k) \ \&\ |f(X_\beta \cap k)| \leq n\}.$$

then, $\{A^{\beta}_{\alpha,n} \colon \alpha, \beta < \mathfrak{c}; n \geq 1\}$ is an ILS w.r.t. $\mathscr{F}_{\mathrm{cof}}$. □

The proof of the next theorem is that in [4] however, in that paper only extensions of $\mathscr{F}_{\mathrm{cof}}$ were considered.

Theorem 2.7. *Every* OK*-friendly filter is* OK*-extendible.*

Proof. (K. Kunen [4]) Fix and enumeration $\{B_\mu \colon \mu < \mathfrak{c} \text{ is even}\}$ of $\mathcal{P}(\omega)$ and a listing $\langle\langle C^{\mu}_{n} \colon n < \omega\rangle \colon \mu < \mathfrak{c} \text{ is odd}\rangle$ of the decreasing sequences in $[\omega]^\omega$ where every sequence appears listed cofinally often. Let $\mathscr{F}$ be OK-friendly and let $\{A^{\beta}_{\alpha,n} \colon \alpha, \beta < \mathfrak{c}; n \geq 1\}$ an ILS w.r.t $\mathscr{F}$. We will construct families $\{\mathscr{F}_\mu \colon \mu < \mathfrak{c}\}$ and $\{K_\mu \colon \mu < \mathfrak{c}\}$ of filters on ω and subsets of $\mathfrak{c}$ respectively satisfying the following conditions:

(1) $\mathscr{F}_0 = \mathscr{F}$ and $K_0 = \mathfrak{c}$.

(2) If $\mu < \nu < \mathfrak{c}$ then, $\mathscr{F}_\mu \subseteq \mathscr{F}_\nu$ and $K_\nu \subseteq K_\mu$.

(3) If $\nu < \mathfrak{c}$ is limit, $\mathscr{F}_\nu = \bigcup_{\mu<\nu} \mathscr{F}_\mu$ and $K_\nu = \bigcap_{\mu<\nu} K_\mu$.

(4) If $\mu < \mathfrak{c}$ then $|K_\mu \setminus K_{\mu+1}| < \omega$.

(5) If $\mu < \mathfrak{c}$ is even then either $B_\mu \in \mathscr{F}_{\mu+1}$ or $\omega \setminus B_\mu \in \mathscr{F}_{\mu+1}$.

(6) If $\mu < \mathfrak{c}$ is odd and $\{C^{\mu}_{n} \colon n < \omega\} \subseteq \mathscr{F}_\mu$ there is a sequence $\langle D^{\mu}_{\alpha} \in \mathscr{F}_{\mu+1} \colon \alpha < \mathfrak{c}\rangle$ which is OK for $\{C^{\mu}_{n} \colon n < \omega\}$.

(7) $\{A^{\beta}_{\alpha,n} \colon \alpha < \mathfrak{c},\ \beta \in K_\mu; n \geq 1\}$ is an ILS w.r.t. $\mathscr{F}_\mu$.

If this construction is possible put $\mathscr{U} = \bigcup_{\mu<\mathfrak{c}} \mathscr{F}_\mu$. Conditions (1) and (5) imply that $\mathscr{U}$ is an ultrafilter extending $\mathscr{F}$ and condition (6) that $\mathscr{U}$ is an OK-point. Thus, we only need to show by induction that this construction can be carried out. By condition (3) this is obvious for the limit step. Therefore, suppose that $\mathscr{F}_\mu$ and K_μ are defined. We want to show how to perform the construction of $\mathscr{F}_{\mu+1}$ and $K_{\mu+1}$. If μ is even, the filter $\langle \mathscr{F}_\mu \cup \{B_\mu\}\rangle$ is proper and $\{A^\beta_{\alpha,n}\colon \alpha < \mathfrak{c}, \beta \in K_\mu; n \geq 1\}$ is an ILS w.r.t. $\langle \mathscr{F}_\mu \cup \{B_\mu\}\rangle$ put $\mathscr{F}_{\mu+1} = \langle \mathscr{F}_\mu \cup \{B_\mu\}\rangle$ and $K_{\mu+1} = K_\mu$. Otherwise, there exist $F \in \mathscr{F}_\mu$, $\tau \in [K_\mu]^{<\omega}$, $n_\beta \geq 1$ and $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$ for each $\beta \in \tau$ such that

$$F \cap B_\mu \cap \bigcap_{\beta\in\tau} \bigcap_{\alpha\in\sigma_\beta} A^\beta_{\alpha,n_\beta} = \emptyset.$$

Then put $K_{\mu+1} = K_\mu \setminus \tau$ and let $\mathscr{F}_{\mu+1}$ be the filter generated by $\mathscr{F}_\mu$ and $\bigcap_{\beta\in\tau}\bigcap_{\alpha\in\sigma_\beta} A^\beta_{\alpha,n_\beta}$. Notice that in this case, $\omega \setminus B_\mu \in \mathscr{F}_{\mu+1}$. If μ is odd and there is a C^μ_n not in $\mathscr{F}_\mu$ then, put $\mathscr{F}_{\mu+1} = \mathscr{F}_\mu$ and $K_{\mu+1} = K_\mu$. Otherwise, by condition (4) $K_\mu \neq \emptyset$. Thus pick $\beta \in K_\mu$. Let

$$D^\mu_\alpha = \left(\bigcap_{n<\omega} C^\mu_n\right) \cup \left(\bigcup_{m\geq 1} A^\beta_{\alpha,m} \cap (C^\mu_m \setminus C^\mu_{m+1})\right) \text{ for every } \alpha < \mathfrak{c}.$$

The union on the right is infinite because it contains $A^\beta_{\alpha,1} \cap C^\mu_1 \in \mathcal{I}^+_{\mathscr{F}_\mu}$. Thus, D^μ_α is infinite for every $\alpha < \mathfrak{c}$. We are going to check that if $F \in [\mathfrak{c}]^n$ and $n \geq 1$ then, $|\bigcap_{\alpha\in F} D^\mu_\alpha \setminus C^\mu_n| < \omega$. This is true if $n = 1$ because $D^\mu_\alpha \setminus C^\mu_1 = \emptyset$. Suppose that $n > 1$. We check that $\bigcap_{\alpha\in F} D^\mu_\alpha \setminus C^\mu_n \subseteq \bigcap_{\alpha\in F} A^\beta_{\alpha,n-1}$. If $x \in \bigcap_{\alpha\in F} D^\mu_\alpha \setminus C^\mu_n$ then, $x \notin C^\mu_n$ and for every $\alpha \in F$ there is a $m_\alpha \geq 1$ such that x is in $A^\beta_{\alpha,m_\alpha} \cap (C^\mu_{m_\alpha} \setminus C^\mu_{m_\alpha+1})$. Notice that $m_\alpha < n$ for every $\alpha \in F$ otherwise, we get a contradiction because $x \notin C^\mu_n$. So, $x \in \bigcap_{\alpha\in F} A^\beta_{\alpha,m_\alpha} \subseteq \bigcap_{\alpha\in F} A^\beta_{\alpha,n-1}$. Since this last intersection is finite by clause (a) in Definition 2.4 we get that the sequence $\langle D^\mu_\alpha \colon \alpha < \mathfrak{c}\rangle$ is OK for $\{C^\mu_n \colon n < \omega\}$. To verify condition (7) it is enough to notice that $A^\beta_{\alpha,m} \cap C^\mu_m \subseteq D^\mu_\alpha$ for every $m \geq 1$. □

Corollary 2.8. *Every* OK*-friendly filter can be extended to* $2^\mathfrak{c}$*-many* OK*-points which are not* P*-points.*

Proof. Let $\mathscr{F}$ be OK-friendly and let $\{A^\beta_{\alpha,n}\colon \alpha,\beta < \mathfrak{c}; n \geq 1\}$ be an ILS w.r.t. $\mathscr{F}$. Fix $Z \subseteq \mathfrak{c}$ such that $|Z| = |\mathfrak{c} \setminus Z| = \mathfrak{c}$ and $Z_0 \in [Z]^\omega$. For each $h\colon Z \to \mathfrak{c}$ let $\mathscr{F}_h$ be the filter generated by $\mathscr{F}$, $\{A^\xi_{h(\xi),1}\colon \xi \in Z\}$ and the family $\{\omega \setminus Y \colon \forall \xi \in Z_0\ |Y \setminus A^\xi_{h(\xi),1}| < \omega\}$. Then, $\mathscr{F} \subseteq \mathscr{F}_h$, the family $\{A^\beta_{\alpha,n}\colon \alpha < \mathfrak{c}, \beta \in \mathfrak{c} \setminus Z; n \geq 1\}$ is an ILS w.r.t. $\mathscr{F}_h$ and, $\mathscr{F}_h$ cannot be extended to a P-point. Notice that if $h_1 \neq h_2$ and $h_1(\xi) \neq h_2(\xi)$ then $|A^\xi_{h_1(\xi),1} \cap A^\xi_{h_2(\xi),1}| < \omega$. Thus, $A^\xi_{h_1(\xi),1} \in \mathscr{F}_{h_1}$ and $\omega \setminus A^\xi_{h_1(\xi),1} \in \mathscr{F}_{h_2}$. Therefore, the extensions of $\mathscr{F}_{h_1}$ and $\mathscr{F}_{h_2}$ must be different. Since there are $\mathfrak{c}^{|Z|} = 2^\mathfrak{c}$-many of such functions h we are done. □

Proposition 2.9 (M. Talagrand [6])**.** *A filter $\mathscr{F}$ on ω is meager if and only if there exists a partition $\{I_i : i < \omega\}$ of ω into finite sets such that every member $\mathscr{F}$ intersects every I_i except for finitely many of them.*

Theorem 2.10. *Every meager filter is* OK*-friendly therefore, it is* OK*-extendible. In particular, every analytic filter is* OK*-friendly.*

Proof. Let $\mathscr{F}$ be a meager filter and let $\{I_i : i < \omega\}$ be a partition as in Proposition 2.9. Let $\{A^\beta_{\alpha,n} : \alpha, \beta < \mathfrak{c}; n \geq 1\}$ be an ILS w.r.t. $\mathscr{F}_{\text{cof}}$ and put $B^\beta_{\alpha,n} = \bigcup\{I_i : i \in A^\beta_{\alpha,n}\}$ for every $\alpha, \beta < \mathfrak{c}$ and $n \geq 1$. Then, $\{B^\beta_{\alpha,n} : \alpha, \beta < \mathfrak{c}; n \geq 1\}$ satisfies clauses (a), (b) and (c) in Definition 2.2 because $\{A^\beta_{\alpha,n} : \alpha, \beta < \mathfrak{c}; n \geq 1\}$ does. $\square$

Corollary 2.11. *Every $<\mathfrak{b}$-generated filter is* OK*-friendly and $\mathfrak{b} = \mathfrak{c}$ implies that every $<\mathfrak{c}$-generated filter is* OK*-friendly.*

Proof. This is because every $<\mathfrak{b}$-generated filter is meager. $\square$

Lemma 2.12. *Let X be a countable set, $\mathscr{U}$ an ultrafilter and $\{S_k : k < \omega\}$ a partition of X into finite subsets such that $\sum_{k<\omega} 2^{-|S_k|} < \infty$. If*

$$\mathscr{F}_{\mathscr{U}} = \{A \subseteq X : \{k < \omega : S_k \subseteq A\} \in \mathscr{U}\}$$

then, $\mathscr{F}_{\mathscr{U}}$ is a filter in $\mathcal{N} \setminus \mathcal{M}$.

Proof. Let $Z_k = \{\chi_A \in 2^X : S_k \subseteq A\}$ for every $k < \omega$. Then, $\mu(Z_k) = 2^{-|S_k|}$ for every $k < \omega$ and, $\mathscr{F}_{\mathscr{U}} \subseteq \bigcap_{n<\omega} \bigcup_{k \geq n} Z_k \in \mathcal{N}$. To see that $\mathscr{F}_{\mathscr{U}}$ is not meager let $\{I_i : i < \omega\}$ be a partition of X into finite sets and let $\{J_r : r < \omega\}$ be a partition of ω into finite sets such that for every $r < \omega$ there is an $i < \omega$ such that $I_i \subseteq \bigcup\{S_k : k \in J_r\}$. Since $\mathscr{U}$ is an ultrafilter either $\bigcup_{r<\omega} J_{2r} \in \mathscr{U}$ or $\bigcup_{r<\omega} J_{2r+1} \in \mathscr{U}$. Say $\bigcup_{r<\omega} J_{2r} \in \mathscr{U}$ and let $X = \bigcup\{S_k : k \in \bigcup_{r<\omega} J_{2r}\}$. Then, $X \in \mathscr{F}_{\mathscr{U}}$ and $X \cap I_i = \emptyset$ for every $I_i \subseteq \bigcup\{S_k : k \in \bigcup_{r<\omega} J_{2r+1}\}$. Thus, $\mathscr{F}_{\mathscr{U}} \notin \mathcal{M}$ by Proposition 2.9. $\square$

Theorem 2.13. *There are $2^{\mathfrak{c}}$-many measurable,* OK*-friendly and non-meager filters on ω.*

Proof. Put $S_k = \{k\} \times \mathcal{P}(\mathcal{P}(k))^{\mathcal{P}(k)}$ and $X = \bigcup\{S_k : k < \omega\}$. Then, $\{S_k : k < \omega\}$ is a disjoint family of finite subsets of X with $|S_k| = 2^{2^{2k}}$ for every $k < \omega$. Thus, $\sum_{k<\omega} 2^{-|S_k|} < \infty$. If $\mathscr{U}$ is arbitrary let $\mathscr{F}_{\mathscr{U}}$ be the filter $\mathscr{F}_{\mathscr{U}} = \{A \subseteq X : \{k < \omega : S_k \subseteq A\} \in \mathscr{U}\}$. Then, $\mathscr{F}_{\mathscr{U}} \in \mathcal{N} \setminus \mathcal{M}$. Consider the family $\{A^\beta_{\alpha,n} : \alpha, \beta < \mathfrak{c}; n \geq 1\}$ defined in the proof of Theorem 2.2. We will check only that condition (c) in definition 2.4 holds. Let $\tau \in [\mathfrak{c}]^{<\omega}$, and for every $\beta \in \tau$, let $1 \leq n_\beta < \omega$ and $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$.

Pick $U \in \mathscr{U}$ and $k_0 \in U$ so big that (i) $\forall \beta, \beta' \in \tau$ such that $\beta \neq \beta'$, $X_\beta \cap k_0 \neq X_{\beta'} \cap k_0$ and (ii) $\forall \beta \in \tau, \forall \alpha \in \sigma_\beta$ $|\{X_\alpha \cap k_0 \colon \alpha \in \sigma_\beta\}| = n_\beta$. Then, for $k \in U$ and $k \geq k_0$ define a function $f_k \colon \mathcal{P}(k) \to \mathcal{P}(\mathcal{P}(k))$ by

$$f_k(Z) = \begin{cases} \{X_\alpha \cap k \colon \alpha \in \sigma_\beta\} & \text{if } Z = X_\beta \cap k \text{ for some } \beta \in \tau \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, $(k, f_k) \in \bigcap_{\beta \in \tau} \bigcap_{\alpha \in \sigma_\beta} A^\beta_{\alpha, n_\beta} \cap S_k$. Therefore, given $X \in \mathscr{F}_{\mathscr{U}}$ then, $U = \{k < \omega \colon S_k \subseteq X\} \in \mathscr{U}$. Thus, $(k, f_k) \in \bigcap_{\beta \in \tau} \bigcap_{\alpha \in \sigma_\beta} A^\beta_{\alpha, n_\beta} \cap S_k$ for all but finitely many $k \in U$. So, $\bigcap_{\beta \in \tau} \bigcap_{\alpha \in \sigma_\beta} A^\beta_{\alpha, n_\beta} \in \mathcal{I}^+_{\mathscr{F}_{\mathscr{U}}}$. Since there are $2^{\mathfrak{c}}$-many nonprincipal ultrafilters on ω we are done. □

Proposition 2.14 (T. Bartoszynski/S. Shelah [1]). *If $M \models$ "$\mathsf{ZFC} + \mathfrak{c} = \omega_2$" then there exists $\mathbb{P} \in M$ such that in M, $\mathbb{P}$ is a support finite iteration of c.c.c forcing notions in ω_1 stages such that if G is a $\mathbb{P}$-generic filter over M then*

$$M[G] \models \text{"}\mathsf{ZFC} + \mathfrak{c} = \omega_2 + \exists\, \mathbf{U} \in [\omega^*]^{\omega_1} \textit{ such that } \bigcap \mathbf{U} \in \mathcal{N}.\text{"}$$

Lemma 2.15. *Let $\mathbf{U}$ be a family of ultrafilters on ω such that $|\mathbf{U}| < \mathfrak{c}$. If $\mathscr{F}$ is arbitrary, then $\mathscr{F} \otimes \bigcap \mathbf{U}$ is not* OK*-extendible.*

Proof. Put $L_n = \bigcup\{\{i\} \times \omega \colon i > n\}$ for every $n < \omega$ and let $\mathscr{V}$ be an ultrafilter extending $\mathscr{F} \otimes \bigcap \mathbf{U}$. Notice that $\{L_n \colon n < \omega\} \subseteq \mathscr{V}$. We claim that no sequence $\langle V_\alpha \in \mathscr{V} \colon \alpha < \mathfrak{c}\rangle$ is OK for $\{L_n \colon n < \omega\}$. In fact, we can find an uncountable $X \subseteq \mathfrak{c}$, an $\mathscr{U} \in \mathbf{U}$ and $n \geq 1$ such that $\forall \alpha \in X$, $(V_\alpha)_n \in \mathscr{U}$. If $F \in [X]^n$ then, $|(\bigcap_{\alpha \in F} V_\alpha)_n| = |\bigcap_{\alpha \in F} (V_\alpha)_n| = \omega$. Thus, $|\bigcap_{\alpha \in F} V_\alpha \setminus L_n| = \omega$ and $\langle V_\alpha \in \mathscr{U} \colon \alpha < \mathfrak{c}\rangle$ is not OK for $\{L_n \colon n < \omega\}$. □

The next lemma is a consequence of a more general theorem by M. Talagrand (see [6], Proposition 15). For the sake of the paper we give a self-contained proof.

Lemma 2.16. *If $\mathscr{G} \in \mathcal{N}$ and $\mathscr{F}$ is any filter then, $\mathscr{F} \otimes \mathscr{G} \in \mathcal{N}$.*

Proof. Let μ_1 and μ_2 be the standard measures on 2^ω and $2^{\omega \times \omega}$ respectively. If $n < \omega$ let $f_n \colon 2^{\omega \times \omega} \to 2^\omega$ be defined by $f_n(\chi_A) = \chi_{(A)_n}$. Then, f_n is continuous and, for every basic open set $O \subseteq 2^\omega$, $f_n^{-1}[O] \subseteq 2^{\omega \times \omega}$ is a basic open set such that $\mu_2(f_n^{-1}[O]) = \mu_1(O)$. Thus, $f_n^{-1}[X]$ is a μ_2-null set provided X is μ_1-null. Since $\mathscr{H} \subseteq \bigcap_{n<\omega} \bigcup_{k \geq n} f_k^{-1}[\mathscr{G}]$ we are done. □

Proposition 2.17 (J. Roitman [5]). *P-points exist in iterated c.c.c forcing extensions whose length has uncountable cofinality.*

Theorem 2.18. *There exists a model N of $\mathsf{ZFC} + \mathfrak{c} = \omega_2$ such that in N there are filters $\mathscr{F}_1, \mathscr{F}_2 \in \mathcal{N}$ with $\mathscr{F}_1$ not* OK*-extendible and $\mathscr{F}_2$* OK*-extendible but not* OK*-friendly.*

Proof. Let $N = M[G]$ and $\{\mathscr{U}_\alpha \colon \alpha < \omega_1\}$ be the model and the family of ultrafilters described in Proposition 2.14. Then, $\bigcap_{\alpha<\omega_1} \mathscr{U}_\alpha \in \mathcal{N}$. If $\mathscr{F}$ is any filter then, $\mathscr{F}_1 = \mathscr{F} \otimes \bigcap_{\alpha<\omega_1} \mathscr{U}_\alpha \in \mathcal{N}$ by Lemma 2.16 and it is not OK-extendible by Lemma 2.15. In order to construct $\mathscr{F}_2$ notice that we are in the situation of Proposition 2.17. Therefore, let $\mathscr{U}_{\omega_1} \in M[G]$ be a P-point. Put $\mathscr{F}_2 = \bigcap_{\alpha\leq\omega_1} \mathscr{U}_\alpha$. Then, $\mathscr{F}_2 \subseteq \mathscr{U}_{\omega_1}$, $\mathscr{F}_2 \in \mathcal{N}$ and $\mathscr{F}_2$ is OK-extendible by Proposition 2.2. To see that $\mathscr{F}_2$ is not OK-friendly notice that if $\{A_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathcal{I}^+_{\mathscr{F}_2}$ then there is an $\alpha \leq \omega_1$ and an uncountable $X \subseteq \mathfrak{c}$ such that $A_\xi \in \mathscr{U}_\alpha$ for every $\xi \in X$. Thus, if $n \geq 1$ and $F \in [X]^n$ then $\bigcap_{\xi\in F} A_\xi \in \mathscr{U}_\alpha$ and $|\bigcap_{\xi\in F} A_\xi| = \omega$. Hence, there is no ILS w.r.t. $\mathscr{F}_2$. □

3. Generic Existence

Definition 3.1 (R. M. Canjar [2])**.** Let $\mathscr{C}$ be a class of ultrafilters. The ultrafilters from $\mathscr{C}$ generically exist provided every $< \mathfrak{c}$-generated filter can be extended to an ultrafilter in $\mathscr{C}$.

We abbreviate $GE(\mathscr{C}, \mathfrak{c})$ the statement "ultrafilters in $\mathscr{C}$ generically exist". The next two propositions relate the generic existence of P-points and selective ultrafilters with certain cardinal invariants. Here, P and S stand for the class of P-points and selective ultrafilters respectively.

Proposition 3.2 (J. Ketonen [3])**.**

$$GE(P, \mathfrak{c}) \Leftrightarrow \mathfrak{d} = \mathfrak{c}.$$

Proposition 3.3 (R. M. Canjar [2])**.**

$$GE(S, \mathfrak{c}) \Leftrightarrow \mathsf{cov}(\mathcal{M}) = \mathfrak{c}.$$

Lemma 3.4. *Given $\mathscr{F}$ and $\mathscr{U}$ the filter $\mathscr{F} \otimes \mathscr{U}$ is not* OK-*extendible. Moreover, if $\mathscr{U}$ is a P-point then $\chi(\mathscr{F} \otimes \mathscr{U}) \leq max\{\chi(\mathscr{F}), \chi(\mathscr{U}), \mathfrak{d}\}$. In particular $\mathscr{U} \otimes \mathscr{U}$ is not an* OK-*point and $\chi(\mathscr{U} \otimes \mathscr{U}) = max\{\chi(\mathscr{U}), \mathfrak{d}\}$ provided $\mathscr{U}$ is a P-point.*

Proof. The first part follows from Lemma 2.15 by taking $\mathbf{U} = \{\mathscr{U}\}$. Let $\{F_\xi \colon \xi < \chi(\mathscr{F})\}$ and $\{U_\eta \colon \eta < \chi(\mathscr{U})\}$ be bases of $\mathscr{F}$ and $\mathscr{U}$ respectively and let $\{f_\gamma \colon \gamma < \mathfrak{d}\}$ be a dominating family in ω^ω. Consider the family $\{V_{\xi,\eta,\gamma} \colon \xi < \chi(\mathscr{F}), \eta < \chi(\mathscr{U}), \gamma < \mathfrak{d}\}$ where $V_{\xi,\eta,\gamma}$ is defined by $V_{\xi,\eta,\gamma} = \bigcup\{\{n\} \times (U_\eta \setminus f_\gamma(n)) \colon n \in F_\xi\}$. This is a basis of $\mathscr{F} \otimes \mathscr{U}$. □

In reference [4] K. Kunen constructed an OK-point and he explained how to modify that construction to get it not P-point and $\mathfrak{c}$-generated. The next lemma shows that we have only to worry about the not P-point condition.

Lemma 3.5. *If $\mathscr{U}$ is an* OK-*point but not a P-point, then, $\chi(\mathscr{U}) = \mathfrak{c}$.*

Proof. If $\mathscr{U}$ is not a P-point then, there is a partition $\mathcal{P} = \{P_n : n < \omega\}$ of ω such that $\mathcal{P} \cap \mathscr{U} = \emptyset$ and for every $U \in \mathscr{U}$, $|U \cap P_n| = \omega$ for infinitely many $n < \omega$. Therefore, $\{L_n : n < \omega\} \subseteq \mathscr{U}$ provided $L_n = \bigcup\{P_i : i > n\}$. Suppose that $\chi(\mathscr{U}) < \mathfrak{c}$ and let $\{U_\xi : \xi < \chi(\mathscr{U})\}$ be a base for $\mathscr{U}$. There exist an uncountable $X \subseteq \mathfrak{c}$ and a $\xi < \chi(\mathscr{U})$ such that $U_\xi \subseteq V_\alpha$ for every $\alpha \in X$. Since $U_\xi \in \mathscr{U}$, there is an $n \geq 1$ such that $|U_\xi \cap P_n| = \omega$. If $F \in [X]^n$ then, $U_\xi \subseteq \bigcap_{\alpha \in F} V_\alpha$ and $|\bigcap_{\alpha \in F} V_\alpha \setminus L_n| = \omega$. Thus, no sequence $\langle V_\alpha \in \mathscr{U} : \alpha < \mathfrak{c}\rangle$ can be OK for $\{L_n : n < \omega\}$. $\square$

Theorem 3.6.

$$\mathfrak{d} = \mathfrak{c} \Rightarrow GE(OK, \mathfrak{c}) \Rightarrow max\{\mathfrak{u}, \mathfrak{d}\} = \mathfrak{c}.$$

Proof. The implication on the left follows from Propositions 2.2 and 3.2. To prove the implication on the right suppose by the way of contradiction, that $GE(OK, \mathfrak{c})$ holds but $\max\{\mathfrak{u}, \mathfrak{d}\} < \mathfrak{c}$. Let $\mathscr{U}$ be such that $\chi(\mathscr{U}) = \mathfrak{u} < \mathfrak{c}$. Then, $\mathscr{U}$ is an OK-point. Moreover, $\mathscr{U}$ must be a P-point because if otherwise, $\chi(\mathscr{U}) = \mathfrak{c}$ by Lemma 3.5 and this is impossible. By Lemma 3.4 $\chi(\mathscr{U} \otimes \mathscr{U}) \leq \max\{\mathfrak{u}, \mathfrak{d}\} < \mathfrak{c}$ therefore, $\mathscr{U} \otimes \mathscr{U}$ is OK and this contradicts Lemma 3.4. $\square$

Corollary 3.7. *$GE(OK, \mathfrak{c})$ is independent of the axioms of* ZFC.

Proof. The identity $\mathfrak{d} = \mathfrak{c}$ holds for example, in any model of $\mathsf{ZFC} + \mathsf{MA} + \mathfrak{c} > \omega_2$. On the other hand, in the model of ZFC obtained by iterating Sacks reals with countable supports over a model of $\mathsf{ZFC} + \mathsf{CH}$ all the cardinal invariants of the continuum are equal to ω_1 but $\mathfrak{c} = \omega_2$. Therefore, in that model "$\max\{\mathfrak{u}, \mathfrak{d}\} = \omega_1 < \mathfrak{c}$". $\square$

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