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TOPOLOGY PROCEEDINGS

Volume 47, 2016

Pages 101–114

http://topology.nipissingu.ca/tp/

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Electronically published on May 2, 2015

Topology Proceedings

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ISSN:	0146-4124

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TOPOLOGY PROCEEDINGS
Volume 47 (2016)
Pages 101-114

E-Published on May 2, 2015

ON CHARMING SPACES AND SOME RELATED SUBCLASSES

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Abstract. The class of charming spaces was introduced by A.V. Arhangel'skii in [Remainders of metrizable spaces and a generalization of Lindelöf Σ-spaces, Fund. Math., 215 (2011), 87–100]. The purpose of this paper is to show some relevant properties of this new class of topological spaces. We present generalizations to some results of V. V. Tkachuk related to the lifting of topological properties. Also we show that for every $\aleph_0$-bounded topological group G, G is a Lindelöf Σ-space iff G is a $(\mathcal{K}, L\Sigma)$-structured space. As a consequence we prove that, for every Tychonoff space X, the function space $C_p(X)$ is Lindelöf-Σ if and only if is $(\mathcal{K}, L\Sigma)$-structured, and that, for X compact, $C_p(X)$ is Lindelöf-Σ if and only if $C_p(X)$ is $(L\Sigma, L\Sigma)$-structured.

1. Introduction

In this paper we study the class of charming spaces. The class of charming spaces (or $(L\Sigma, L\Sigma)$-structured spaces) was introduced by A. V. Arhangel'skii in [1] as an extension of the class of Lindelöf Σ-spaces. Our "basic conjecture" is that many results in the class of the Lindelöf Σ-spaces, can be "transformed" into results in this new class of spaces or some intermediate class between the class of Lindelöf Σ-spaces and the $(L\Sigma, L\Sigma)$-structured spaces.

Among other things, we prove that the class of $(L\Sigma, L\Sigma)$-structured spaces has nice categorical properties. For example, we prove that this class is closed under countable unions, closed subspaces and continuous images.

2010 *Mathematics Subject Classification.* Primary 54D20, 54E20; Secondary 54H11.

Key words and phrases. Charming spaces, Topological Function Spaces, Lindelöf-Σ spaces, monotonically monolithic spaces, Collins-Roscoe property.

Research supported by Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica (PAPIIT), Grant IN115312.

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Besides, it is well-known that if a topological space X condenses (that is, there is a continuous bijection) onto a topological space Y with some property $\mathcal{P}$, this does not imply that the topological space X has the property $\mathcal{P}$. In this sense we say that $\mathcal{P}$ is "not lifted" by condensations. In [4], V. V. Tkachuk made a first systematic study of topological properties that are lifted by condensations in Lindelöf Σ-spaces. In this paper we are going to prove that some results of Tkachuk can be extended to classes of $(L\Sigma, L\Sigma)$-structured spaces. For example, we prove that the property of being a monotonically κ-monolithic space is lifted by condensations in $(L\Sigma, L\Sigma)$-structured spaces. Also, we prove that if X is a $(\mathcal{K}, L\Sigma)$-structured space such that X condenses onto a κ-monolithic space, then X is κ-monolithic.

In the last part of this paper we show that for every $\aleph_0$-bounded topological group G, G is a Lindelöf Σ-space iff G is a $(\mathcal{K}, L\Sigma)$-structured space. As a consequence we prove that, for every Tychonoff space X, the function space $C_p(X)$ is Lindelöf Σ if and only if it is $(\mathcal{K}, L\Sigma)$-structured, and that, for X compact, $C_p(X)$ is Lindelöf Σ if and only if $C_p(X)$ is $(L\Sigma, L\Sigma)$-structured.

2. Notation and terminology

All spaces are assumed to be Tychonoff, unless otherwise is stated. Given a space X we denote by $\tau(X)$ its topology. If X is a topological space and $A \subseteq X$ then $\tau(A, X) = \{U \in \tau(X) : A \subseteq U\}$; given a point $x \in X$ we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The space $\mathbb{R}$ is the real line with the usual order topology and $\mathbb{N}$ will denote the set of natural numbers with the subspace topology. $\mathbb{N}^+ = \omega \setminus \{0\}$.

For any infinite cardinal κ, we denote by D_κ the discrete space of cardinality κ, and by L_κ the *one-point Lindelöfication of* D_κ; thus, L_κ is the set $D_\kappa \cup \{\infty\}$ with the topology in which all points of the set D_κ are isolated, and a set U is a neighborhood of the point ∞ if and only if $\infty \in U$ and the complement of U in D_κ is at most countable.

For any space X we denote by $C_p(X)$ the set of all real-valued continuous functions on X endowed with the topology of pointwise convergence. A map $f : X \to Y$ is called a *condensation* if it is a continuous bijection; in this case we say that X *condenses onto* Y.

A family $\mathcal{N}$ of subsets of a space X is a *network with respect to a cover* $\mathcal{C}$ (or a network modulo $\mathcal{C}$) of X if for every set $C \in \mathcal{C}$ and every $U \in \tau(C, X)$ there is an element $N \in \mathcal{N}$ such that $C \subset N \subset U$.

A topological space X is a *Lindelöf Σ-space* (or $L\Sigma$) if there is a countable family $\mathcal{N}$ of subsets of X such that $\mathcal{N}$ is a network with respect to a compact cover $\mathcal{C}$ of X.

We denote by $\mathcal{K}$ the class of compact spaces, $\sigma\mathcal{K}$ the class of σ-compact spaces, $\mathcal{M}$ the class of metrizable separable spaces, $\mathcal{L}$ the class of Lindelöf spaces, and $L\Sigma$ the class of Lindelöf Σ-spaces. $\mathcal{P}$ and $\mathcal{Q}$ will denote classes of topological spaces. When Y is a subspace of X such that Y belongs to $\mathcal{P}$, we say that Y is a $\mathcal{P}$-subspace of X.

Given a space X assume that, for every point $x \in X$, a countable family $\mathcal{G}(x)$ of subsets of X is chosen. We say that $\{\mathcal{G}(x) : x \in X\}$ is a *Collins-Roscoe collection* if, for any $x \in X$ and for any $U \in \tau(x, X)$, we can find an open set V such that $x \in V \subset U$ and for any $y \in V$ there exists a set $P \in \mathcal{G}(y)$ with $x \in P \subset U$. If a space X has a Collins-Roscoe collection then we will say that X has the Collins-Roscoe property.

The rest of our notation is standard and follows [2].

3. $(\mathcal{P}, \mathcal{Q})$-structured spaces

The following concept was introduced by A. V. Arhangel'skii in [1].

Definition 3.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be some classes of topological spaces. A topological space X will be called $(\mathcal{P}, \mathcal{Q})$-*structured* if there exists a subspace Y of X (called a $\mathcal{P}$-kernel of X) such that $Y \in \mathcal{P}$ and for every open neighborhood U of Y in X, the subspace $X \setminus U$ belongs to $\mathcal{Q}$.

The $(L\Sigma, L\Sigma)$-structured spaces are also called *charming spaces*.

It is clear that if $\mathcal{P}_0$ is a subclass of $\mathcal{P}$ and $\mathcal{Q}_0$ is a subclass of $\mathcal{Q}$, then every $(\mathcal{P}_0, \mathcal{Q}_0)$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space.

It is easy to see that every Lindelöf Σ-space is $(\mathcal{K}, L\Sigma)$-structured space. Indeed, any compact subset of the space is an $L\Sigma$-kernel.

It follows that all metrizable separable spaces and all the Lindelöf p-spaces are $(\mathcal{K}, L\Sigma)$-structured. Nevertheless, there exist $(L\Sigma, L\Sigma)$-structured spaces that are not Lindelöf Σ-spaces. Indeed, let $\kappa \geq \aleph_1$ and L_κ be the one-point Lindelöfication of the discrete space of cardinality κ, then L_κ is a $(\mathcal{K}, L\Sigma)$-structured space. To verify it, we can take as $L\Sigma$-kernel the subspace $Z = \{\infty\}$, where ∞ the non-isolated point of L_κ; let U be an arbitrary open neighborhood of Z. It is clear that the set $L_\kappa \setminus U$ is a countable subspace of L_κ, and then $L\Sigma$. It is known that L_κ is not Lindelöf Σ-space.

Recall that a space X is *simple* if it has at most one non-isolated point. Observe that the last argument shows also that every simple Lindelöf space is a $(\mathcal{K}, L\Sigma)$-structured space.

The following facts are immediate from the definition. We give a proof of (2), because it will be helpful in the rest of the paper.

Proposition 3.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ classes of topological spaces closed under continuous images, closed subspaces, and such that $\mathcal{M} \subseteq \mathcal{P} \subseteq \mathcal{Q} \subseteq L\Sigma$.*

Then

(1) *every continuous image of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space;*
(2) *every closed subspace of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space;*
(3) *if $\mathcal{P}$ and $\mathcal{Q}$ are closed under perfect preimages, then every perfect preimage of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space;*
(4) *every $(\mathcal{P}, \mathcal{Q})$-structured space is Lindelöf.*

Proof. (2). Let X be a $(\mathcal{P}, \mathcal{Q})$-structured space, Z a $\mathcal{P}$-kernel of X and $F \subset X$ a closed subspace of X. If $F \cap Z = \emptyset$ then F belongs to $\mathcal{Q}$ and therefore is a $(\mathcal{P}, \mathcal{Q})$-structured space.

If $F \cap Z \neq \emptyset$, then $Z_F = F \cap Z$ is a $\mathcal{P}$-subspace of F. We claim that Z_F is a $\mathcal{P}$-kernel of F. Indeed, let U be an open neighborhood of Z_F in F, then there exists an open set $W \subset X$ such that $W \cap F = U$. Let $V = W \cup (X \setminus F)$, note that V is an open neighborhood of Z in X and given that Z is a $\mathcal{P}$-kernel of X, $X \setminus V$ is a $\mathcal{Q}$-subspace of X. Now $X \setminus V = F \setminus U$, and therefore $F \setminus U$ is a $\mathcal{Q}$-subspace of F, and F is $(\mathcal{P}, \mathcal{Q})$-structured with $\mathcal{P}$-kernel Z_F. □

Remark 3.3. If $\mathcal{P}$ and $\mathcal{Q}$ are as in Proposition 3.2, it is easy to see that the product $X \times K$ of a $(\mathcal{P}, \mathcal{Q})$-structured space X and a compact space K is a $(\mathcal{P}, \mathcal{Q})$-structured space. Besides, since a multivalued mapping $p : X \to Y$ is compact-valued upper semicontinuous iff it is a composition of the inverse of a perfect mapping onto a closed subspace of X and a continuous function (see, e.g., [3]), we have that the image of a $(\mathcal{P}, \mathcal{Q})$-structured space under a compact-valued upper semicontinuous mapping is a $(\mathcal{P}, \mathcal{Q})$-structured space.

In the rest of this section we assume that $\mathcal{P}$ and $\mathcal{Q}$ are subclasses of the class of Lindelöf Σ-spaces invariant with respect to countable unions.

Our next result is related to topological sums.

Proposition 3.4. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of $(\mathcal{P}, \mathcal{Q})$-structured spaces. Then $X = \bigoplus\{X_n : n \in \mathbb{N}\}$ is $(\mathcal{P}, \mathcal{Q})$-structured.*

Proof. Since every X_n is $(\mathcal{P}, \mathcal{Q})$-structured, we can take a $\mathcal{P}$-kernel $Z_n \subset X_n$ to define $Z = \bigcup\{Z_n : n \in \mathbb{N}\}$. Then Z is a $\mathcal{P}$-subspace of X. Now consider an open neighborhood U of Z in X. Note that $Z_n \subset U_n = X_n \cap U$ and U_n is an open neighborhood of Z_n in X_n. Since, for all $n \in \mathbb{N}$, Z_n is a $\mathcal{P}$-kernel of X_n it follows that $X_n \setminus U_n$ is a subspace of X_n that belong to $\mathcal{Q}$.

Now it is enough to observe that $X \setminus U = \bigcup\{X_n \setminus U_n : n \in \mathbb{N}\}$ and make use of the fact that the countable union of $\mathcal{Q}$-subspaces is again a $\mathcal{Q}$-space to conclude that X is a $(\mathcal{P}, \mathcal{Q})$-structured space. □

Corollary 3.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be closed under countable unions, X be a space and $\{X_n : n \in \mathbb{N}\}$ a family of $(\mathcal{P}, \mathcal{Q})$-structured subspaces of X. Then $\bigcup\{X_n : n \in \mathbb{N}\}$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.*

Proof. It is a consequence of the following facts: the topological sum of a countable family of $(\mathcal{P}, \mathcal{Q})$-structured spaces is $(\mathcal{P}, \mathcal{Q})$-structured, the function $f : \bigoplus_{n\in\mathbb{N}} X_n \to \bigcup_{n\in\mathbb{N}} X_n$ defined by $f(x) = x$ is continuous and onto, and the continuous image of a $(\mathcal{P}, \mathcal{Q})$-structured space is a $(\mathcal{P}, \mathcal{Q})$-structured space. □

Corollary 3.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be closed under countable unions, and X be a $(\mathcal{P}, \mathcal{Q})$-structured space. Then every F_σ subspace of X is a $(\mathcal{P}, \mathcal{Q})$-structured space.*

As we mentioned before, the product of a $(\mathcal{P}, \mathcal{Q})$-structured space and a compact space is a $(\mathcal{P}, \mathcal{Q})$-structured space. This fact can be generalized in the following way.

Proposition 3.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be closed under countable unions and perfect preimages. If X is a $(\mathcal{P}, \mathcal{Q})$-structured space and Z a σ-compact space, then $X \times Z$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.*

Proof. Let bZ be a compactification of Z, then the projection $\pi_1 : X \times bZ \to X$ is perfect, and since the preimage under a perfect mapping of a $(\mathcal{P}, \mathcal{Q})$-structured space is a space that belongs to the same class, it follows that $X \times bZ$ is a $(\mathcal{P}, \mathcal{Q})$-structured space.

Since Z is σ-compact, we can represent $Z = \bigcup\{K_n : n \in \mathbb{N}\}$ with K_n compact, for every $n \in \mathbb{N}$. Then $X \times K_n$ is a closed subspace of $X \times bZ$, and therefore a $(\mathcal{P}, \mathcal{Q})$-structured space.

Finally, $X \times Z$ is the countable union of $(\mathcal{P}, \mathcal{Q})$-structured spaces, because $X \times Z = \bigcup_{n\in\mathbb{N}}(X \times K_n)$. So, $X \times Z$ is a $(\mathcal{P}, \mathcal{Q})$-structured space. □

A natural question that arises in connection with Proposition 3.7 is the following: Is the product of $(\mathcal{P}, \mathcal{Q})$-structured spaces a $(\mathcal{P}, \mathcal{Q})$-structured space? The next example answers this question in the negative.

Example 3.8. Let $Y = L_\kappa \times L_\kappa$ (with $\kappa \geq \omega_1$) the square of the Lindelöfication of the discrete space of cardinality κ, then Y is not $(L\Sigma, L\Sigma)$-structured space. Indeed, suppose that there exists an $L\Sigma$-kernel Z^* of Y. Since the Lindelöf Σ property is preserved by continuous functions, it follows that the projection of Z^* in the first factor, $Z_1 = \pi_1(Z^*)$, is an $L\Sigma$-subspace of L_κ, so it is a countable subset of L_κ. Let $y \in L_\kappa \setminus Z_1$, and take as U the set $Y \setminus (\{y\} \times L_\kappa)$. Then U is an open neighborhood of Z^* and the complement is homeomorphic to L_κ, but L_κ is not a Lindelöf Σ-space. Therefore, Y is not an $(L\Sigma, L\Sigma)$-structured space.

Note that this example also shows that even the subclass of the $(\mathcal{K}, \mathcal{M})$-structured spaces is not closed under finite products. However we have some positive results about products.

Proposition 3.9. *Let $\mathcal{P} \subseteq L\Sigma$, κ be an infinite cardinal and X be a space in $\mathcal{P}$. Then $L_\kappa \times X$ is a $(\mathcal{P}, L\Sigma)$-structured space.*

Proof. Let ∞ the non-isolated point of L_κ, the subspace $Z = \{\infty\} \times X \subset L_\kappa \times X$ is a space in $\mathcal{P}$. We claim that Z is a $\mathcal{P}$-kernel of $L_\kappa \times X$. Let W be an open neighborhood of Z in $L_\kappa \times X$. Since Z is Lindelöf we can assume that W contains a countable union of basic open sets of $L_\kappa \times X$, i.e. $W \supseteq \bigcup\{U_n \times V_n : n \in \mathbb{N}\}$. Now, for every $n \in \mathbb{N}$, we have that U_n is an open neighborhood of ∞ in L_κ and therefore the set $U = \cap\{U_n : n \in \mathbb{N}\}$ also is an open neighborhood of ∞ in L_κ. So the set $(L_\kappa \setminus U) \times X$ is an $L\Sigma$-space and $(L_\kappa \times X) \setminus W$ is a closed subspace of it, therefore an $L\Sigma$-space. □

Remark 3.10. Note that Proposition 3.9 allows us to construct $(L\Sigma, L\Sigma)$-structured spaces that are not Lindelöf Σ-spaces. Indeed, by 3.9, given a Lindelöf Σ-space X, we have that $L_{\omega_1} \times X$ is an $(L\Sigma, L\Sigma)$-structured space with an $L\Sigma$-kernel homeomorphic to X and is not an $L\Sigma$-space.

In [3] the authors introduce the classes of $L\Sigma(\leq \kappa)$-spaces. We can use these classes to introduce other subclasses of the class of $(L\Sigma, L\Sigma)$-stuctured spaces.

We recall that, given a cardinal κ, finite or infinite, a space X is called an $L\Sigma(< \kappa)$-space if there is a compact cover $\mathcal{C}$ of X such that $w(C) < \kappa$ for every $C \in \mathcal{C}$ and a countable network modulo $\mathcal{C}$ in X.

A space X is an $L\Sigma(\leq \kappa)$-space if it is an $L\Sigma(< \kappa^+)$-space. X is an $L\Sigma(\kappa)$-space if it is an $L\Sigma(\leq \kappa)$-space and not an $L\Sigma(< \kappa)$-space. Of course, for finite κ, the weights of the elements of the compact covers in the above definition can be replaced by the cardinalities.

It is easy to see (see Remark 3.10) that, for every $n \in \mathbb{N}^+$, the class of the $(L\Sigma(\leq n), L\Sigma)$-structured spaces is non-empty and there exists $(L\Sigma(\leq n), L\Sigma)$-structured spaces that are not $L\Sigma$-spaces.

4. Lifting properties

If a topological space X condenses onto a topological space Y with some property $\mathcal{P}$, this does not imply that the topological space X has the property $\mathcal{P}$. In this sense we say that $\mathcal{P}$ is "not lifted" by condensations. For example, the connectedness is not lifted by condensations. Indeed, the Sorgenfrey line condenses onto $\mathbb{R}$, but the Sorgenfrey line is zero-dimensional and $\mathbb{R}$ is connected. In [4], V. V. Tkachuk made a first

systematic study of topological properties that are lifted by condensations in Lindelöf Σ-spaces. In this section we are going to prove that some results of Tkachuk can be extended to class of $(L\Sigma, L\Sigma)$-structured spaces.

Let X be a topological space, recall that a family $\mathcal{G}$ of subsets of X is *a network at a point* $x \in X$ if for any $U \in \tau(x, X)$ there exists $G \in \mathcal{G}$ such that $x \in G \subset U$.

Lemma 4.1. *Let X be a topological space such that there exist a compact cover $\mathcal{C}$ and a network $\mathcal{N}$ modulo $\mathcal{C}$. Suppose that $f : X \to Y$ is a condensation and $\mathcal{F}$ is a network for $y = f(x)$ in Y. Then the family $\mathcal{E} = \{f^{-1}(F) \cap N : F \in \mathcal{F}, N \in \mathcal{N}\}$ is a network at the point $x = f^{-1}(y)$ in X.*

Proof. Let $x = f^{-1}(y)$ and take any $U \in \tau(x, X)$. Given that $\mathcal{C}$ is a compact cover of X, there exists $C_x \in \mathcal{C}$ such that $x \in C_x$. Since the family $\mathcal{F}$ is a network at y, we can choose a set $F \in \mathcal{F}$ such that $y \in F$ and $\overline{F} \cap f(C_x \setminus U) = \emptyset$. If $G = f^{-1}(F)$, then $\overline{G} \cap (C_x \setminus U) = \emptyset$. Since $C_x \setminus U$ is a compact subspace of X, $\overline{G}$ and $C_x \setminus U$ are completely separated subsets of X. Then the closures of $\overline{G}$ and $C_x \setminus U$ in the Stone-Čech compactification of X, βX, are disjoint ([2, 3.6.2]). Hence there exists a set $V \in \tau(C_x \setminus U, X)$ such that $\overline{V} \cap G = \emptyset$. The set $U \cup V$ is an open neighborhood of C_x in X so we can find $N \in \mathcal{N}$ for which $C_x \subset N \subset (U \cup V)$.

The set $E = G \cap N$ is an element of $\mathcal{E}$ such that $x \in E \subset U$. Therefore $\mathcal{E}$ is a network for the point $x = f^{-1}(y)$ in X. □

We are going to prove that the property of being a monotonically κ-monolithic space is lifted by condensations in $(L\Sigma, L\Sigma)$-structured spaces (c.f. Corollary 4.5). First recall the definition of monotonically κ-monolitic space.

Definition 4.2. (1) Given a set $A \subset X$ we say that a family $\mathcal{N}$ of subsets of X is an *external network of A in X* if $\mathcal{N}$ is a network at every $x \in A$.

(2) For an infinite cardinal κ, a space X is *monotonically κ-monolithic* if, for any set $A \subset X$ with $|A| \leq \kappa$, we can assign an external network $\theta(A)$ to the set $\overline{A}$ in such a way that the following conditions are satisfied:

(a) $|\theta(A)| \leq \kappa$;

(b) if $A \subset B$, then $\theta(A) \subset \theta(B)$;

(c) if $\lambda \leq \kappa$ is an ordinal and we have a family $\{A_\alpha : \alpha < \lambda\}$ of subsets of X such that $\alpha < \beta < \lambda$ implies $A_\alpha \subset A_\beta$, then

$$\theta\Big(\bigcup\{A_\alpha : \alpha < \lambda\}\Big) = \bigcup\{\theta(A_\alpha) : \alpha < \lambda\}.$$

(3) A space X is *monotonically monolitic* if it is monotonically κ-monolitic for every infinite cardinal κ.

Theorem 4.3. *Let X be a topological space, $\mathcal{C}$ a compact cover of X and $\mathcal{N}$ a network modulo $\mathcal{C}$ of cardinality less or equal to κ. If there exists a condensation of X onto a monotonically κ-monolithic space Y, then X is monotonically κ-monolithic.*

Proof. Let $f : X \to Y$ be a condensation, θ be a κ-monolithity operator in Y. Take any set $A \subset X$ of cardinality less or equal than κ and put the family $G(A) = \{f^{-1}(B) \cap N : N \in \mathcal{N}, B \in \theta(f(A))\}$. Then $|G(A)| \leq \kappa$, and since f is a condensation, the properties (b) and (c) of the operator θ also hold for the operator G.

Now, if $x \in \overline{A}$, let $y = f(x)$. Then $\theta(f(A))$ is a network at the point y, so we can apply the Lemma 4.1 to conclude that $G(A)$ is a network at the point x. So we have that G is a monotonic κ-monolithity operator on X. □

Let X be a topological space and $Z \subseteq X$, the *character of Z in X* is the cardinal $\chi(Z, X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \tau(Z, X)$ is a base of Z in $X\}$, where $\mathcal{U} \subseteq \tau(Z, X)$ is a *base* of Z in X if for every $W \in \tau(Z, X)$ there is a $U \in \mathcal{U}$ such that $Z \subseteq U \subseteq W$.

Remark 4.4. It is easy to see that if an $(L\Sigma, L\Sigma)$-structured space X has an $L\Sigma$-kernel Z such that $\chi(Z, X) \leq \kappa$, then there is a compact cover $\mathcal{C}$ of X and a network modulo $\mathcal{C}$ of cardinality less or equal than κ.

The next corollary is a slight generalization of a result of Tkachuk in [4].

Corollary 4.5. *Let X be a $(L\Sigma, L\Sigma)$-structured space such that there exists an $L\Sigma$-kernel Z with $\chi(Z, X) \leq \kappa$. If X condenses onto a monotonically κ-monolithic space Y, then X is monotonically κ-monolithic.*

Corollary 4.6 (Tkachuk, [4])**.** *If a Lindelöf Σ-space X condenses onto a monotonically monolithic space, then X is monotonically monolithic.*

Now, we are going to prove that the Collins-Roscoe property is lifted by condensations in certain subclasses of the $(L\Sigma, L\Sigma)$-structured spaces to an adequate and natural variation of the Collins-Roscoe property (c.f. Proposition 4.7)

Let κ be an infinite cardinal. If to every $x \in X$ we assign a collection of cardinality less or equal than κ so that the conditions of the Collins-Roscoe property are satisfied (i.e., for each $x \in X$ and every $U \in \tau(x, X)$, we can find a set V such that $x \in V \subset U$ and, for every $y \in V$, there exists a set $P \in \mathcal{G}(y)$ such that $x \in P \subset U$), we say that X has the κ-Collins-Roscoe property.

Proposition 4.7. *Let X be an $(L\Sigma, L\Sigma)$-structured space, Z an $L\Sigma$ kernel of X with $\chi(Z, X) \leq \kappa$. If X condenses onto a Collins-Roscoe space Y, then X has the κ-Collins-Roscoe property.*

Proof. Let $\mathcal{C}$ be a compact cover of X and $\mathcal{N}$ a network modulo $\mathcal{C}$ of cardinality less or equal than κ. Let $f : X \to Y$ a condensation, $\{\mathcal{G}(y) : y \in Y\}$ a Collins-Roscoe collection in Y. For each $x \in X$, we define the family $\mathcal{G}(x) = \{f^{-1}(F) \cap N : N \in \mathcal{N}, F \in \mathcal{G}(f(x))\}$. It is clear that the cardinality of $\mathcal{G}(x)$ is less or equal than κ.

Take a set $A \subset X$. If $x \in \overline{A}$ and $f(x) = y$, then $\varepsilon = \bigcup\{\mathcal{G}(f(z)) : z \in A\}$ is a network at the point $y = f(x) \in f(\overline{A})$. By Lemma 4.1, we can conclude that the family

$$\mathcal{G}(x) = \{f^{-1}(F) \cap N : N \in \mathcal{N}, F \in \mathcal{G}(f(x))\} = \bigcup\{\mathcal{G}(f(z)) : z \in A\}$$

is a network at the point x. That is $\{\mathcal{G}(x) : x \in X\}$ is a κ- Collins-Roscoe collection in X. □

The following is a well-known fact; nevertheless, for the sake of completeness, we give the proof.

Lemma 4.8. *Let X be a space of weight κ and $K \subset X$ a compact subspace. Then $\chi(K, X) \leq \kappa$.*

Proof. Let $\beta = \{U_\alpha : \alpha \in I\}$ be a base for X of cardinality less or equal than κ and K a compact subspace of X. Let β_K be the family of all finite unions of elements of β that cover K. This family satisfies $|\beta_K| \leq |[\beta]^\omega| = |\beta| \leq \kappa$, therefore $|\beta_K| \leq \kappa$. The family β_K is a base of K in X. Indeed, let $U \in \tau(K, X)$. For each $x \in K$ let U_x be an element of β such that $x \in U_x \subset U$. Let $V_K = \bigcup\{U_x : x \in K\}$, then $V_K \subset U$ and there exists $\{x_1, x_2, \ldots, x_n\} \subset K$ such that $K \subset \bigcup_{i=1}^n U_{x_i} \subset V_K$. Clearly $\bigcup_{i=1}^n U_{x_i} \in \beta_K$ and it shows that β_K is a base for K in X. Therefore, $\chi(K, X) \leq \kappa$. □

Recall that a topological space X is κ-stable if for every continuous image Y of X, if Y condenses onto a space Z with $w(Z) \leq \kappa$ we have that $nw(Y) \leq \kappa$. A space is *stable* if it is κ-stable for every infinite cardinal κ.

An important property of the $L\Sigma$-spaces is that they are stable. Now we can prove a slight generalization of this fact. We show that the $(\mathcal{K}, L\Sigma)$-structured spaces are also stable.

Proposition 4.9. *Let X be a $(\mathcal{K}, L\Sigma)$-structured space. Then X is stable.*

Proof. Let K be a compact kernel of X, $f : X \to Y$ a continuous function and $g : Y \to Z$ a condensation of Y onto a space of weight κ. We need to prove that $nw(Y) \leq \kappa$. Observe that Y is a $(\mathcal{K}, L\Sigma)$-structured space and $f(K)$ is a compact kernel of Y.

Since K is compact, $K^* = g(f(K)) \subset Z$ is compact and, by Lemma 4.8, we have that $\chi(K^*, Z) \leq \kappa$. Let $\mathcal{U} = \{U_\alpha : \alpha \leq \kappa\}$ be a base for K^* in Z. Let $V_\alpha = g^{-1}(U_\alpha)$ and consider the family $\mathcal{V} = \{V_\alpha : \alpha \leq \kappa\}$, then $\cap\mathcal{V} = f(K)$. If we define the subspaces $Y_\alpha = Y \setminus V_\alpha$ we can represent Y in the following form: $Y = \bigcup_{\alpha\leq\kappa}(Y_\alpha \cup f(K))$. Observe that every space Y_α is $L\Sigma$, and since $f(K)$ is compact, every Y_α and $f(K)$ have i-weight less or equal than κ, so we have that $nw(Y) \leq \kappa$ and we can conclude that X is κ-stable. Since κ is arbitrary, it follows that X is stable. □

A space X is called *κ-monolithic* if $nw(\overline{B}) \leq \kappa$ for every subset B of X with $|B| \leq \kappa$. X is called *monolithic* if X is κ-monolithic for every cardinal κ. Arhangel'skii proved that $C_p(X)$ is monolithic if and only if X is stable. Since every $(\mathcal{K}, L\Sigma)$-structured space is stable, we arrive at the following.

Corollary 4.10. *Let X be a $(\mathcal{K}, L\Sigma)$-structured space. Then $C_p(X)$ is monolithic.*

Tkachuk proved in [4, Proposition 2.1] that if a Lindelöf Σ-space X condenses onto a κ-monolithic space then X itself is a κ-monolithic space. With the help of Proposition 4.9 we can give a slight generalization of this result of Tkachuk.

Proposition 4.11. *Let κ be an infinite cardinal and X be a $(\mathcal{K}, L\Sigma)$-structured space. If X condenses onto a κ-monolithic space, then X is κ-monolithic.*

Proof. Let $f : X \to Y$ be a condensation of X onto a κ-monolithic space Y. Let $A \subset X$ a subset of cardinality less or equal than κ, then $\overline{A}$ is a $(\mathcal{K}, L\Sigma)$-structured space that condenses onto the space $Z = f(\overline{A})$. Using that $nw(Z) \leq \kappa$ and the stability of $\overline{A}$, we can conclude that $nw(\overline{A}) \leq \kappa$. Therefore, X is κ-monolithic. □

5. $\aleph_0$-bounded topological groups and $(L\Sigma, L\Sigma)$-structured spaces

The purpose in this section is to study when the properties of being $L\Sigma$-space and being $(L\Sigma, L\Sigma)$-structured space (or belonging to some subclass of the class of $(L\Sigma, L\Sigma)$-structured spaces) coincide in some class of topological spaces. We show, for example, that an $\aleph_0$-bounded topological group is $(\mathcal{K}, L\Sigma)$-structured if and only if it is a Lindelöf Σ-space. As a consequence we prove that, for every Tychonoff space X, the function space $C_p(X)$ is Lindelöf-Σ if and only if it is $(\mathcal{K}, L\Sigma)$-structured.

Recall that a topological group G is $\aleph_0$-bounded if for every neighborhood of the identity element of G there exists a countable subset $K \subseteq G$ such that $G = K \cdot U$.

Proposition 5.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be some classes of topological spaces. If $\mathcal{Q}$ is closed under countable unions and G is an $\aleph_0$-bounded topological group such that G is a $(\mathcal{P}, \mathcal{Q})$-structured space with a non-dense $\mathcal{P}$-kernel, then G belongs to the class $\mathcal{Q}$.*

Proof. Let $K \subset G$ a non-dense $\mathcal{P}$-kernel. Then there exists a point $g \in U = G \setminus \overline{K}$. Using the regularity of G, we can find an open neighborhood V of g in G such that $g \in V \subset \overline{V} \subset U$. Observe that $G \setminus \overline{V}$ is an open neighborhood of K, therefore $\overline{V}$ belongs to $\mathcal{Q}$ and, without loss of generality, we can assume that V is a neighborhood of the identity element of G. Finally, given that G is $\aleph_0$-bounded, there exists a countable set $N \subset G$ such that $G = N \cdot \overline{V}$, so we can write G as a countable union of subspaces of G such that each subspace belongs to $\mathcal{Q}$. Therefore, G belongs to $\mathcal{Q}$. □

Corollary 5.2. *Let $\mathcal{P}$ be a subclass of $\mathcal{L}$ and let $\mathcal{Q}$ be one of the following classes: $\sigma\mathcal{K}, L\Sigma(< n), L\Sigma(\leq n), L\Sigma(< \omega), L\Sigma(\leq \omega), L\Sigma$, $\mathcal{L}$. If G is an $\aleph_0$-bounded topological group such that G is a $(\mathcal{P}, \mathcal{Q})$-structured space with a non-dense $\mathcal{P}$-kernel, then G belongs to the class $\mathcal{Q}$.*

Since every topological group of countable cellularity is $\aleph_0$-bounded, we have the following corollary.

Corollary 5.3. *Let $\mathcal{Q}$ be one of the following classes: $\sigma\mathcal{K}, L\Sigma(< n)$, $L\Sigma(\leq n), L\Sigma(< \omega), L\Sigma(\leq \omega), L\Sigma$, $\mathcal{L}$. Suppose that $C_p(X)$ is a $(\mathcal{K}, \mathcal{Q})$-structured space, then $C_p(X)$ belongs to class $\mathcal{Q}$.*

Since every $L\Sigma$-space is a $(\mathcal{K}, L\Sigma)$-structured space, we have the following result.

Corollary 5.4. *Let G be an $\aleph_0$-bounded topological group. Then G is a $(\mathcal{K}, L\Sigma)$-structured space if and only if it is a Lindelöf Σ-space.*

Corollary 5.5. *Let X be a Tychonoff space. Then $C_p(X)$ is a $(\mathcal{K}, L\Sigma)$-structured space if and only if it is a Lindelöf Σ-space.*

When the space X is compact, we have an improvement to Corollary 5.5.

Theorem 5.6. *Let $\mathcal{P}$ be a subclass of $L\Sigma$ that includes the compact spaces. Let X be a compact space. Then $C_p(X)$ is $(\mathcal{P}, L\Sigma)$-structured space if and only if it is a Lindelöf Σ-space.*

Proof. Suppose that $C_p(X)$ is a $(\mathcal{P}, L\Sigma)$-structure space. Then $C_p(X)$ has a dense subgroup that is a Lindelöf Σ-space ([1, Theorem 5.2]). Since X is compact, the fact that $C_p(X)$ contains a dense Lindelöf Σ-subspace implies that it is Lindelöf Σ.

Besides, if $C_p(X)$ is a Lindelöf Σ-space it is a $(\mathcal{P}, L\Sigma)$-structured space. □

In particular, we have the following corollaries.

Corollary 5.7. *Let X be a compact space. Then $C_p(X)$ is $(\sigma\mathcal{K}, L\Sigma)$-structured space if and only if it is a Lindelöf Σ-space.*

Corollary 5.8. *Let X be a compact space. Then $C_p(X)$ is $(L\Sigma, L\Sigma)$-structured space if and only if it is a Lindelöf Σ-space.*

Corollary 5.9. *Suppose that $X = \bigoplus_{n\in\mathbb{N}} K_n$ is a topological sum of compact spaces. If $C_p(X)$ is a $(L\Sigma, L\Sigma)$-structured spaces then $C_p(X)$ is a Lindelöf Σ-space.*

Proof. Since $\prod_{n\in\mathbb{N}} C_p(K_n)$ is homeomorphic to $C_p(\bigoplus_{n\in\mathbb{N}} K_n) = C_p(X)$, we have that $\prod_{n\in\mathbb{N}} C_p(K_n)$ is a $(L\Sigma, L\Sigma)$-structured space. Now, for every $m \in \mathbb{N}$ the space $C_p(K_m)$ is a $(L\Sigma, L\Sigma)$-structured space because $C_p(K_m)$ is a continuous image of $\prod_{n\in\mathbb{N}} C_p(K_n)$. Since K_m is compact, by the Corollary 5.8 we have that $C_p(K_m)$ is a Lindelöf Σ. It follows that $C_p(X)$ is a Lindelöf Σ-space. □

Considering the Corollaries 5.7, 5.8 and 5.9, the following questions are natural:

Question 5.10. Let X be a non-compact space. Suppose that $C_p(X)$ is a $(L\Sigma, L\Sigma)$-structured space. Is it true that $C_p(X)$ is a Lindelöf Σ-space?

Question 5.11. Let X be a non-compact space. Suppose that $C_p(X)$ is a $(\sigma\mathcal{K}, L\Sigma)$-structured space. Must $C_p(X)$ be a Lindelöf Σ-space?

We do not know the answers to above questions but our Proposition 5.16 gives a result in connection with question 5.11.

Definition 5.12. Let κ be an infinite cardinal.

(1) A κ-Baire space is a space such that the intersection of less than κ dense open sets, is a dense set.
(2) A space X belongs to the $(2, \kappa^+)$-Baire category, if the intersection of a family of at most κ dense open sets is non-empty.

It is not difficult to show that a space X belongs to the $(2, \kappa^+)$-Baire category iff X cannot be represented as the union of at most κ nowhere dense subsets. Neither is it difficult to show that for a homogeneous space X, it is a κ-Baire space iff it belongs to $(2, \kappa^+)$-Baire category. Nevertheless, for the sake of completeness, we give proof of the last claim. For that we recall first the following:

Remark 5.13. Let U and V be open subsets of the space X, with $V \subseteq U$. Then, if V belongs to the $(2, \kappa^+)$-Baire category, U belongs to the $(2, \kappa^+)$-Baire category. In fact, if $\{G_\alpha : \alpha < \kappa\}$ is a family of open dense subsets of U and we define the sets $V_\alpha = G_\alpha \cap V$, then $\{V_\alpha : \alpha < \kappa\}$ is a family of open dense subsets in V. Therefore, $\emptyset \neq \cap\{V_\alpha : \alpha < \kappa\} \subseteq \cap\{G_\alpha : \alpha < \kappa\}$ and this implies that U belongs to the $(2, \kappa^+)$-Baire category.

Now, it is easy to show the next result.

Proposition 5.14. *A non-empty space X is κ^+-Baire if and only if every non-empty open subset of X belongs to the $(2, \kappa^+)$-Baire category.*

Proof. $\Rightarrow \rfloor$ Let V be an open non-empty subset of X and $\{V_\alpha : \alpha < \kappa\}$ a family of open dense subsets in V. Define the sets G_α as $G_\alpha = V_\alpha \cup (X \setminus \overline{V})$, for each $\alpha < \kappa$. Since $\{G_\alpha : \alpha < \kappa\}$ is a family of dense open subsets in X, and X is a κ^+-Baire space, it follows that $\cap\{G_\alpha : \alpha < \kappa\}$ is a dense subset of X and given that V is an open non-empty set, we have that $\cap\{G_\alpha : \alpha < \kappa\} \cap V = \cap\{V_\alpha : \alpha < \kappa\}$ is dense in V. Therefore, V is κ^+-Baire and belongs to the $(2, \kappa^+)$-Baire category.

$\Leftarrow \rfloor$ Let $\{G_\alpha : \alpha < \kappa\}$ be a family of open and dense subsets in X and U an open subset, non-empty, of X. It is easy to see that $\{G_{\alpha,U} : \alpha < \kappa\}$ is a family of dense open subsets of U, where $G_{\alpha,U} = G_\alpha \cap U$. By hypothesis, U belongs to the $(2, \kappa^+)$-Baire category, so

$$\emptyset \neq \cap\{G_\alpha \cap U : \alpha < \kappa\} = (\cap\{G_\alpha : \alpha < \kappa\}) \cap U.$$

Therefore, $\cap\{G_\alpha : \alpha < \kappa\}$ is a dense subset of X. □

Proposition 5.15. *Let X be a homogeneous space, then X belongs to the $(2, \kappa^+)$-Baire category if and only if X has the κ^+-Baire property.*

Proof. Suppose that X belongs to the $(2, \kappa^+)$-Baire category. Suppose X does not have the κ^+-Baire property. By Proposition 5.14, there exists an open non-empty set $U \subseteq X$ that does not belong to $(2, \kappa^+)$-Baire category. Let $x \in U$, then x has a neighborhood base of sets that do not belong to $(2, \kappa^+)$-Baire category. Since X is homogeneous we can take a neighborhood base $\mathcal{B}$, for X, of sets that do not belong to the $(2, \kappa^+)$-Baire category. By the Zorn's lemma, there exists a maximal family $\gamma \subseteq \mathcal{B}$ of disjoint open sets. We have that $\overline{(\cup\gamma)} = X$. Since every $U \in \gamma$ does not belong to the $(2, \kappa^+)$-Baire category, we can choose a family $\{G_{\alpha,U} : \alpha < \kappa\}$ of open dense subsets in U such that $\cap\{G_{\alpha,U} : \alpha < \kappa\} = \emptyset$. Let $G_\alpha = \cup\{G_{\alpha,U} : U \in \gamma\}$, for each $\alpha < \kappa$. Then $\{G_\alpha : \alpha < \kappa\}$ is a family of open dense subsets of X. However

$$\bigcap\{G_\alpha : \alpha < \kappa\} = \bigcap_{\alpha<\kappa} \cup\{G_{\alpha,U} : U \in \gamma\} = \bigcup_{U\in\gamma} \cap\{G_{\alpha,U} : \alpha < \kappa\} = \emptyset,$$

a contradiction. Therefore X has the κ^+-Baire property. □

Since every topological group is a homogeneous space, we have the following result.

Proposition 5.16. *Let G be a $(\sigma\mathcal{K}, L\Sigma)$-structured and $\aleph_0$-bounded topological group. Suppose that Z is a σ-compact kernel of G of character α in X and let $\kappa = max\{\alpha, 2^{\aleph_0}\}$. If G has the κ^+-Baire property, then G is a Lindelöf Σ-space.*

Proof. Suppose that G is a $(\sigma\mathcal{K}, L\Sigma)$-structured topological group. Let $Z = \bigcup_{n\in\mathbb{N}} K_n$ be a σ-compact kernel of G. We can represent G in the following form: $G = \left(\bigcup_{n\in\mathbb{N}} K_n\right) \cup \bigcup_{V\in\mathcal{B}}(G \setminus V)$, where $\mathcal{B}$ is a base for Z in G of minimum cardinality. Given that all the elements in the union are closed subspaces and Lindelöf Σ, and G has the κ^+-Baire property, there is some element in the union with non-empty interior. Since G is homogeneous and $\aleph_0$-bounded, we can apply the method used in Proposition 5.1 to show that G is a Lindelöf Σ-space. □

Acknowledgement.

The authors are grateful to the referee for many valuable remarks, corrections that improved the paper.

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