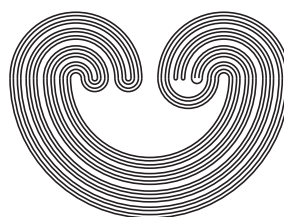


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BOXES AND NON-NORMALITY SETS

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BOXES AND NON-NORMALITY SETS

ANDRZEJ SZYMANSKI

ABSTRACT. We discuss normality and non-normality of spaces obtained by removing a box from the Cartesian product of spaces. Our results enable us, among others, to characterize non-normality points of Cartesian products of compact Hausdorff spaces.

1. INTRODUCTION

Preservation of normality by Cartesian products and their subspaces has been one of the most explored research topics in general topology. A vast literature exists on this subject (the reader may consult a bit outdated but still an excellent survey article by T. Przymusiński [8]). In our note, we discuss normality and non-normality of subspaces obtained by removing a box from a Cartesian product of spaces, i.e., subspaces of the form $\left(\prod_{s \in S} X_s\right) - B$, where $B = \prod_{s \in S} F_s$ and $F_s \subseteq X_s$ is closed for each $s \in S$. Two classical results, non-normality of the Tychonoff plank and non-normality of the space $\kappa \times (\kappa + 1)$ for any regular uncountable cardinal κ , are very special cases of our considerations.

A well known result of M. Katětov [4] asserts that if X and Y are compact Hausdorff spaces such that $X \times Y$ is hereditary normal, then the spaces X and Y are perfectly normal. We show that the same conclusion holds true assuming only that the subspaces of the form $X \times Y - E \times F$ are normal for all closed sets $E \subseteq X$ and $F \subseteq Y$. In fact, we show the converse statement to be true as well (see Theorem 6).

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An old and not yet completely solved problem (that harks back to L. Gillman [1]) is to determine non-normality points in compact Hausdorff spaces, i.e., points whose removal from a compact Hausdorff space yields a non-normal subspace. In the setting of Cartesian products, singletons are boxes. We provide an answer to that problem for products of (compact) Hausdorff spaces (see Remark 1, Corollary 4, and Proposition 2). For example, non-normality points of the Cartesian product of two compact Hausdorff crowded spaces can be characterized as those having uncountable character (cf. Corollary 3).

2. PRODUCTS OF TWO FACTORS

Our preliminary results are of general interest. Let us recall some basic definitions.

Sets $E, F \subseteq X$ are *separated in the space X* if there exist disjoint open subsets V, W of X such that $E \subseteq V$ and $F \subseteq W$. Thus a space X is *normal* if its singletons are closed and any two disjoint closed subsets of X can be separated.

Set $E \subseteq X$ has *open tightness* $< \kappa$ if for each open family \mathcal{P} such that $X - E \subseteq \bigcup \mathcal{P}$ there exists a subfamily $\mathcal{S} \subseteq \mathcal{P}$ such that $|\mathcal{S}| < \kappa$ and $\text{cl}(\bigcup \mathcal{S}) \cap E \neq \emptyset$. Trivially, if $A \subseteq X - E$, $|A| < \kappa$, and $\text{cl}A \cap E \neq \emptyset$, then E has open tightness $< \kappa$.

Set $E \subseteq X$ is a P_κ -set in X , if E is contained in the interior of the intersection of fewer than κ its open neighborhoods; P_{ω_1} -sets are usually called P -sets.

Lemma 1. (a) *If X is a regular space and E is a closed subset of X that has open tightness $< \kappa$, then E is not a P_κ -set in X .*

(b) *If E is not a P_κ -set in a compact Hausdorff space X , then E has open tightness $< \kappa$.*

Proof. Part (a) of the lemma has an almost obvious proof. Part (b) is obvious as well but we're going to prove it anyway.

Pick a family \mathcal{R} of open neighborhoods of E in X such that $|\mathcal{R}| < \kappa$ and $E \not\subseteq \text{int} \bigcap \mathcal{R}$. Let \mathcal{P} be any open family such that $X - E \subseteq \bigcup \mathcal{P}$. For each $W \in \mathcal{R}$ there exist $U_1, U_2, \dots, U_n \in \mathcal{P}$, $n < \omega$, such that $W \cup U_1 \cup U_2 \cup \dots \cup U_n = X$. Hence there exists a subfamily $\mathcal{S} \subseteq \mathcal{P}$ such that $|\mathcal{S}| < \kappa$ and $\text{cl}(\bigcup \mathcal{S}) \cap E \neq \emptyset$. \square

The following well known fact was proved first by P. Urysohn in 1925, [9].

Lemma 2. *Let X be a normal space. If H is G_δ subset of X , then $X - H$ is normal.*

A closed subset F of a space X is called a *non-normality set* of X if the space X is normal and the subspace $X - F$ is not normal. A point $p \in X$ is a *non-normality point* of X if $\{p\}$ is a non-normality subset of X . The lemma above, shows that any non-normality set must be a non G_δ set. The converse statement is not true generally. Take, e.g., a trivial example of the one-point compactification of an uncountable discrete space. However we are going to show that this property characterizes (almost all) boxes as non-normality sets for Cartesian products of compact Hausdorff spaces. Gearing up towards this goal, we will first study boxes in the Cartesian product of two spaces.

If $H \subseteq X \times Y$ and $x \in X$, then $H(x) = \{y \in Y : (x, y) \in H\}$ is called the *vertical cross-section of H at x* .

Lemma 3. *Let $H \subseteq X \times Y$, let $E = clE \subseteq X$, and let $F = clF \subseteq Y$. If for each $a \in X - E$ there exists an open neighborhood U of a such that $F \cap cl\left(\bigcup \{H(x) : x \in U\}\right) = \emptyset$, then $clH \cap (X \times F) \subseteq E \times F$.*

Proof. Let $(a, b) \in (X \times F) - E \times F$. Since $a \notin E$, let U be such an open neighborhood of a that $F \cap cl\left(\bigcup \{H(x) : x \in U\}\right) = \emptyset$. If V is an open neighborhood of F such that $V \cap \left(\bigcup \{H(x) : x \in U\}\right) = \emptyset$, then $U \times V$ is an open neighborhood of the point (a, b) disjoint from H . \square

Lemma 4. *Let $H \subseteq X \times Y$, let $E = clE \subseteq X$, and let $F \subseteq Y$ be compact. If $clH \cap (X \times F) \subseteq E \times F$, then for each $a \in X - E$ there exists an open neighborhood U of a such that $F \cap cl\left(\bigcup \{H(x) : x \in U\}\right) = \emptyset$.*

Proof. Let $a \notin E$ and let $y \in F$. Since $(a, y) \in (X \times F) - E \times F$, there is an open neighborhood U_y of a and an open neighborhood V_y of y such that $H \cap (U_y \times V_y) = \emptyset$. Take $V_{y_1}, V_{y_2}, \dots, V_{y_n}$ that cover F and set $U = U_{y_1} \cap U_{y_2} \cap \dots \cap U_{y_n}$ and $V = V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$. Clearly, $V \cap \left(\bigcup \{H(x) : x \in U\}\right) = \emptyset$. \square

The cardinal number $\psi(H, X) =$

$$\inf \left\{ |\mathcal{U}| : \mathcal{U} \text{ is an open family in } X \text{ and } \bigcap \mathcal{U} = H \right\}$$

is called the *pseudocharacter* of H . Notice that a set is G_δ if and only if it is of countable pseudocharacter.

Theorem 1. *Let X be any space and let $E = clE \subseteq X$. Let Y be any space and let $F \subseteq Y$ be compact. If E has open tightness $< \kappa$, where $\kappa = \psi(F, Y)$, then the sets $K = (X \times F) - (E \times F)$ and $L = (E \times Y) - (E \times F)$ are disjoint and cannot be separated in $X \times Y$.*

In particular, if $X \times Y$ is normal, then $E \times F$ is a non-normality set of $X \times Y$.

Proof. Let U and V be open subsets of $X \times Y$ such that $K \subseteq U$ and $L \subseteq V$. By compactness of F , we may assume that $U = \bigcup \{G_\alpha \times H_\alpha : \alpha \in A\}$, where G_α is an open subset of X , H_α is an open neighborhood of F in Y , and $\bigcup \{G_\alpha : \alpha \in A\} = X - E$. Hence there exists $B \subseteq A$ such that $|B| < \kappa$ and $E \cap \text{cl}\left(\bigcup \{G_\alpha : \alpha \in B\}\right) \neq \emptyset$. Pick any $x \in E \cap \text{cl}\left(\bigcup \{G_\alpha : \alpha \in B\}\right)$ and $y \in \left(\bigcap \{H_\alpha : \alpha \in B\}\right) - F$. Then $(x, y) \in L$. Take any open neighborhood $M \times N$ of (x, y) in $X \times Y$ such that $M \times N \subseteq V$. There exists $\alpha \in B$ such that $M \cap G_\alpha \neq \emptyset$. Since $y \in H_\alpha$, $(M \times N) \cap (G_\alpha \times H_\alpha) \neq \emptyset$. Hence $U \cap V \neq \emptyset$. \square

Let κ be an infinite cardinal. An open family $\{U_\xi : \xi < \kappa\}$ in X is said to be a κ -tower in X if $\text{cl}U_\beta \subseteq U_\alpha$ whenever $\alpha < \beta < \kappa$.

A set $E \subseteq X$ is said to be a κ -tower set in X if there exists a κ -tower $\{U_\xi : \xi < \kappa\}$ in X such that $E = \bigcap \{U_\xi : \xi < \kappa\}$.

Theorem 2. Let X and Y be any spaces and let $E \subseteq X$, and let $F \subseteq Y$ be κ -tower sets. Then the sets $K = (X \times F) - (E \times F)$ and $L = (E \times Y) - (E \times F)$ can be separated in $X \times Y$.

Proof. Let $\{U_\xi : \xi < \kappa\}$ and $\{V_\xi : \xi < \kappa\}$ be κ towers in X and Y , respectively, so that $E = \bigcap \{U_\xi : \xi < \kappa\}$ and $F = \bigcap \{V_\xi : \xi < \kappa\}$. Set

$$U = \bigcup \{(X - \text{cl}U_\alpha) \times V_\alpha : \alpha < \kappa\}, \text{ and } V = \bigcup \{U_\alpha \times (Y - \text{cl}V_\alpha) : \alpha < \kappa\}.$$

It is obvious that $K \subseteq U$ and that $L \subseteq V$. Notice that $[(X - \text{cl}U_\alpha) \times V_\alpha] \cap [U_\beta \times (Y - \text{cl}V_\beta)] = [(X - \text{cl}U_\alpha) \cap U_\beta] \times [V_\alpha \cap (Y - \text{cl}V_\beta)]$. Thus, $(X - \text{cl}U_\alpha) \cap U_\beta = \emptyset$ in case $\alpha \leq \beta$, and $V_\alpha \cap (Y - \text{cl}V_\beta) = \emptyset$ in case $\beta \leq \alpha$. Hence $U \cap V = \emptyset$. \square

The following theorem may be considered as a converse statement of Theorem 1.

Theorem 3. Let X and Y be normal spaces and let $E = \text{cl}E \subseteq X$, and $F = \text{cl}F \subseteq Y$ be such that E is a P_κ -set in X , where $\kappa = \psi(F, Y)$, and F is a P_λ -set in Y , where $\lambda = \psi(E, X)$. Then the sets $K = (X \times F) - (E \times F)$ and $L = (E \times Y) - (E \times F)$ can be separated in $X \times Y$.

Proof. Since E is a P_κ -set in X , $\kappa \leq \psi(E, X) = \lambda$. By the same token, $\lambda \leq \kappa = \psi(F, Y)$. In consequence, $\kappa = \lambda$ is a regular cardinal. Since X and Y are normal spaces, E and F are both κ -tower sets. Therefore Theorem 2 applies. \square

Let κ be an infinite cardinal. A transfinite sequence $\{y_\xi : \xi < \kappa\}$ of points of the space Y is said to be *converging to a set* $F \subseteq Y$ if:

(1) For each open neighborhood V of F there is an $\alpha < \kappa$ such that $y_\xi \in V$ for each $\xi \geq \alpha$;

If, additionally,

(2) $q \notin \text{cl}\{y_\xi : \xi \leq \alpha\}$ for each $\alpha < \kappa$,

then the sequence $\{y_\xi : \xi < \kappa\}$ is said to be *strongly converging to* F .

Let $\{U_\xi : \xi < \kappa\}$ be a κ -tower in the space X . For each $0 < \alpha < \kappa$ the set $L_\alpha = \bigcap \{\text{cl}U_\xi : \xi < \alpha\} - U_\alpha$ is called the α^{th} layer of the tower.

Theorem 4. *Let X be a compact Hausdorff space and let $\{U_\xi : \xi < \kappa\}$ be a κ -tower in X such that $E = \bigcap \{U_\xi : \xi < \kappa\}$. Let Y be any space and let $\{y_\xi : \xi < \kappa\}$ be a transfinite sequence of points of the space Y that converges to a compact set $F \subseteq Y$. If κ is an uncountable regular cardinal, then the sets $K = (X \times F) - (E \times F)$ and $L = \bigcup \{L_\alpha \times \{y_\alpha\} : 0 < \alpha < \kappa\}$ cannot be separated in $X \times Y$.*

Proof. Let U and V be open subsets of $X \times Y$ such that $K \subseteq U$ and $L \subseteq V$. We may assume that $V = \bigcup \mathcal{R}$, where \mathcal{R} consists of open sets of the form $G \times H$. For each $0 < \alpha < \kappa$, let $\mathcal{R}_\alpha = \{G : \text{There is an } H \text{ such that } G \times H \in \mathcal{R} \text{ and } G \times H \cap L_\alpha \times \{y_\alpha\} \neq \emptyset\}$. Since $L \subseteq \bigcup \mathcal{R}$, $L_\alpha \subseteq \bigcup \mathcal{R}_\alpha$ for each $0 < \alpha < \kappa$. If β is a limit ordinal $< \kappa$, then, since X is compact and $L_\beta = \bigcap \{\text{cl}U_\xi : \xi < \beta\} - U_\beta = \bigcap \{\text{cl}U_\xi - U_\beta : \xi < \beta\} \subseteq \bigcup \mathcal{R}_\beta$, there exists an $\alpha < \beta$ such that $\text{cl}U_\alpha - U_\beta \subseteq \bigcup \mathcal{R}_\beta$. This fact enables us to define a regressive function f on the set of all limit ordinals of κ by setting $f(\beta) = \alpha$ whenever $\alpha < \beta$ and $\text{cl}U_\alpha - U_\beta \subseteq \bigcup \mathcal{R}_\beta$. Let S be a stationary set of limit ordinals such that f is constant on S , say $f(\beta) = \gamma$ for each $\beta \in S$. Consider the layer $L_{\gamma+1} = \text{cl}U_\gamma - U_{\gamma+1}$. Since both $L_{\gamma+1}$ and F are compact and $L_{\gamma+1} \times F \subseteq K \subseteq U$, there exist open sets $M \subseteq X$ and $N \subseteq Y$ such that $L_{\gamma+1} \subseteq M$, $F \subseteq N$, and $L_{\gamma+1} \times F \subseteq M \times N \subseteq U$. So there is an $\alpha < \kappa$ such that $y_\xi \in N$ for each $\xi \geq \alpha$. Take a $\beta \in S$ with $\alpha < \beta$. By the definition of S , $L_{\gamma+1} \subseteq \text{cl}U_\gamma - U_\beta \subseteq \bigcup \mathcal{R}_\beta$. Consequently, there must be a $G \in \mathcal{R}_\beta$ intersecting the layer $L_{\gamma+1}$. Take an open set H in Y such that $G \times H \in \mathcal{R}$ and $(G \times H) \cap (L_\beta \times \{y_\beta\}) \neq \emptyset$. Hence $y_\beta \in H \cap N$ and so $\emptyset \neq (G \times H) \cap (M \times N) \subseteq V \cap U$. \square

Theorem 5. *Let X be a compact Hausdorff space and let $\{U_\xi : \xi < \kappa\}$ be a κ -tower in X such that $E = \bigcap \{U_\xi : \xi < \kappa\}$. Let Y be any space and let $\{y_\xi : \xi < \kappa\}$ be a transfinite sequence of points of the space Y that strongly converges to a compact set $F \subseteq Y$. If κ is an uncountable regular cardinal, then the sets $K = (X \times F) - (E \times F)$ and*

$L = \text{cl}\left(\bigcup \{L_\alpha \times \{y_\alpha\} : 0 < \alpha < \kappa\}\right) - (E \times F)$ are disjoint closed subsets of the space $(X - E) \times Y$ that cannot be separated. In particular, $(X - E) \times Y$ is not normal. Moreover, if $X \times Y$ is normal, then $E \times F$ is going to be a non-normality set of $X \times Y$.

Proof. Clearly, K and L are closed subsets of the space $(X \times Y) - (E \times F)$, so, they are closed in $(X - E) \times Y$ as well. To show that $K \cap L = \emptyset$, we are going to apply Lemma 3 with $H = \bigcup \{L_\alpha \times \{y_\alpha\} : 0 < \alpha < \kappa\}$.

Take any $a \in X - E$. There exists $\alpha < \kappa$ such that $a \notin \text{cl}U_\alpha$. Since $L_\beta \subseteq \text{cl}U_\alpha$ for each $\alpha \leq \beta < \kappa$, $H(x) \subseteq \{y_\xi : \xi < \alpha\}$ whenever $x \in X - \text{cl}U_\alpha$. Hence $F \cap \text{cl}\left(\bigcup \{H(x) : x \in X - \text{cl}U_\alpha\}\right) = \emptyset$. By Lemma 3, $\text{cl}\left(\bigcup \{L_\alpha \times \{y_\alpha\} : 0 < \alpha < \kappa\}\right) \cap (X \times F) \subseteq E \times F$.

The fact that K and L cannot be separated in $X \times Y$ follows immediately from Theorem 4. \square

Remark 1. In dealing with the problem of determining which closed boxes $E \times F$ could be non-normality sets of the product $X \times Y$, one has to consider the case of one of the sets E or F being clopen (i.e., closed and open). For assuming, e.g., that F is a clopen subset of the space Y , the non-normality of $X \times Y - E \times F = (X - E) \times F \cup X \times (Y - F)$ boils down to the non-normality of $(X - E) \times F$ since $X \times (Y - F)$ is a clopen subset of $X \times Y$. This, in turn, leads to studying non-normality of the Cartesian product of two spaces, one of which is compact and the other one is locally compact (see Theorem 5). We do not deal with this general problem in our note.

Theorem 6. *Let X and Y be compact Hausdorff spaces. Let E and F be closed and non-open subsets of X and Y , respectively. Then $E \times F$ is a non-normality set of $X \times Y$ if and only if $\max\{\psi(E, X), \psi(F, Y)\} > \aleph_0$.*

Proof. That this condition is necessary follows immediately from Lemma 2.

To show it is also sufficient, assume that $\max\{\psi(E, X), \psi(F, Y)\} > \aleph_0$, say $\kappa = \psi(F, Y) > \aleph_0$. If E is not a P_κ -set in X , then Theorem 1 applies and we are done. So assume that E is a P_κ -set in X . Since E is not open, $\lambda = \psi(E, X) \geq \kappa > \aleph_0$. Now if F is not a P_λ -set in Y , then Theorem 1 applies again. If neither of the two is the case, then $\kappa \leq \psi(E, X) = \lambda$ and $\lambda \leq \kappa = \psi(F, Y)$. It follows that $\kappa = \lambda$ is a regular uncountable cardinal and E, F are both κ -tower sets. Assume that $\{V_\xi : \xi < \kappa\}$ is a κ -tower in Y such that $F = \bigcap \{V_\xi : \xi < \kappa\}$. Clearly, if $y_\xi \in V_\xi$ for each $\xi < \kappa$, then $\{y_\xi : \xi < \kappa\}$ is a transfinite sequence of points of the space Y that strongly converges to the compact set $F \subseteq Y$. But now Theorem 5 applies, and we are done. \square

Corollary 1. *Let X and Y be compact Hausdorff spaces. The spaces X and Y are perfectly normal if and only if $X \times Y - E \times F$ is normal for all closed sets $E \subseteq X$ and $F \subseteq Y$.*

Proof. Let X and Y be perfectly normal spaces and let $E \subseteq X$ and $F \subseteq Y$ be closed. Since $\psi(E \times F, X \times Y) = \aleph_0$, $X \times Y - E \times F$ is normal by Lemma 2.

The converse follows immediately from Theorem 6. \square

Corollary 2. *Let X be a compact Hausdorff space. The space X is perfectly normal if and only if $X^2 - E^2$ is normal for each closed subset E of X .*

Proof. It follows immediately from Theorem 6 (or Corollary 1). \square

Katětov's question asks whether hereditary normality of the square of a compact Hausdorff space implies the metrizability of the space itself. It is now known that the answer to Katětov's problem is independent of ZFC. P. Larson and S. Todorcevic [5] (see also [6]) gave a consistent positive answer whereas P. Nyikos [7] and G. Gruenhage and P. Nyikos [3] gave a consistent negative answer. Since a space is hereditarily normal if and only if every open subspace is normal, one wonders whether normality of some special open subspaces of the square would yield a consistent positive solution to the Katětov problem as well. The double arrow space shows that assuming the normality of complements of boxes won't be enough.

Corollary 3. *Let $p \in X$ and $q \in Y$ be non-isolated points of the compact Hausdorff spaces X and Y . Then the point (p, q) is a non-normality point of $X \times Y$ if and only if $\max\{\psi(p, X), \psi(q, Y)\} > \aleph_0$.*

Proof. Apply Theorem 6 to the box $\{p\} \times \{q\}$. \square

3. ARBITRARY PRODUCTS

The results for Cartesian products of two factors can be generalized and carried out to Cartesian products of arbitrary number of factors. Since some of our results entail special notation, we will introduce some technicalities first.

The *Cartesian product* of spaces X_s , $s \in S$, denoted by the symbol

$$\prod_{s \in S} X_s, \text{ is}$$

$$\prod_{s \in S} X_s = \{x : x \text{ is a function on } S \text{ and } x_s = x(s) \in X_s \text{ for each } s \in S\}$$

endowed with the topology generated by sets of the form $\prod_{s \in S} U_s$, where $U_s \subseteq X_s$ is open for each $s \in S$ and $U_s \neq X_s$ for finitely many $s \in S$.

Let us recall that a set $B \subseteq \prod_{s \in S} X_s$ is called a *box* if $B = \prod_{s \in S} F_s$, where F_s is a closed subset of X_s for each $s \in S$. Notice that for each $x \in \prod_{s \in S} X_s$, $\{x\}$ is a box.

Clearly, $\prod_{s \in S} X_s = \prod_{s \in A} X_s \times \prod_{s \in S-A} X_s$ for any $\emptyset \neq A \subset S$.

In what follows, ν stands for a cardinal number, and X_ξ , $\xi < \nu$, denotes a space of cardinality at least 2, and E_ξ , $\xi < \nu$, denotes a non-empty closed subset of X_ξ .

Proposition 1. *Suppose that $\nu \leq \aleph_0$ and that X_ξ is a compact Hausdorff space for every $\xi < \nu$. If at least two of the sets E_ξ , $\xi < \nu$, are not open, then the box $\prod_{\xi < \nu} E_\xi$ is a non-normality set of the space $\prod_{\xi < \nu} X_\xi$ if and only if $\sup \{\psi(E_\xi, X_\xi) : \xi < \nu\} > \aleph_0$.*

Proof. Without loss of generality we may assume that E_0 and E_1 are closed and non-open subsets of X_0 and X_1 , respectively. Consider the space $\prod_{\xi < \nu} X_\xi$ as $X \times Y$, where $X = X_0$ and $Y = \prod_{0 < \xi < \nu} X_\xi$. Setting $E = E_0$ and $F = \prod_{0 < \xi < \nu} E_\xi$, we can apply Corollary 6. Since $\nu \leq \aleph_0$, $\max \{\psi(E, X), \psi(F, Y)\} = \sup \{\psi(E_\xi, X_\xi) : \xi < \nu\}$. The corollary now follows since $E \times F = \prod_{\xi < \nu} E_\xi$. \square

Corollary 4. *Suppose that $\nu \leq \aleph_0$ and that X_ξ is a compact Hausdorff space for every $\xi < \nu$. Let $(x_\xi)_{\xi < \nu} = x \in \prod_{\xi < \nu} X_\xi$ be such that $|\{\xi < \nu : x_\xi \text{ is non-isolated}\}| \geq 2$. Then x is a non-normality point of the space $\prod_{\xi < \nu} X_\xi$ if and only if $\sup \{\psi(x_\xi, X_\xi) : \xi < \nu\} > \aleph_0$.*

We shall embark on discussing the case of ν being an uncountable cardinal.

Let $\emptyset \neq A \subseteq \nu$ and let $\mathcal{R} = \{E_\xi : \xi < \nu\}$. Set

$$\prod(\mathcal{R}, A) = \left\{ x \in \prod_{\xi < \nu} X_\xi : x_\xi \in E_\xi \text{ for each } \xi \in A \right\}$$

Clearly, $\prod(\mathcal{R}, A)$ is a closed subset of $\prod_{\xi < \nu} X_\xi$. The following are other trivial observations about the sets $\prod(\mathcal{R}, A)$:

- (1) If $A \subseteq B$, then $\prod(\mathcal{R}, B) \subseteq \prod(\mathcal{R}, A)$;
- (2) $\prod(\mathcal{R}, A) \cap \prod(\mathcal{R}, B) = \prod(\mathcal{R}, A \cup B)$;

(3) $\prod(\mathcal{R}, \nu) = \prod_{\xi < \nu} E_\xi$; Thus $\prod(\mathcal{R}, A) \cap \prod(\mathcal{R}, B) = \prod_{\xi < \nu} E_\xi$; whenever $A \cup B = \nu$;

Lemma 5. *If $|\{\xi \in A : E_\xi \neq X_\xi\}| \geq \aleph_0$, then there exists a sequence $\{x_n : n < \omega\} \subset \prod\{X_\xi : \xi < \nu\} - \prod(\mathcal{R}, A)$ that converges to the set $\prod(\mathcal{R}, A)$. In particular, $\prod(\mathcal{R}, A)$ is not a P -set of $\prod\{X_\xi : \xi < \nu\}$.*

Proof. Pick any point $p = (p_\xi) \in \prod(\mathcal{R}, A)$ and a faithfully indexed set $\{\xi_n : n < \omega\} \subseteq A$ such that $E_{\xi_n} \neq X_{\xi_n}$ for each $n < \omega$. We define $x_n = (x_{n\xi})$ as follows:

$$x_{n\xi} = \begin{cases} p_\xi & \text{if } \xi \neq \xi_n \\ q_n \in X_{\xi_n} - E_{\xi_n} & \text{if } \xi = \xi_n. \end{cases}$$

The defined sequence $\{x_n : n < \omega\}$ lies outside the set $\prod(\mathcal{R}, A)$. To get the second part of the lemma, we are going to show even more, namely, that it converges to p .

Let $\prod_{\xi < \nu} U_\xi$ be a basic open neighborhood of p . Thus $B = \{\xi < \nu : U_\xi \neq X_\xi\}$ is a finite set. By the definition, if $\xi_n \notin B$, then $x_n \in \prod_{\xi < \nu} U_\xi$. \square

Lemma 6. *If $|\{\xi \in A : E_\xi \neq X_\xi\}| \geq \aleph_1$, then the pseudocharacter of the set $\prod(\mathcal{R}, A)$ in the space $\prod_{\xi < \nu} X_\xi$ is uncountable.*

Proof. Let $W_i, i < \omega$, be open neighborhoods of $\prod(\mathcal{R}, A)$ in the space $\prod_{\xi < \nu} X_\xi$. Pick any point $r \in \prod(\mathcal{R}, A)$. There exists a countable set $B \subset \nu$ such that the set

$$\prod(r, B) = \left\{ x \in \prod_{\xi < \nu} X_\xi : x_\xi = r_\xi \text{ for each } \xi \in B \right\}$$

is contained in the intersection of the open sets $W_i, i < \omega$. Let $\alpha \in A - B$ and let $a_\alpha \in X_\alpha - E_\alpha$. Since the set

$$\left\{ x \in \prod_{\xi < \nu} X_\xi : x_\alpha = a_\alpha \right\}$$

is disjoint from the set $\prod(\mathcal{R}, A)$ and intersects the set $\prod(r, B)$, the intersection of the open sets $W_i, i < \omega$ cannot be equal to $\prod(\mathcal{R}, A)$. \square

Theorem 7. *Let X_ξ be a compact Hausdorff space for each $\xi < \nu$. Let $B = \{\xi < \nu : E_\xi \neq X_\xi\}$ be uncountable. Let $A \subseteq \nu$ be such that both $A \cap B$ and $(\nu - A) \cap B$ are infinite sets. Then the sets $K = \prod (\mathcal{R}, A) - \prod (\mathcal{R}, \nu)$ and $L = \prod (\mathcal{R}, \nu - A) - \prod (\mathcal{R}, \nu)$ are disjoint closed subsets of the space $\prod_{\xi < \nu} X_\xi - \prod (\mathcal{R}, \nu)$ that cannot be separated. In particular, $\prod (\mathcal{R}, \nu)$ is a non-normality set of $\prod_{\xi < \nu} X_\xi$.*

Proof. Consider the Cartesian product $\prod_{\xi < \nu} X_\xi$ as $X \times Y$, where $X = \prod_{\xi \in A} X_\xi$ and $Y = \prod_{\xi \in B} X_\xi$. We may assume that, e.g., $|(\nu - A) \cap B| \geq \aleph_1$. We set $E = \prod_{\xi \in A} E_\xi$ and $F = \prod_{\xi \notin A} E_\xi$. By Lemma 5, E is not a P -set of X , and by Lemma 6, the pseudocharacter of the set F in the space Y is uncountable. Thus our corollary follows from Theorem 1. \square

Proposition 2. *Let X_ξ , $\xi < \nu$, be a Hausdorff space of cardinality at least 2, where $\nu > \aleph_0$. Then the subspace $\prod_{\xi < \nu} X_\xi - \{x\}$ is not normal for each $x \in \prod_{\xi < \nu} X_\xi$.*

Proof. Let $x \in \prod_{\xi < \nu} X_\xi$. For every $\xi < \nu$ pick $y_\xi \in X_\xi$ distinct from x_ξ and set $Y_\xi = \{x_\xi, y_\xi\}$. Then $\prod_{\xi < \nu} Y_\xi$ is a compact Hausdorff space and so it is a closed subset of the space $\prod_{\xi < \nu} X_\xi$. By Theorem 7, x is a non-normality point of the space $\prod_{\xi < \nu} Y_\xi$, which in turn implies that $\prod_{\xi < \nu} X_\xi - \{x\}$ is not normal. \square

W. Fleissner, J. Kulesza, and R. Levy showed (cf. [2], Proposition 11) that removing any non-empty subset of cardinality less than 2^ν from the space 2^ν is ANIC, i.e., a space whose any continuous normal image is compact. Our Proposition 2 is a very special case of this result. We thank Ronnie Levy for pointing that out to us.

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