http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 47, 2016

Pages 147–161

http://topology.nipissingu.ca/tp/

Decompositions of function spaces

by

David Guerrero Sánchez

Electronically published on July 23, 2015

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 47 (2016)
Pages 147-161

E-Published on July 23, 2015

DECOMPOSITIONS OF FUNCTION SPACES

DAVID GUERRERO SÁNCHEZ

Abstract. In this article we generalize a known result of Velichko by proving that a space $C_p(X)$ is the union of less than $\mathfrak{d}$ of its countably compact subspaces if and only if X is finite. We present an example of a space X which is not a P-space and $C_p(X,[0,1])$ admits a closure-preserving cover by countably compact subspaces. It is also proved that $C_p(X,[0,1])$ is contained in the closure of a second countable space $M \subset C_p(X)$ and for some $f \in C_p(X,[0,1])$ the space $M \cup \{f\}$ has a countable local base at f, then X is countable.

1. Introduction

In this note we continue with the work started in [8] and [10] by studying different kind of decompositions of function spaces. In section 3 we look at decompositions that yield covers of function spaces by compact-like subspaces. In this case, we will extend a result of Velichko by showing that $C_p(X)$ is the union of less than $\mathfrak{d}$ of its compact subspaces if and only if X is finite. We will also provide an example of a space X such that $C_p(X,\mathbb{I})$ is not countably compact, but has a closure-preserving cover by countably compact spaces.

Section 4 is devoted to the study of certain topological games in function spaces. Let $\mathcal{C}(\mathcal{P})$ be the class of topological spaces with a certain topological property $\mathcal{P}$. We will see that for many topological properties $\mathcal{P}$ there is a very similar behavior between spaces $C_p(X)$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}(\mathcal{P}),C_p(X))$ and those with a closure-preserving cover by subspaces in the class $\mathcal{C}(\mathcal{P})$. However, substantial differences will be observed for the Lindelöf and Lindelöf Σ properties.

2010 *Mathematics Subject Classification.* 54C35, 54A25, 54D20 54D30, 49N75, 91A05.

Key words and phrases. Compact space, closure-preserving cover, function space, Lindelöf Σ space, topological game.

The author was supported by FAPESP Grant 2013/093378 at IME-USP.

2. Notation and terminology

Every topological space in this text is assumed to be Tychonoff. For a space X, the family of all subsets of X is denoted by $\exp(X)$. The topology of X is denoted by $\tau(X)$ and $\tau^*(X)$ is the family of non-empty open subsets of X. For $C \subset X$ the family of all open sets of X that contain C is denoted by $\tau(C, X)$; if $x \in X$ then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The set of real numbers with the natural topology is denoted by $\mathbb{R}$ and the interval $[0, 1] \subset \mathbb{R}$ is represented by $\mathbb{I}$.

For every space X we denote by υX the Hewitt real-compactification of the space X. A map $f : X \to Y$ is compact covering if every compact subset of Y is the image under f of some compact subset of X. A space X is called scattered if every non-empty subspace of X has an isolated point.

A family $\mathcal{F}$ of subspaces of a space X is closed if every $F \in \mathcal{F}$ is closed in X. The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X, Y)$. The space $C_p(X, \mathbb{R})$ will be abbreviated by $C_p(X)$. Given the points $x_1, \ldots, x_m \in X$, and the open sets $O_1, ..., O_m \in \tau(\mathbb{R})$ let $[x_1, ..., x_m, O_1, ..., O_m] = \{f \in C_p(X) : f(x_i) \in O_i \text{ for } i = 1 \ldots, m\}$. For every $U = [x_1, ..., x_m, O_1, ..., O_m]$ we define $Supp(U) = \{x_1, \ldots, x_m\}$. For every $f \in C_p(X, Y)$, define the dual map $f^* : C_p(Y) \to C_p(X)$ by $f^*(g) = g \circ f$ for every $g \in C_p(Y)$. Given a space X, a function $f \in C_p(X)$, and a number $\varepsilon > 0$, let $I(f, \varepsilon) = \{g \in C_p(X) : |g(x) - f(x)| \leq \varepsilon \text{ for all } x \in X\}$.

A continuous bijection is called a condensation. If there is a condensation $\varphi : X \longrightarrow Y$ we say that X condenses onto Y. If Y is a subspace of X we denote by $\pi_Y : C_p(X) \longrightarrow C_p(Y)$ the restriction map defined by $\pi_Y(f) = f|Y$ for any $f \in C_p(X)$.

On the other hand, $C_u(X)$ is the space of all continuous real-valued functions on a space X, with the topology of uniform convergence.

If X is a space and $\mathcal{C}$ is a cover of X then a family $\mathcal{F}$ is called a network modulo $\mathcal{C}$ if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there is $F \in \mathcal{F}$ with $C \subset F \subset U$. A family $\mathcal{N}$ of subsets of a space X is a network in X if it is a network modulo of the cover $\{\{x\} : x \in X\}$. The network weight $nw(X)$ of a space X is the minimal cardinality of a network in X. A space X is cosmic if $nw(X) = \omega$. The space X is a P-space if every G_δ subset of X is onpen in X

A map $\varphi : Y \longrightarrow \exp(X)$ is called upper semicontinuous if for every $U \in \tau(X)$ the set $\bigcup\{\varphi^{-1}(C) : C \subset U\} \in \tau(Y)$ and φ is onto if $\bigcup\{\varphi(y) : y \in Y\} = X$. If each $\varphi(y)$ is a compact subspace of X then φ is called compact-valued.

A space X is Lindelöf Σ if it has a countable network modulo a compact cover of X. The number of K-determination of a space X is denoted by $\ell\Sigma(X)$ and is defined in [CMO] as $\min\{w(M) : M$ is a metric space and there is a compact-valued upper semicontinuous onto map $\varphi : M \to \exp(X)\}$.

Given a space Z the family $\mathcal{K}(Z)$ consists of all compact subsets of Z. A family $\mathcal{A}$ is called fundamental if for every $K \in \mathcal{K}(Z)$ there is $A \in \mathcal{A}$ such that $K \subset A$. If all elements of a cover $\mathcal{C}$ of X are compact then the family $\mathcal{C}$ is called compact. Whereas a family $\mathcal{B}$ is M-ordered for some space M if $\mathcal{B} = \{B_K : K \in \mathcal{K}(M)\}$ where $K \subset L$ implies $B_K \subset B_L$. A space X is dominated by a space M if it has an M-ordered compact cover. The metric domination index of a space X is denoted $dm(X)$ as defined in [9] as $\min\{w(M) : M$ is a metric space that dominates $X\}$.

The cardinal $iw(X) = min\{\kappa :$ the space X has a weaker topology of weight $\kappa\}$ is called i-weight of X, observe that it coincides with the minimum of the set $\{w(Y) :$ the space X condenses onto $Y\}$. Recall that $iw(X) \leq nw(X)$ for any space X. The rest of our notation is standard and follows [7]; our reference book on C_p-theory is [19].

3. Covering Function Spaces by compact-like subspaces

As we can observe from [15] or [8], decomposing $C_p(X)$ spaces by compact-like spaces imply strong restrictions on X whenever the decomposition is countable or closure-preserving. In both cases the following fact seems crucial: $\mathbb{R}^\omega$ does not embed in $C_p(X)$ as a closed subspace which implies X is pseudocompact. Since the minimum amount of compact spaces needed to cover $\mathbb{R}^\omega$ is $\mathfrak{d}$, we decided to address the corresponding question in this section.

The following theorem summarizes some known results (by Velichko, Shakmatov and Tkachuk, and Guerrero) that characterize function spaces by expressing them as a union of compact-like subspaces (see for example [8, Corollary 2.7]).

Theorem 3.1. *For a space X the following conditions are equivalent:*
(a) The space X is finite.
(b) The space $C_p(X)$ is σ-compact.
(c) The space $C_p(X)$ is σ-countably compact.
(d) The space $C_p(X) = \bigcup \mathcal{F}$ where $\mathcal{F}$ is a closure-preserving closed σ-countably compact family.

To generalize $((a) \Leftrightarrow (b))$ we will use the following fact that will be helpful later.

Lemma 3.2. *If X is a pseudocompact infinite space then there is a closed subspace of $C_p(X)$ that maps continuously onto ω^ω.*

Proof. Since X is infinite, by Lemma [8, Lemma 2.6] we can find $f \in C_p(X)$ such that $Y = f(X)$ is an infinite compact subspace of $\mathbb{R}$. It follows that there is a countable infinite compact $Z \subset Y$. We can identify $C_p(Y)$ with a closed subspace of $C_p(X)$. Apply [19, Problem 152] to see that the restriction map $\pi_Z : C_p(Y) \longrightarrow C_p(Z)$ is continuous. Since Z is compact and countable the space $C_p(Z)$ is analytic but not σ-compact; by [12, Theorem 3.5.3] we can deduce that it contains a closed subspace T homeomorphic to ω^ω. It follows that $\pi_Z^{-1}(T)$ is homeomorphic to a closed subspace of $C_p(X)$ that maps continuously onto ω^ω. □

Recall that $\mathfrak{d}$ is the minimum amount of compact sets needed to cover ω^ω or equivalently $\mathbb{R}^\omega$.

Theorem 3.3. *Let $\kappa < \mathfrak{d}$. For a space X the following conditions are equivalent:*

(a) The space X is finite.

(b) The space $C_p(X) = \bigcup_{\alpha<\kappa} K_\alpha$ where K_α is a compact subspace of $C_p(X)$ for every $\alpha < \kappa$.

(c) The space $C_p(X) = \bigcup_{\alpha<\kappa} K_\alpha$ where K_α is a countably compact subspace of $C_p(X)$ for every $\alpha < \kappa$.

Proof. We will show that $(c) \Leftrightarrow (a)$. By [8, Lemma 2.6] it suffices to show that every continuous real image of X is finite. Let $f \in C_p(X)$ and suppose that $f(X)$ is infinite. Condition (c) implies that $\mathbb{R}^\omega$ does not embed in $C_p(X)$ as a closed subspace which means X is pseudocompact (see [19, S 186, Fact 1]). Thus, by Lemma 3.2 $C_p(X)$ contains a closed subspace Z that maps continuously onto ω^ω. This implies that Z and therefore ω^ω can be covered by κ many countably compact subsets. This contradiction shows that $f(X)$ is finite and so is X. □

In [8, Corollary 3.8 and Corollary 2.4] it is proved that if $C_p(X)$ is equal to the union of a closure-preserving (not necessarily closed) family of countably compact subspaces then X is finite. Whereas in [10, Theorem 2.11] the authors show that if $C_p(X, \mathbb{I})$ admits a closure-preserving closed cover by σ-countably compact subspaces then $C_p(X, \mathbb{I})$ is countably compact. The following example shows that it is not possible to replace $C_p(X)$ by $C_p(X, \mathbb{I})$ in the statement of [10, Theorem 2.11]. Furthermore this example evinces that it is essential to assume that the elements of the closure-preserving cover in [10, Theorem 2.11] are closed.

Example 3.4. There exists a space X such that $C_p(X, \mathbb{I})$ contains a countably compact dense subspace but $C_p(X, \mathbb{I})$ is not countably compact.

Proof. By [19, S.480 Fact 2] there exists a space X such that

(a) X condenses onto a P-space.
(b) X condenses onto $\mathbb{R}$ and every open subset of X has cardinality $\mathfrak{c}$, in particular, X does not contain isolated points.

We will show that the space X is the one we are looking for. By condition (a) we can find a P-space Y for which it is possible to find a condensation $r : X \to Y$. From [19, Problem 397] it follows that $C_p(Y, \mathbb{I})$ is countably compact. The image of $C_p(Y, \mathbb{I})$ under the dual map $r^* : C_p(Y) \to C_p(X)$ is a dense subspace of $C_p(X, \mathbb{I})$. Besides, by [19, Problem 133] we can see that $D = r^*(C_p(Y, \mathbb{I}))$ is a countably compact subspace of $C_p(X, \mathbb{I})$.

To verify that $C_p(X, \mathbb{I})$ is not countably compact it suffices to show that X is not a P-space (see [19, Problem 397]). Indeed, condition (b) implies that there is a condensation $t : X \to \mathbb{R}$, thus the set $\{x\} = t^{-1}(t(x))$ is G_δ for every $x \in X$. If X were a P-space, then each of its points would be isolated which cannot happen by condition (b). □

Corollary 3.5. *There is a space X such that $C_p(X, \mathbb{I}) = \bigcup \mathcal{F}$ where $\mathcal{F}$ is a closure-preserving family and each $F \in \mathcal{F}$ is countably compact, however $C_p(X, \mathbb{I})$ is not countably compact.*

Proof. The space X from Example 3.4 has the property that the space $C_p(X, \mathbb{I})$ is not countably compact and contains a countably compact dense subspace F. The family $\{F \cup \{f\} : f \in C_p(X, \mathbb{I})\}$ is a closure-preserving cover of $C_p(X, \mathbb{I})$ by countably compact subspaces of $C_p(X, \mathbb{I})$. □

In [10, Problem 4.1] the authors ask if the presence of a closure-preserving closed cover by Lindelöf subspaces of $C_p(X)$ implies that $C_p(X)$ is Lindelöf. We still do not know if this is so. However, recalling that $C_p(X)$ is paracompact if and only if it is Lindelöf we can provide a partial answer to the problem in [10] for the case of locally finite covers that consist of paracompact subspaces of $C_p(X)$.

Proposition 3.6. *Suppose that $\mathcal{P}$ is a property preserved by subsets of type F_σ. If $C_p(X) = \bigcup \mathcal{F}$ and $\mathcal{F}$ is locally finite and each $F \in \mathcal{F}$ has $\mathcal{P}$ then $C_p(X)$ is the union of finitely many subspaces with $\mathcal{P}$.*

Proof. It follows easily from the fact that $C_p(X)$ embeds as an F_σ subset of any of its non-empty open subspaces. □

Corollary 3.7. *Let $\mathbb{F} = \{$realcompletness, monolithicity, paracompactness$\}$. If a property $\mathcal{P}$ is in the list $\mathbb{F}$ and $C_p(X)$ has a locally finite closed cover $\mathcal{C}$ such that $C \in \mathcal{C}$ has $\mathcal{P}$ then $C_p(X)$ also has $\mathcal{P}$.*

Proof. Is an immediate consequence of Proposition 3.6 and Theorems 2.2, 2.5 and 2.9 of [4]. □

The most important problem yet unsolved in [8] and [10] is to determine whether the space $C_p(\mathbb{I})$ can be covered by a closure-preserving family of second countable subspaces. It has already been shown that if $C_p(X)$ has a closure-preserving cover $\mathcal{F}$ then a homeomorphic copy of $C_p(X, \mathbb{I})$ is contained in $\overline{F}$ for some $F \in \mathcal{F}$. Considering the case when $C_p(X, \mathbb{I})$ is contained in the closure of a second countable space $M \subset C_p(X)$ we have the following:

Theorem 3.8. *Suppose $C_p(X, \mathbb{I})$ is contained in the closure of a space $M \subset C_p(X)$ and for some $f \in C_p(X, \mathbb{I})$ the space $M' = M \cup \{f\}$ has a countable local base at f, then X is countable.*

Proof. Take a countable local base $\mathcal{B} = \{U_n : n \in \omega\}$ of M' at f. For each $U_n \in \mathcal{B}$ there is $V_n = [x_1, ..., x_m, O_1, ..., O_m] \cap M'$ such that $f \in V_n \subset U_n$. Let $A = \bigcup\{Supp(V_n) : n \in \omega\}$. Suppose there is $x \in X \setminus A$. We can find $O_x \in \tau(f(x), \mathbb{R})$ with the property that there is $O \in \tau^*(\mathbb{R})$ such that $O \subset \mathbb{I} \setminus \overline{O_x}$. Since the function $f \in [x, O_x] \cap M'$, there is $n \in \omega$ for which $f \in V_n \subset [x, O_x] \cap M'$ with $V_n = [y_1, ..., y_k, O_1, ...O_k] \cap M'$. Let $U = [y_1, ..., y_k, x, O_1, ...O_k, O]$. It is possible to find $g \in C_p(X, \mathbb{I})$ such that $g(y_i) = f(y_i)$ for $i = 1, ..., k$ and $g(x) \in O$. This shows that the set $C_p(X, \mathbb{I}) \cap U$ is not empty, so there exists $h \in M \cap C_p(X, \mathbb{I}) \cap U$ which implies $h(y_i) \in O_i$ for $i = 1, ..., k$ hence $h \in V_n$. On the other hand $h(x) \in O$ implies $h \notin [x, O_x]$ a contradiction. □

4. Topological games on $C_p(X)$ and $C_p(X, \mathbb{I})$

If a space Z has a compact closure-preserving cover then a topological game on Z can be defined in a natural manner; in this game the first player has a winning strategy. Therefore studying analogous games in function spaces gives a possibility to strengthen some results of the previous section. The following game is a slight variation of the one presented by R. Telgarsky in [14]. It is worth to mention that studying properties of function spaces by means of topological games is a procedure that has already proven fruitful as shown in [7].

Definition 4.1. On a Tychonoff space Y, consider a family $\mathcal{C} \subset exp(Y)$. We define the game $\mathcal{G}(\mathcal{C}, Y)$ of two players I and II who take turns in the following way: at the move number n, Player I chooses $C_n \in \mathcal{C}$ and Player II chooses a set $U_n \in \tau(C_n, Y)$. The game ends after the n-th move of each player has been made for every $n \in \omega$ and Player I wins if $Y = \bigcup\{U_n : n \in \omega\}$; otherwise the winner is Player II.

Definition 4.2. A strategy t for the first Player in the game $\mathcal{G}(\mathcal{C}, Y)$ on a space Y is defined inductively in the following way. First the set $t(\varnothing) = F_0 \in \mathcal{C}$ is chosen. An open set $U_0 \in \tau(X)$ is legal if $F_0 \subset U_0$. For every legal set U_0 the set $t(U_0) = F_1 \in \mathcal{C}$ has to be defined. Let us assume that legal sequences $(U_0, ..., U_i)$ and sets $t(U_0, ...U_i)$ have been defined for each $i \leq n$. The sequence $(U_0, ...U_{n+1})$ is legal if the sequence $(U_0, ..., U_i)$ for each $i \leq n$ and $F_{n+1} = t(U_0, ..., U_n) \subset U_{n+1}$ is too. A strategy t for Player I is winning if it ensures victory for I in every play it is used.

Definition 4.3. A strategy s for Player II in the game $\mathcal{G}(\mathcal{C}, Y)$ on a space X is a function that assigns to every finite sequence $(F_0, \ldots, F_n)$ of elements of $\mathcal{C}$ an open set $U \in \tau(F_n, X)$. Such a strategy for Player II is winning if it ensures victory for II in every play where it is used.

The following facts appeared first in [8]. Since we will use them extensively we formulate them here.

Theorem 4.4. [8, Theorem 3.4] *Given a non-empty space X, if $Y = C_p(X, \mathbb{I})$ and*

$$\mathcal{F} = \{F \subset Y : F \text{ is nowhere dense in } C_u(X, \mathbb{I})\},$$

then Player II has a winning strategy in the game $\mathcal{G}(\mathcal{F}, Y)$.

Remark 4.5. [8, Remark 3.5] It is possible to reformulate Theorem 4.4 for the set $Y = C_p(X)$ and the family $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$, applying the same method to prove that Player II has a winning strategy in the game $\mathcal{G}(\mathcal{F}, Y)$.

Remark 4.6. [8, Remark 3.6] Given a space X consider the set $Y = C_p(X, \mathbb{I})$ (or $Y = C_p(X)$), and let $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X, \mathbb{I})\}$ (or $\mathcal{F} = \{F \subset Y : F$ is nowhere dense in $C_u(X)\}$). If $\mathcal{C}$ is a family of non-empty closed subsets of Y for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, Y)$ then $\mathcal{C} \nsubseteq \mathcal{F}$.

It is standard to verify that a space X is Lindelöf if and only if Player I has a winning strategy for the game $\mathcal{G}(\mathcal{L}, X)$ where $\mathcal{L}$ is the family of all the Lindelöf not necessarily closed subspaces of X. Is it possible to characterize other topological properties of function spaces in an analogous way?

In Remark 4.6 it is established that if X is non-empty and $\mathcal{F} \subset \exp(C_p(X, \mathbb{I}))$ and player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ then there is $F \in \mathcal{F}$ that is not nowhere dense in $C_u(X, \mathbb{I})$. An analogous fact is also established for $C_p(X)$.

Proposition 4.7. *For a non-empty space X, if $\mathcal{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X, \mathbb{I})$ and player I has a winning strategy for the game $\mathcal{G}(\mathcal{C}, C_p(X)$ or the game $\mathcal{G}(\mathcal{C}, C_p(X, \mathbb{I}))$ then there exists $C \in \mathcal{C}$ such that $U \subset C$ for some non-empty open subset U of the space $C_u(X)$.*

Proof. Indeed, $\mathcal{C}$ is also a closed family of subsets of $C_u(X)$ or $C_u(X,\mathbb{I})$ so by Remark 4.6 the interior of some $C \in \mathcal{C}$ in $C_u(X)$ must be non-empty. $\square$

Corollary 4.8. *For a non-empty X, if $\mathcal{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ and player I has a winning strategy for the game $\mathcal{G}(\mathcal{C}, C_p(X)$ or the game $\mathcal{G}(\mathcal{C}, C_p(X,\mathbb{I}))$ then there exist $C \in \mathcal{C}$ and $f \in C$ such that $I(f,\varepsilon) \subset \mathcal{C}$ for some $\varepsilon > 0$.*

Proof. Apply [10, Proposition 2.1] and Proposition 4.7. $\square$

Corollary 4.9. *Suppose X is a space and $\mathcal{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$. If player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, C_p(X))$ or $\mathcal{G}(\mathcal{C}, C_p(X,\mathbb{I}))$, then some $C \in \mathcal{C}$ contains a homeomorphic copy of $C_p(X)$.*

Proof. Apply Corollary 4.8 and [10, Proposition 2.1] $\square$

Corollary 4.10. *Suppose that $\mathcal{P}$ is a hereditary topological property and $\mathcal{C}$ is a closed family of subsets of $C_p(X)$ or $C_p(X,\mathbb{I})$ such that every $C \in \mathcal{C}$ has $\mathcal{P}$. If player I has a winning strategy in the game $\mathcal{G}(\mathcal{C}, C_p(X)$ or $\mathcal{G}(\mathcal{C}, C_p(X,\mathbb{I}))$ then $C_p(X)$ also has $\mathcal{P}$.*

Proof. By Corollary 4.8, there exists $C \in \mathcal{C}$ such that some $I \subset C$ is homeomorphic to $C_p(X)$; since C has $\mathcal{P}$, the space I and hence $C_p(X)$ must have $\mathcal{P}$. $\square$

Remark 4.11. Suppose that κ is an infinite cardinal. Notice that Corollary 4.10 applies, for instance, to the following properties: weight $\leq \kappa$, network weight $\leq \kappa$, i-weight $\leq \kappa$, diagonal number $\leq \kappa$, character $\leq \kappa$, pseudocharacter $\leq \kappa$, tightness $\leq \kappa$, spread $\leq \kappa$, hereditary Lindelöf number $\leq \kappa$, hereditary density $\leq \kappa$, κ-monolithicity, metrizability, Fréchet-Urysohn property, small diagonal, hereditary realcompactness, Whyburn property, being perfect, being functionally perfect.

In [17, Example 15] it is proved that if K is the Cantor set then $C_p(K)$ has a countable family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup_{n\in\omega} F_n = C_p(K)$ and every F_n has a countable π-base but $C_p(K)$ does not have a countable π-base. It is easy to see that this implies that the first player has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(K))$ where $\mathcal{F}$ is the family of all the closed subspaces of $C_p(K)$ with countable π-weight. We can conclude that if a property $\mathcal{P}$ is not hereditary and $\mathcal{F}$ is the family of all the subspaces of $C_p(X)$ that have $\mathcal{P}$, the existence of a winning strategy for Player I in the game $\mathcal{G}(\mathcal{F}, C_p(X)$ does not necessarily imply that $C_p(X)$ has $\mathcal{P}$.

Nevertheless, for properties that are inherited by closed subspaces we can proceed in a similar way as in [10] and use Corollary 4.8 to observe the following.

Remark 4.12. Given a non-empty space X and a closed-hereditary property $\mathcal{P}$, call $\mathcal{F}$ the family of all the closed subspaces of $C_p(X, \mathbb{I})$ that have $\mathcal{P}$. If Player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ then $C_p(X, \mathbb{I})$ also has the property $\mathcal{P}$. We can name some of these properties: extent $\leq \kappa$, Nagami number $\leq \kappa$, K-analyticity, $\ell\Sigma \leq \kappa$, $dm \leq \kappa$, $mg \leq \kappa$, $mi \leq \kappa$, normality, sequentiality.

Again following the arguments presented in Section 3 of [10] we notice that for some properties we can say even more.

Remark 4.13. If $\mathcal{F}$ is a closed family of subsets of $C_p(X, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ and every $F \in \mathcal{F}$ is realcompact then $C_p(X)$ is realcompact. If it is the case that every $F \in \mathcal{F}$ is a Čech-complete subspace, then X is discrete. Given a space X, if it happens that every $F \in \mathcal{F}$ is σ-countably compact, then $C_p(X, \mathbb{I})$ is countably compact. Whereas if the elements of $\mathcal{F}$ are σ-compact then X is discrete.

Theorem 4.14. *Given a space X and a property $\mathcal{P}$ that is preserved by quotient images, if $\mathcal{F}$ is a closed family of subsets of $C_p(X, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ and every $F \in \mathcal{F}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ also has the property $\mathcal{P}$.*

Proof. Apply 4.8 to find $F \in \mathcal{F}$ such that $I(f, \varepsilon) \subset F$ for some $f \in F$ and $\varepsilon > 0$. By [10, Proposition 2.1] the set $I(f, \varepsilon)$ is a retract of $C_p(X)$ and so it is also a retract of F which implies it has the property $\mathcal{P}$ and so does $C_p(X, \mathbb{I})$. □

Remark 4.15. Suppose that κ is an infinite cardinal. Theorem 4.14 applies to properties such as weak functional tightness $\leq \kappa$, functional tightness $\leq \kappa$.

Theorem 4.14 applies also to the property of κ-stability, but in this case not only $C_p(X, \mathbb{I})$ is κ-stable but the whole $C_p(X)$ is.

Remark 4.16. Indeed, since $C_p(X, \mathbb{I})$ is κ-stable then $C_p(C_p(X, \mathbb{I}))$ is κ-monolithic by [1, Theorem II.6.8]; the space X embeds in $C_p(C_p(X, \mathbb{I}))$, thus X is also κ-monolithic so $C_p(X)$ is κ-stable by [1, Theorem II.6.9].

Theorem 4.17. *Given a space X and a topological property $\mathcal{P}$ that is invariant under continuous images, if $\mathcal{F}$ is a family of subsets (not necessarily closed) of either $C_p(X)$ or $C_p(X, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X))$ or $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ and every $F \in \mathcal{F}$ has $\mathcal{P}$ then $C_p(X, \mathbb{I})$ contains a dense subspace that has $\mathcal{P}$.*

Proof. Apply Corollary 4.8 to find a function $f \in C_p(X, \mathbb{I})$ and $\varepsilon > 0$ such that $I(f, \varepsilon) \subset \overline{C}$ for some $C \in \mathcal{C}$. The set $R = I(f, \varepsilon)$ is a retract of $C_p(X)$ homeomorphic to $C_p(X, \mathbb{I})$. Consequently, there exists a retraction $r : \overline{C} \to R$. The set $r(C)$ is dense in $r(\overline{C}) = R$ and has the property $\mathcal{P}$. Since R is homeomorphic to $C_p(X, \mathbb{I})$, the latter also has a dense subspace with the property $\mathcal{P}$. $\square$

Theorem 4.18. *Suppose that X is a space and $\mathcal{P}$ is a σ-additive topological property preserved by continuous images. If $\mathcal{F}$ is a family of subsets (not necessarily closed) of either $C_p(X)$ or $C_p(X, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X))$ or $\mathcal{G}(\mathcal{F}, C_p(X, \mathbb{I}))$ and every $F \in \mathcal{F}$ has $\mathcal{P}$ then $C_p(X)$ contains a dense subspace that has $\mathcal{P}$.*

Proof. Apply Theorem 4.17 to deduce that $C_p(X, \mathbb{I})$ has $\mathcal{P}$. Since the property $\mathcal{P}$ is σ-additive the space $\bigcup_{n \in \mathbb{N}} C_p(X, [-n, n])$ has $\mathcal{P}$ and is dense in $C_p(X)$. $\square$

Remark 4.19. For an infinite cardinal κ Theorem 4.18 applies to the following properties: network weight $\leq \kappa$, spread $\leq \kappa$, hereditary density $\leq \kappa$. Furthermore when applying Theorem 4.18 to k-separability, caliber κ, point-finite cellularity $\leq \kappa$, or density $\leq \kappa$ then it is possible to ensure the presence of the corresponding property in $C_p(X)$ and in $C_p(X, \mathbb{I})$.

In the case of pseudocompactness it is not possible to obtain this property for $C_p(X)$ with non-empty X, yet we obtain σ-pseudocompactness of $C_p(X)$ and $C_p(X, \mathbb{I})$ is pseudocompact.

Remark 4.20. Assume $\mathcal{F}$ is a family of pseudocompact subspaces of $C_p(X)$ and Player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(X))$. Then the space $C_p(X)$ is σ-pseudocompact and $C_p(X, \mathbb{I})$ is pseudocompact.

Proof. Since the closure of every element of $\mathcal{F}$ is pseudocompact we do not lose generality if we consider that $\mathcal{F}$ is a closed family. If the space X is not pseudocompact, then there is a retraction $C_p(X) \to \mathbb{R}^\omega$. Let $\mathcal{C} = \{r(F) : F \in \mathcal{F}\}$; it is standard to verify that the first player has a winning strategy for the game $\mathcal{G}(\mathcal{C}, \mathbb{R}^\omega)$. This is a contradiction with Corollary 3.11 of [8] which shows that X is pseudocompact. Apply now Theorem 4.17 to conclude that $C_p(X, \mathbb{I})$ has a dense pseudocompact subspace and therefore it is pseudocompact and $C_p(X)$ is σ-pseudocompact. $\square$

This method of studying topological games in function spaces can also provide characterizations of important classes of compact spaces.

Corollary 4.21. *Let K be a compact space and let $\mathcal{F}$ be the class of σ-compact spaces. If Player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(K, \mathbb{I}))$ then K is Eberlein compact.*

Proof. Recall (see e.g. [1, Theorem IV.1]) dense σ-compact subspace and apply Theorem 4.17. □

Clearly Corollary 4.21 implies that if Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(K))$ then K is Eberlein, however [8, Corollary 3.12] implies that the converse is not true if K is infinite.

Arhangel'skii [1, Section IV.2] defined ω-perfect classes $\mathcal{P}$ as closed-hereditary, invariant under continuous images and such that $Z \in \mathcal{P}$ implies $(Z \times \omega)^\omega \in \mathcal{P}$. It turns out that ω-perfect classes are also relevant to the topic of this section.

Proposition 4.22. *If K is a compact space and $\mathcal{P}$ is an ω-perfect class then the following are equivalent:*

(a) Player I has a winning strategy for the game $\mathcal{G}(\mathcal{P}, C_p(K))$.
(b) Player I has a winning strategy for the game $\mathcal{G}(\mathcal{P}, C_p(K, \mathbb{I}))$.
(c) $C_p(X)$ belongs to $\mathcal{P}$.

Proof. The implications $(c) \implies (a)$ and $(c) \implies (b)$ are trivial. If (a) or (b) holds then we can apply Theorem 4.17 to convince ourselves that $C_p(X, \mathbb{I})$ has a dense subspace Z that belongs to $\mathcal{P}$. Therefore Z separates the points of X and hence we can apply [2, Proposition IV.3.3] to conclude that $C_p(X)$ belongs to $\mathcal{P}$. □

Corollary 4.23. *Suppose that K is a compact space and $\mathcal{P}$ is either K-analitycity or $\ell\Sigma(\cdot) \leq \kappa$, or even $dm(\cdot) \leq \kappa$. Then the following conditions are equivalent:*

(a) Player I has a winning strategy for the game $\mathcal{G}(\mathcal{P}, C_p(K))$.
(b) Player I has a winning strategy for the game $\mathcal{G}(\mathcal{P}, C_p(K, \mathbb{I}))$.
(c) $C_p(X)$ belongs to $\mathcal{P}$.

Proof. Observe that K-analitycity, $\ell\Sigma(\cdot) \leq \kappa$, and $dm(\cdot) \leq \kappa$, are ω-perfect properties and apply Proposition 4.22. □

Remark 4.24. Suppose that K is a compact space and $\mathcal{F}$ is a family of subsets of $C_p(K, \mathbb{I})$ for which Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(K, \mathbb{I}))$. If the elements of $\mathcal{F}$ are K-analytic then K is a Talagrand compact space whereas if every $F \in \mathcal{F}$ is Lindelöf Σ then K is Gul'ko compact.

In the rest of this section we will consider the situation when Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X))$ where $\mathcal{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$. It is not clear at all if in this case the space $C_p(X)$ has to be Lindelöf Σ.

Proposition 4.25. *If $dm(X) \leq \omega$ and Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X))$ where $\mathcal{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then $C_p(X)$ is a Lindelöf Σ-space.*

Proof. From Theorem 4.14 we obtain that $C_p(X, \mathbb{I})$ is Lindelöf Σ. Since $dm(X) \leq \omega$ we can apply [6, Proposition 2.14] to conclude that $C_p(X)$ Lindelöf Σ-framed (i.e. there is a Lindelöf Σ space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$) and therefore υX is Lindelöf Σ by [11, Theorem 3.6]. Let $\pi_X : C_p(\upsilon X) \to C_p(X)$ be the restriction map. The space $C_p(\upsilon X, \mathbb{I}) = \pi_X^{-1} C_p(X, \mathbb{I})$ is Lindelöf Σ by [19, Theorem 2.6]. Therefore, $C_p(\upsilon X)$ is Lindelöf Σ by [11, Theorem 3.6]. Hence, $C_p(X)$ is a Lindelöf Σ space for it is a continuous image of $C_p(\upsilon X)$. $\square$

Corollary 4.26. *If ω_1 is a caliber of a scattered space X and Player I has a winning strategy in the game $\mathcal{G}(\mathcal{F}, C_p(X))$ where $\mathcal{F}$ is a closed family of Lindelöf Σ-subspaces of $C_p(X)$ then X is cosmic.*

Proof. Since ω_1 is a caliber of X and the set of isolated is dense in X, we have that the space X must be separable and therefore $iw(C_p(X)) \leq \omega$. This implies that every Lindelöf Σ subspace of $C_p(X)$ has countable i-weight and hence countable network weight. Apply Remark 4.11 to conclude that $C_p(X)$ is cosmic and so is X. $\square$

5. Open Problems

Problem 5.1. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf space. But must the whole $C_p(X)$ be Lindelöf?*

Problem 5.2. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.*

The existence of a topological property in $C_p(C_p(X))$ usually implies stronger restrictions on X than having this property in $C_p(X)$. Therefore there is hope that the following question has a positive answer.

Problem 5.3. *Suppose that $C_p(C_p(X))$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must the space $C_p(C_p(X))$ be Lindelöf Σ?*

If $C_p(X)$ is a Lindelöf Σ-space and has the Baire property then X must be countable. This is the motivation for the following question.

Problem 5.4. *Suppose that X is a space such that $C_p(X)$ has the Baire property and can be represented as the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X be countable?*

If a space X has countable spread and $C_p(X)$ is a Lindelöf Σ-space then X must be cosmic. However it is not clear whether we could replace $C_p(X)$ by $C_p(X, \mathbb{I})$ in this result.

Problem 5.5. *Suppose that X is a space such that $s(X) \leq \omega$ and $C_p(X)$ is the union of a closure-preserving family of its closed Lindelöf Σ-subspaces. Must X have a countable network?*

Problem 5.6. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed K-analytic subspaces. We know that in this case $C_p(X, \mathbb{I})$ is a K-analytic space. But must the whole $C_p(X)$ be K-analytic?*

Problem 5.7. *Suppose that X is a space such that $C_p(X)$ is the union of a closure-preserving family of its closed sequential subspaces. We know that in this case $C_p(X, \mathbb{I})$ must be sequential. But must the whole $C_p(X)$ be sequential?*

Problem 5.8. *Suppose that X is a space such that $C_p(X, \mathbb{I})$ is sequential. Must $C_p(X, \mathbb{I})$ (or equivalently $C_p(X)$) be Fréchet-Urysohn?*

Problem 5.9. *Is the space $C_p(\mathbb{I})$ representable as the union of a closure-preserving family of its second countable subspaces?*

With respect of characterizing function spaces by means of the topological games described here the most important remaining problem on this topic is the following.

Problem 5.10. *Suppose that X is a space such that Player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(X))$ where $\mathcal{F}$ is the family of the closed Lindelöf Σ-subspaces of $C_p(X)$. We know that in this case $C_p(X, \mathbb{I})$ is a Lindelöf Σ-space. But must the whole $C_p(X)$ be Lindelöf Σ? The answer is not clear even if X has a unique non-isolated point.*

Not only do we not know the answer to the previous problem, but we do not even know if in that case, the space $C_p(X)$ has the properties that it would if it were a Lindelöf Σ space.

Problem 5.11. *Suppose that X is a space such that Player I has a winning strategy for the game $\mathcal{G}(\mathcal{F}, C_p(X))$ where $\mathcal{F}$ is the family of the closed Lindelöf Σ-subspaces of $C_p(X)$. Must the whole $C_p(X)$ be ω-monolithc?*

The author wishes to thank the referee for her/his most valuable comments and remarks. In particular the ones regarding Corollary 4.26.

References

[1] A. V. Arhangel'skii, *Topological function spaces,* Mathematics and its Applications (Soviet Series), **78**, Kluwer Acad. Publ., Dordrecht, 1992.

[2] ———, *On Lindelöf property and spread in $C_p(X)$-theory,* Topology Appl., **74:1** (1996), 83–90.

[3] A. Avilés, D. Guerrero Sánchez, *Are Eberlein-Grothendieck scattered spaces σ-discrete?* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **108:2** (2014), 849–859.

[4] F. Casarrubias-Segura, *Realcompactness and monolithity are finitely additive in Cp(X),* Proceedings of the 14th Summer Conference on General Topology and its Applications (Brookville, NY, 1999). Topology Proc. **24** (1999), Summer, 89–102.

[5] B. Cascales, M. Muñoz, and J. Orihuela, *The number of K-determination of topological spaces,* Topology Appl., Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM (2012).

[6] B. Cascales, J. Orihuela, and V. Tkachuk, *Domination by second countable spaces and Lindelöf Σ-property*, Topology Appl. **158:2** (2011), 204–214.

[7] J. Gerlits, Z. Nagy, *Some properties of C(X), I*, Topol. Appl., **14:2** (1982), 151–161.

[8] D. Guerrero Sánchez, *Closure-preserving covers in function spaces,* Comment. Math. Univ. Carolinae, **51:4** (2010), 693–703.

[9] ———, *Domination by metric spaces,* Topology Appl. **160:13** (2013), 1652–1658.

[10] D. Guerrero Sánchez, V. V. Tkachuk, *Dense subspaces vs closure-preserving covers of function spaces,* Top. Proc. **39** (2012), 219–234.

[11] O. G. Okunev, *On Lindelöf Σ-spaces of continuous functions in the pointwise convergence topology,* Topology Appl., **49:2** (1993), 149–166.

[12] C. A. Rogers, J. E. Jayne, *K-analytic Sets,* Analytic Sets, Academic Press, London, 1980, 2–181.

[13] D. B. Shakmatov, V. V. Tkachuk, *When is the space $C_p(X)$ σ-countably compact?* Vestnik Mosk. Univ., Mat., **41:1** (1986), 73–75.

[14] R. Telgarsky, *Spaces defined by topological games,* Fund. Math., **88:3** (1975), 193–223.

[15] V. V. Tkachuk, *Calibers of spaces of functions and the metrization problem for compact subsets of $C_p(X)$,* Vestnik Mosk. Univ., Mat., **43:3** (1988), 25–29.

[16] ———, *The spaces $C_p(X)$: decomposition into a countable union of bounded subspaces and completeness properties,* Topol. Appl., **22:3** (1986), 241–254.

[17] ———, *The decomposition of $C_p(X)$ into a countable union of subspaces with "good" properties implies "good" properties of $C_p(X)$,* Trans. Moscow Math. Soc., **55** (1994), 239–248.

[18] ———, *A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic*, Acta Math. Hung., **107:4** (2005), 253–265.

[19] ———, *A C_p-Theory Problem Book, Topological and Function Spaces*, Springer, New York, 2011.

Universidad Autónoma Metropolitana San Rafael Atlixco 186, Vicentina, Iztapalapa, 09340 Ciudad de México, D.F., México.

E-mail address: `dgs@ciencias.unam.mx`