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TOPOLOGY
PROCEEDINGS

Volume 47, 2016
Pages 163–176

http://topology.nipissingu.ca/tp/

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Electronically published on August 9, 2015

Topology Proceedings

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Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 47 (2016)
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E-Published on August 9, 2015

STRUCTURE SPACES OF INTERMEDIATE RINGS OF ORDERED FIELD VALUED CONTINUOUS FUNCTIONS

SUDIP KUMAR ACHARYYA AND PRITAM ROOJ

Abstract. Let F be a totally ordered field equipped with its order topology and X, a Hausdorff Completely F-regular topological space(CFR space in short) in the sense that, points and closed sets in X could be separated by F-valued continuous functions on X. Suppose $C(X,F)$ is the ring of all F-valued continuous functions on X and $B(X,F) = \{f \in C(X,F) : |f| < \lambda \text{ for some } \lambda > 0 \text{ in } F\}$. We call any ring $A(X,F)$ lying between $B(X,F)$ and $C(X,F)$ an intermediate ring. Given an intermediate ring $A(X,F)$ it is shown that, there is a one-to-one correspondence between the set $\mathcal{M}_F(A)$ of all maximal ideals in this ring and the set $\beta_F X$ of all z_F-ultrafilters on X. If $\mathcal{M}_F(A)$ is endowed with the Hull-Kernel topology and $\beta_F X$ with the Stone topology, then these two spaces become homeomorphic. This extends a result of Byun and Watson [3] which says on choosing $F = \mathbb{R}$ that, the structure space of any ring lying between $C^*(X)$ and $C(X)$ is βX, the Stone-Čech compactification of X. The Hausdorff compactification $\beta_F X$ of X thus obtained enjoys a kind of extension property similar to that of βX described as follows: any continuous map from X to a compact Hausdorff CFR space Y extends to a continuous map from $\beta_F X$ to Y. Using this extension property, we have shown that the ring $C_K(X,F)$ of all functions in $C(X,F)$ with compact support becomes identical to the set $\bigcap_{p\in\beta_F X - X} O^p_F$, where for $p \in \beta_F X$, $O^p_F = \{f \in C(X,F)$: the closure in $\beta_F X$ of the zero-set of f in X is a neighborhood of p in the space $\beta_F X$ $\}$. A special case of this result with $F = \mathbb{R}$ yields the standard formula $C_K(X) = \bigcap_{p\in\beta X - X} O^p$ in the classical situation. This exemplifies a further similarity between $\beta_F X$ and βX.

2010 *Mathematics Subject Classification.* Primary 54C40; Secondary 46E25.
Key words and phrases. Totally ordered field, zero-dimensional space, compact support, intermediate ring, hull-kernel topology, stone topology, extension property.

The first author is the corresponding author.

The second author thanks the UGC, New Delhi-110002, India, for financial support.

1. Introduction

Let F be a totally ordered field equipped with its order topology. For any topological space X, the set $C(X,F) = \{f : X \to F \mid f$ is continuous on $X\}$ makes a commutative lattice ordered ring with 1, if the relevant operations are defined pointwise on X. The set $B(X,F) = \{f \in C(X,F) :$ there exists $\lambda > 0$ in F with $|f| \leq \lambda$ on $X\}$ and $C^*(X,F) = \{f \in C(X,F) : \mathrm{cl}_F\, f(X)$ is compact$\}$ turn out to be subrings and sublattices of $C(X,F)$ with the inclusion relation $C^*(X,F) \subseteq B(X,F) \subseteq C(X,F)$. With $F = \mathbb{R}$, $C^*(X,F)$ is the same as $B(X,F)$. However with $F \neq \mathbb{R}$, it may well happen that these two rings are different. This can be illustrated by choosing $X = F = \mathbb{Q}$ and observing that the function $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = \frac{x}{1+|x|}$ where $x \in \mathbb{Q}$, belongs to $B(X,F)$, without belonging to $C^*(X,F)$. Indeed for this function f, $\mathrm{cl}_F\, f(X)$ is the set $\{x \in \mathbb{Q} : -1 \leq x \leq 1\}$, which is never compact. It is well known that, there is a nice interaction between the topological structure of X and the algebraic ring and order structure of $C(X)$ and $C^*(X)$ both. An excellent account of this interplay can be found in [4]. It is worth mentioning in this context that a good many results related to this interaction are still valid if $C(X)$ (respectively $C^*(X)$) is replaced by $C(X,F)$ (respectively $B(X,F)$ and $C^*(X,F)$) for any totally ordered field F and this is best realized if one sticks to the completely F-regular spaces. X is called completely F-regular if it is Hausdorff and given a point $x \in X$ and a closed set K in X with $x \notin K$, there is an $f \in B(X,F)$ such that $f(x) = 0$ and $f(K) = 1$. Thus complete F-regularity reduces to complete regularity in case $F = \mathbb{R}$. Incidentally if $F \neq \mathbb{R}$ then complete F-regularity of X and zero-dimensionality of X are equivalent conditions. Problems of this kind are already investigated by Acharyya, Chattopadhyay and Ghosh in an earlier paper [1]. A seemingly similar kind of problem, albeit treated differently is also addressed by Bachman, Beckenstein, Narici and Warner in [2]. For brevity, completely F-regular Hausdorff spaces will be termed as CFR spaces. By following the terminology of Sack and Watson [7], we call a ring lying between $B(X,F)$ and $C(X,F)$ an intermediate ring. Further by adapting closely the techniques of Byun and Watson [3], we have shown that for a typical intermediate ring $A(X,F)$, there exists a one to one correspondence between the set $\mathcal{M}_F(A)$ of all maximal ideals of $A(X,F)$ and the set $\beta_F X$ of all z_F-ultrafilters on X. A z_F-ultrafilter on X stands for a family of zero-sets of F-valued continuous functions on X, which is maximal with respect to having finite intersection property. The just mentioned bijective correspondence culminates to a homeomorphism, if $\mathcal{M}_F(A)$ is endowed with hull-kernel topology and $\beta_F X$ with the stone topology. Thus the structure spaces of all the intermediate rings are the same and identical to $\beta_F X$. This extends a result of Byun and Watson [3], which reads on choosing $F = \mathbb{R}$ that the structure space of any ring lying between $C^*(X)$ and $C(X)$ is βX. For a large class of zero-dimensional spaces X, viz, when X is strongly zero-dimensional it is realized that $\beta_F X = \beta X$ for any totally ordered field F (see [1]). Essentially therefore for all strongly zero-dimensional spaces X and for all choices of F, the structure space of all the intermediate rings are simply βX. X is called strongly zero-dimensional if every open cover of X by co-zero sets in X has a finite open refinement of mutually disjoint sets. Indeed X is strongly zero-dimensional if and only if βX is zero-dimensional (see Prop. 3.34, p.85 [8]). This implies in particular that, every strongly zero-dimensional space is zero-dimensional. But there do exist zero-dimensional spaces which are not strongly zero-dimensional (see Example 3.39, p.87 [8]). It is not known to us, whether for all zero-dimensional spaces X, $\beta_F X$ is still the same as βX for every choice of F. Nevertheless $\beta_F X$ possesses the following extension property akin to that of βX: any continuous map from X to a compact Hausdorff CFR space Y extends to a unique continuous map from $\beta_F X$ to Y. We have exploited this extension property of $\beta_F X$ to establish that if $C_K(X, F)$ is the ring of all functions in $C(X, F)$ with compact support then $C_K(X, F) = \bigcap_{p \in \beta_F X - X} O^p_F$, where for $p \in \beta_F X$, $O^p_F = \{f \in C(X, F)\colon cl_{\beta_F X} Z(f)$ is a neighborhood of p in $\beta_F X\}$. The well known standard formula $C_K(X) = \bigcap_{p \in \beta X - X} O^p$, results as a special case of this representation on putting $F = \mathbb{R}$ (see 7E, [4]). Thus we see that the last two properties of $\beta_F X$ put the analogous properties enjoyed by βX on a more general setting.

It may appear in this context that, any ring lying between $C^*(X, F)$ and $C(X, F)$ would have been also a natural candidate to be designated as an intermediate ring. But the structure spaces of these 'intermediate' rings may not be identical. Indeed it was established in ([1], Theorem 3.12) that, for a CFR space X, the structure space of $C^*(X, F)$ is $\beta_0 X$, the Banaschewski compactification of X. We note that $\beta_0 X$ is necessarily CFR though it is not known to us whether the structure space of $C(X, F)$ is really CFR (see [1], Remark 3.2). Since the main aim of this article is to look for those subrings of $C(X, F)$, which have identical structure space as that of $C(X, F)$, it just happened in the general case that, $B(X, F)$, rather than $C^*(X, F)$ turns out to be the right analogue of $C^*(X)$. Apart from this, $C(X, F)$ and $B(X, F)$ yield the same family of zero subsets of X, a fact which we have used several times in this paper without explicit notice to prove our main results. On the contrary, it is not known to us, whether in general $C^*(X, F)$ and $C(X, F)$ produce the same family of zero subsets in X. In the classical situation, i.e., with $F = \mathbb{R}$, to show that some chosen f in $C(X, F)$ is also a function in $C^*(X, F)$, it requires only to check that, f is bounded on X, which is often quite easy (see the proof of Theorems 2.13, 2.14). In the general case however (i.e., with $F \neq \mathbb{R}$), one has to verify in addition that the range of such an f is a pre compact subset of F, which may not be the case, as closed and bounded subsets of an arbitrary F need not be compact (see the example considered earlier in this section).

2. A few technicalities

We reproduce a few preliminary results from our paper [1] which we will use from time to time to establish the main theorems in this paper.

Theorem 2.1 (see [9], 1978)**.** *Any topological field is either connected or totally disconnected.*

Theorem 2.2. *Any totally ordered field F is either connected, in which case it is isomorphic to $\mathbb{R}$ or else zero-dimensional.*

Proof. If F is connected, then it is Dedekind-complete (indeed an ordered set is connected if and only if it is Dedekind-complete (see [4], Problem 3O, p.52)), and hence Archimedian. Therefore F is isomorphic either to $\mathbb{R}$ or to a proper sub-field of $\mathbb{R}$. Since no proper sub-field of $\mathbb{R}$ is Dedekind-complete, it follows that F is isomorphic to $\mathbb{R}$. On the other hand if F is not connected, then it is totally disconnected by Theorem 2.1. Let F^* be the Dedekind-completion of F and of course $F \subsetneqq F^*$. It is easy to check that $\{(\alpha,\beta)\cap F : \alpha,\beta \in (F^* \setminus F)\}$ constitutes a clopen base for the order topology on F, where $(\alpha,\beta) = \{\gamma \in F^* : \alpha < \gamma < \beta\}$. Hence F is zero-dimensional. □

For any $f \in C(X,F)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero set of f and it is clear that $Z(f) = Z(g)$, if one chooses $g = (-1 \vee f) \wedge 1$, so that $B(X,F)$ and $C(X,F)$ produce the same family of zero sets in X (with respect to F). Let $Z(X,F) = \{Z(f) : f \in C(X,F)\}$

Theorem 2.3. *For $F \neq \mathbb{R}$, X is a CFR space if and only if X is zero-dimensional.*

Proof. By following the classical technique adopted in the Chapter 3 of [4], one can easily see that, X is a CFR space if and only if its topology is the same as the weak topology on X, induced by $C(X,F)$. This implies in view of the fact that, F is zero-dimensional and the inverse image under a continuous map of a clopen set is clopen, that X is zero-dimensional if it is a CFR space.

It is easy to verify that a zero-dimensional space is completely F-regular for any choice of F. □

Remark 2.4. Zero-dimensionality of a topological space X is realized as a kind of separation axiom effected by ordered field ($\neq \mathbb{R}$) valued continuous functions on X.

Definition 2.5. A z_F-filter on X is a subfamily $\mathfrak{F}$ of $Z(X,F)$ with the following three conditions:

(1) $\phi \notin \mathfrak{F}$.
(2) If $Z_1,\ Z_2 \in \mathfrak{F}$, then $Z_1 \cap Z_2 \in \mathfrak{F}$.
(3) If $Z \in \mathfrak{F}$ and $Z \subseteq Z' \in Z(X,F)$, then $Z' \in \mathfrak{F}$.

A z_F-ultrafilter on X is a z_F-filter on X, which is not properly contained in any other z_F-filter on X. By an ideal I in $C(X,F)$ or in $B(X,F)$ or in any ring between $B(X,F)$ and $C(X,F)$, we shall always mean a proper ideal.

Theorem 2.6. *The following two results describe the relation between ideals (resp. maximal ideals) of $C(X,F)$ and z_F-filters (resp. z_F-ultrafilters) on X:*

(1) *If I is an ideal of $C(X,F)$, then $Z_F[I] = \{Z(f) : f \in I\}$ is a z_F-filter on X. Conversely for any z_F-filter $\mathfrak{F}$ on X, $Z_F^{-1}[\mathfrak{F}] = \{f \in C(X,F) : Z(f) \in I\}$ is an ideal in $C(X,F)$.*
(2) *If M is a maximal ideal of $C(X,F)$ then $Z_F[M]$ is a z_F-ultrafilter on X and conversely for any z_F-ultrafilter $\mathfrak{U}$ on X, $Z_F^{-1}[\mathfrak{U}]$ is a maximal ideal in $C(X,F)$.*

The map $M \mapsto Z_F[M]$ establishes a bijection on the set of all maximal ideals of $C(X,F)$ onto the collection of all z_F-ultrafilters on X.

Theorem 2.7. *Every prime ideal in $C(X,F)$ extends to a unique maximal ideal of $C(X,F)$, equivalently a prime z_F-filter $\mathfrak{F}$ on X is extendable to a unique z_F-ultrafilter on X. (A z_F-filter $\mathfrak{F}$ on X is called prime if for any $Z_1, Z_2 \in Z(X,F)$, $Z_1 \cup Z_2 \in \mathfrak{F} \Rightarrow Z_1 \in \mathfrak{F}$ or $Z_2 \in \mathfrak{F}$). It is easy to see that a z_F-ultrafilter on X is also a prime z_F-filter on X.*

Theorem 2.8. *For a CFR space X, the following statements are equivalent:*

(1) *X is compact.*
(2) *Every maximal ideal M of $C(X,F)$ is fixed in the sense that there exists $x \in X$ for which $f(x) = 0$ for any $f \in M$.*
(3) *Every maximal ideal of $B(X,F)$ is fixed.*

Remark 2.9. The proof of the last three theorems viz Theorems 2.6, 2.7, 2.8 can be done by a simple adaptation of the proof of the corresponding Theorems with $F = \mathbb{R}$ as given in Chapter 2 and Chapter 4 of [4].

Intermediate rings $\Sigma(X,F)$

In what follows, we introduce the rings that lie between $B(X,F)$ and $C(X,F)$ and call these intermediate rings. Let $\Sigma(X,F) = \{A(X,F) : A(X,F)$ is a ring with $B(X,F) \subseteq A(X,F) \subseteq C(X,F)\}$. We explore into some natural duality existing between the ideals of a typical intermediate ring and the z_F-filters on X.

Definition 2.10. As in Byun and Watson [3] we call an $f \in A(X,F) \in \sum(X,F)$, E-regular, where $E \subseteq X$, if there exists $g \in A(X,F)$ such that $(f.g)\mid_E = 1$. We set for $f \in A(X,F)$, $\mathcal{Z}_{A,F}(f) = \{E \in Z(X,F) : f$ is $(X \setminus E)-$regular$\}$ and for any ideal I of $A(X,F)$, $\mathcal{Z}_{A,F}[I] = \bigcup\{\mathcal{Z}_{A,F}(f) : f \in I\}$.

It is easy to check that, if $f \in A(X,F)$ satisfies $f \geq c > 0$ on some $E \subseteq X$, then f is E-regular.

Theorem 2.11. *The following results are straight forward adaptations from Byun and Watson's paper* [3]. *We give a sketch of proof of a few of these only.*

(1) *For $f \in A(X,F)$,$\mathcal{Z}_{A,F}(f)$ is a z_F-filter on X if and only if f is noninvertible in $A(X,F)$.*
(2) *For any ideal I of $A(X,F)$, $\mathcal{Z}_{A,F}[I]$ is a z_F-filter on X.*
(3) *For any z_F-filter $\mathfrak{F}$ on X, $\mathcal{Z}^{-1}_{A,F}[\mathfrak{F}] = \{f \in A(X,F) : \mathcal{Z}_{A,F}(f) \subseteq \mathfrak{F}\}$ is an ideal in $A(X,F)$.*
(4) *For $f \in A(X,F)$, $\bigcap \mathcal{Z}_{A,F}(f) = Z(f) \equiv \{x \in X : f(x) = 0\}$.*

Proof. Observe that for each $\epsilon > 0$ in F, $E_\epsilon(f) = \{x \in X : |f| \leq \epsilon\}$ is a zero set in X(with respect to F) and f is $[X \setminus E_\epsilon(f)]$-regular because $|f| \geq \epsilon > 0$ on this last set, this means that $E_\epsilon(f) \in \mathcal{Z}_{A,F}(f)$. Hence $Z(f) = \bigcap_{\epsilon>0} E_\epsilon(f) \supseteq \bigcap \mathcal{Z}_{A,F}(f)$. On the other hand if $E \in \mathcal{Z}_{A,F}(f)$ then f is $(X \setminus E)$-regular, consequently f cannot vanish on $(X \setminus E)$, i.e., $Z(f) \subseteq E$. Therefore $\bigcap \mathcal{Z}_{A,F}(f) \supseteq Z(f)$. □

(5) *For any ideal I of $A(X,F)$, $\mathcal{Z}_{A,F}[I] \subseteq Z_F[I]$.*

Proof. Let $E \in \mathcal{Z}_{A,F}[I]$. Then there exists $f \in I$ such that $E \in \mathcal{Z}_{A,F}(f)$. This implies in view of (4) that, $E \supseteq Z(f)$ and hence $E \in Z_F[I]$, as $Z_F[I]$ is a z_F-filter on X. □

Theorem 2.12. *Let I be an ideal of $A(X,F) \in \Sigma(X,F)$ and $\mathfrak{F}$ be a z_F-filter on X. Then the following relation holds: $\mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[I]] \supseteq I$ and therefore if M is a maximal ideal of $A(X,F)$, then $\mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[M]] = M$.*

Theorem 2.13. *If $A_0(X,F) \supseteq B_0(X,F) \supseteq B(X,F)$, where each $A_0(X,F)$, $B_0(X,F) \in \Sigma(X,F)$, then for any ideal I of $A_0(X,F), \mathcal{Z}_{A_0,F}[I] = \mathcal{Z}_{B_0,F}[I \cap B_0(X,F)]$.*

Proof. Enough to prove that, $\mathcal{Z}_{A_0,F}[I] = \mathcal{Z}_{B,F}[I \cap B(X,F)]$. It is trivial that, $\mathcal{Z}_{B,F}[I \cap B(X,F)] \subseteq \mathcal{Z}_{A_0,F}[I]$. Let $E \in \mathcal{Z}_{A_0,F}[I]$. Then there exists $f \in I$ and $g \in A_0(X,F)$ such that $fg \mid_{(X \setminus E)} = 1$. Set $h = \frac{2fg}{1+|fg|}$, then $h \in I \cap B(X,F)$ and $h \mid_{(X \setminus E)} = 1$. This shows that $E \in \mathcal{Z}_{B,F}(h)$ and hence $h \in \mathcal{Z}_{B,F}[I \cap B(X,F)]$. □

Theorem 2.14. *Let a zero set $E \in Z(X,F)$ meet every member of the z_F-filter $\mathcal{Z}_{C,F}(N)$, where N is a maximal ideal of $C(X,F)$. Then $E \in Z_F[N] \equiv$ the z_F-ultrafilter on X, corresponding to the maximal ideal N of $C(X,F)$.*

Proof. If possible, let $E \notin Z_F[N]$. Then there exists $f \in N$ such that, $E \cap Z(f) = \phi$. Since $f \in C(X,F)$ is not invertible, it follows that $Z(f) \neq \phi$. Consequently there is a $g \in B(X,F)$ such that $g(X) \subset [0,1]$, $g(E) = 1$ and $g(Z(f)) = \{0\}$. Since $Z(g) \supseteq Z(f) \in Z_F[N]$, it is clear that $g \in N$. Let $h \colon X \to F$ be defined as follows:

$$h(x) = \begin{cases} g(x), & \text{if } g(x) \leq \frac{1}{2} \\ \frac{1}{4g(x)}, & \text{if } g(x) \geq \frac{1}{2}, \end{cases}$$

then h is continuous on X and of course $h \in B(X,F)$. Let $G = \{x \in X : g(x) \leq \frac{1}{2}\}$. Then $G \in Z(X,F)$, $E \cap G = \phi$. Furthermore $4gh \mid_{(X \setminus G)} = 1$. This indicates that g is $(X \setminus G)$-regular, which means that, $G \in \mathcal{Z}_{C,F}(g) \subset \mathcal{Z}_{C,F}[N]$. This contradicts the hypothesis that E meets every member of $\mathcal{Z}_{C,F}[N]$. □

Theorem 2.15. *If M is a maximal ideal of $A(X,F) \in \Sigma(X,F)$, then $\mathcal{Z}_{A,F}[M]$ is contained in a unique z_F-ultrafilter on X.*

Proof. It follows from Theorem 2.6 that, $\mathcal{Z}_{A,F}[M]$ is a z_F-filter on X and there exists a maximal ideal M' of $C(X,F)$ such that $\mathcal{Z}_{A,F}[M] \subset Z_F[M']$. This yields in view of Theorems 2.11, 2.12, 2.13 that, $M = \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[M]] \subseteq \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_F[M']] =$ an ideal(proper) of $A(X,F)$ and consequently $M = \mathcal{Z}^{-1}_{A,F}[Z_F[M']] \supseteq \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{C,F}[M']] = \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[M' \cap A(X,F)]] \supseteq M' \cap A(X,F)$. Hence by Theorem 2.13, applied once again it follows that, $\mathcal{Z}_{C,F}[M'] = \mathcal{Z}_{A,F}[M' \cap A(X,F)] \subseteq \mathcal{Z}_{A,F}[M] \subseteq Z_F[M']$. Now suppose that there is another maximal ideal N of $C(X,F)$ such that, $\mathcal{Z}_{A,F}[M] \subseteq Z_F[N]$. Then a simple repetition of the above arguments yield: $\mathcal{Z}_{C,F}[N] \subseteq \mathcal{Z}_{A,F}[M] \subseteq Z_F[N]$. Now any zero set $E \in Z_F[N]$ meets each zero set in the family $\mathcal{Z}_{A,F}[M]$ and hence it meets every zero set in the family $\mathcal{Z}_{C,F}[M']$. It follows from Theorem 2.14 that, each such $E \in Z_F[N]$ is a member of $Z_F[M']$, i.e., $Z_F[N] \subseteq Z_F[M']$ and hence $Z_F[N] = Z_F[M']$ because each is a z_F-ultrafilter on X. □

Theorem 2.16. *Let $\mathfrak{U}$ be a z_F-ultrafilter on X. Then for any $A(X,F) \in \Sigma(X,F)$, $\mathcal{Z}^{-1}_{A,F}[\mathfrak{U}] \equiv \{f \in A(X,F) : \mathcal{Z}_{A,F}(f) \subseteq \mathfrak{U}\}$ is a maximal ideal of $A(X,F)$.*

Proof. There is a maximal ideal M' of $C(X,F)$ such that, $\mathfrak{U} = Z_F[M'] \equiv \{Z(f) : f \in M'\}$. So we can write by using Theorems 2.11, 2.13 that, $\mathcal{Z}^{-1}_{A,F}[\mathfrak{U}] = \mathcal{Z}^{-1}_{A,F}[Z_F[M']] \supseteq \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{C,F}[M']] = \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[M' \cap A(X,F)]] \supseteq M' \cap A(X,F)$. Now the prime ideal $M' \cap A(X,F)$ of the ring $A(X,F)$ extends to a maximal ideal M of $A(X,F)$, i.e., $M' \cap A(X,F) \subseteq M$. The theorem will be finished if we can show that, $\mathcal{Z}^{-1}_{A,F}[\mathfrak{U}] = M$. We see that $\mathcal{Z}_{A,F}[M]$ is a z_F-filter on X, therefore $\mathcal{Z}_{A,F}[M] \subseteq Z_F[N]$ for some maximal ideal N of $C(X,F)$. This yields: $\mathcal{Z}_{C,F}[M'] = \mathcal{Z}_{A,F}[M' \cap A(X,F)] \subseteq \mathcal{Z}_{A,F}[M] \subseteq Z_F[N]$. It is clear that, each zero set in $Z_F[N]$ meets every zero set in the family $\mathcal{Z}_{C,F}[M']$ and hence by Theorem 2.14, $Z_F[N] \subseteq Z_F[M']$, which yields $Z_F[N] = Z_F[M']$ as each is a z_F-ultrafilter on X. Now $\mathcal{Z}_{A,F}[M] \subseteq Z_F[N] \Rightarrow M = \mathcal{Z}^{-1}_{A,F}[\mathcal{Z}_{A,F}[M] \subseteq \mathcal{Z}^{-1}_{A,F}[Z_F[N]] = \mathcal{Z}^{-1}_{A,F}[Z_F[M']] =$ a proper ideal of $A(X,F)$. This implies $M = \mathcal{Z}^{-1}_{A,F}[Z_F[M']] = \mathcal{Z}^{-1}_{A,F}[\mathfrak{U}]$, (as M is a maximal ideal of $A(X,F)$). □

Remark 2.17. For any $A(X,F) \in \Sigma(X,F)$, there is a bijective correspondence on the set of all maximal ideals of $A(X,F)$ onto the family of all z_F-ultrafilters on X.

3. Structure spaces of Intermediate rings, realized as Stone-Čech like compactification of X

Let $\beta_F X$ be the set of all z_F-ultrafilters on X, we recall that, X is a CFR space. For any $E \in Z(X,F)$, set $\beta_F X_E = \{\mathfrak{U} \in \beta_F X : E \in \mathfrak{U}\}$. Then it is easy to verify that, $\mathcal{B} = \{\beta_F X_E : E \in Z(X,F)\}$ is a base for the closed sets of some topology, which we wish to call the Stone-topology on $\beta_F X$.

Definition 3.1. For each point $p \in X$, set $A_{F,p} = \{Z \in Z(X,F) : p \in Z\} \equiv$ the family of all zero sets in X (with respect to F-valued continuous functions on X), which contain the point 'p', which is obviously a z_F-ultrafilter on X.

One can easily check for any $E \in Z(X,F)$ that, $\beta_F X_E \cap \{A_{F,p} : p \in X\} = \{A_{F,p} : p \in E\}$. This simple fact can be written using different notations.

Let us define a map $\eta_{X,F}\colon X \to \beta_F X$ by the rule: $\eta_{X,F}(p) = A_{F,p}$. Therefore we can write: for any $E \in Z(X,F)$, $\beta_F X_E \cap \eta_{X,F}(X) = \eta_{X,F}(E)$. This means that, $\eta_{X,F}\colon X \to \beta_F X$ enchanges the basic closed sets of X and the basic closed sets of the subspace $\eta_{X,F}(X)$ of $\beta_F X$. Also the fact that, X is CFR ensures that, $\eta_{X,F}$ is a one-to-one map. Altogether, we can write the following result.

Theorem 3.2. *The map $\eta_{X,F}\colon X \to \beta_F X$ defined by the rule: $\eta_{X,F}(p) = A_{F,p}$ is a topological embedding.*

By using a few standard properties of z_F-ultrafilters on X and taking note of the construction of the Stone-topology on $\beta_F X$, we can establish the following results without any difficulty.

Theorem 3.3. *For all $E \in Z(X,F)$, $\overline{\eta_{X,F}(E)} = \beta_F X_E$, where $\overline{\eta_{X,F}(E)}$ means taking closure with respect to the Stone-topology on $\beta_F X$. In particular therefore $\overline{\eta_{X,F}(X)} = \beta_F X_X = \beta_F X$.*

Theorem 3.4. *For all E_1, $E_2 \in Z(X,F)$,*

$$\overline{\eta_{X,F}(E_1 \cap E_2)} = \overline{\eta_{X,F}(E_1)} \cap \overline{\eta_{X,F}(E_2)}.$$

Using the last fact, one can easily prove that, $\beta_F X$ is a compact space. Also given a pair of disjoint zero sets Z_1, Z_2 in X (with respect to F-valued continuous functions on X), there always exist a pair of zero sets Z_1',Z_2' in X with the following property: $Z_1 \cap Z_1' = \phi = Z_2 \cap Z_2'$, $Z_1' \cup Z_2' = X$. This fact can be used to prove that $\beta_F X$ is a Hausdorff space. Altogether we can write:

Theorem 3.5. *$\beta_F X$ or more formally the pair $(\eta_{X,F}, \beta_F X)$ is a Hausdorff compactification of X.*

It is clear that, with $F = \mathbb{R}$, $\beta_F X$ is the same as βX. It is not known to us whether there exists an $F \neq \mathbb{R}$ and a CFR space X, for which $\beta_F X \neq \beta X$. In view of the conclusive remarks made in [1] it follows that, any such space X must be zero-dimensional without being strongly zero-dimensional. In this connection it may be mentioned that, there exists a zero-dimensional space which is not strongly zero-dimensional, [8, Example 3.39, p. 87]. But it is not known to us whether for that space X, $\beta_F X \neq \beta X$ with $F \neq \mathbb{R}$. In general, however, $\beta_F X$ enjoys the following property which we call the F-extension property similar to that of βX.

Theorem 3.6. *Any continuous map $f\colon X \to Y$, where Y is a compact Hausdorff CFR space, can be extended to a continuous map $f^\beta\colon \beta_F X \to Y$ with the following property: $f^\beta \circ \eta_{X,F} = f$, i.e., which renders the* *following diagram commutative:*

$$\begin{tikzcd} X \arrow[r, "\eta_{X,F}"] \arrow[d, "f"'] & \beta_F X \arrow[dl, "f^\beta"] \\ Y & \end{tikzcd}$$

Proof. Choose $\mathfrak{U}$ from the set $\beta_F X$. Then it is not hard to check that, $\widetilde{f}(\mathfrak{U}) \equiv \{Z \in Z(Y,F) : f^{-1}(Z) \in \mathfrak{U}\}$ is a prime z_F-filter on Y, because if $Z_1, Z_2 \in Z(Y,F)$ are such that, $Z_1 \cup Z_2 \in \widetilde{f}(\mathfrak{U})$ then $f^{-1}(Z_1) \cup F^{-1}(Z_2) \in \mathfrak{U}$; this yields in view of the fact that, the z_F-ultrafilter $\mathfrak{U}$ on X is prime that either $f^{-1}(Z_1) \in \mathfrak{U}$ or $f^{-1}(Z_2) \in \mathfrak{U}$. Consequently $Z_1 \in \widetilde{f}(\mathfrak{U})$ or $Z_2 \in \widetilde{f}(\mathfrak{U})$. It follows from Theorem 2.7 that, $\widetilde{f}(\mathfrak{U})$ extends to a unique z_F-ultrafilter on Y. As Y is compact CFR space, by Theorem 2.8, each z_F-filter on X is fixed, consequently $\bigcap \widetilde{f}(\mathfrak{U}) = \{y\}$, a singleton. We set $f^\beta(\mathfrak{U}) = \bigcap \widetilde{f}(\mathfrak{U}) = \{y\}$. Then $f^\beta : \beta_F X \to Y$ is a well defined map. It is clear that, if $p \in X$ and $Z \in \widetilde{f}(A_{F,p})$, then $f^{-1}(Z) \in A_{F,p}$. Consequently $p \in f^{-1}(Z)$, i.e., $f(p) \in Z$. This indicates that $f^\beta(A_{F,p}) = f(p)$, i.e., $f^\beta \circ \eta_{X,F}(p) = f(p)$ for all $p \in X$. This settles the commutativity of the diagram. To establish the continuity of f^β, let U be a neighborhood of $f^\beta(\mathfrak{U})$, ($\mathfrak{U} \in \beta_F X$) in the space Y. Since both the zero set neighborhoods (with respect to F-valued continuous functions on Y) and also the co-zero set neighborhoods of a point in the CFR space Y can generate independently the entire neighborhood system of the same point, we can therefore write: $f^\beta(\mathfrak{U}) \in (Y - Z_1) \subseteq Z_2 \subseteq U$ for some Z_1, Z_2 $\in Z_F(Y)$. Now $f^\beta(\mathfrak{U}) \notin Z_1 \Rightarrow Z_1 \notin \widetilde{f}(\mathfrak{U}) \Rightarrow f^{-1}(Z_1) \notin \mathfrak{U} \Rightarrow \mathfrak{U} \notin \beta_F X_{f^{-1}(Z_1)} \Rightarrow \mathfrak{U} \in \beta_F X - \beta_F X_{f^{-1}(Z_1)} =$ an open neighborhood of the point $\mathfrak{U}$ in the space $\beta_F X$. We assert that $f^\beta(\beta_F X - \beta_F X_{f^{-1}(Z_1)}) \subseteq U$, indeed $\mathfrak{U}^* \in (\beta_F X - \beta_F X_{f^{-1}(Z_1)}) \Rightarrow f^{-1}(Z_1) \notin \mathfrak{U}^* \Rightarrow f^{-1}(Z_2) \in \mathfrak{U}^*$ (because $f^{-1}(Z_1) \cup f^{-1}(Z_2) = f^{-1}(Y) = X \in \mathfrak{U}^*$) $\Rightarrow Z_2 \in \widetilde{f}(\mathfrak{U}^*) \Rightarrow f^\beta(\mathfrak{U}^*) \in Z_2 \subseteq U$. □

Remark 3.7. In the classical situation with $F = \mathbb{R}$ the CFR condition on Y in the above theorem is redundant as a compact Hausdorff space is completely regular. But in general with $F \neq \mathbb{R}$, a compact Hausdorff space Y need not be CFR, equivalently need not be zero-dimensional; in view of Theorem 2.3.

We shall now exploit the above extension property of $\beta_F X$ to show that, the ring $C_K(X,F)$ of all F-valued continuous functions on X with compact support can be realized as the intersection of a suitable family of ideals emerging from the growth $\beta_F X - X$ of X. Surely this exemplifies a further similarity between $\beta_F X$ and βX. Recall that $C^*(X,F)=$ $\{f \in C(X,F)$: $cl_F f(X)$ is compact$\}$. Since F is completely F-regular and complete F-regularity is an hereditary topological property (see Theorem 2.2 and Theorem 2.3), it is clear that, every subspace of F is completely F-regular. Hence we can use the result of Theorem 3.6 to ensure that each $f \in C^*(X,F)$ can be extended to a unique continuous $f_F^\beta\colon \beta_F X \to F$. Let $Z_{\beta_F X}(f_F^\beta)$ stand for the zero set of the function f_F^β in the space $\beta_F X$. Then the following fact emerges.

Theorem 3.8. *For any CFR space X, $C_K(X,F) =$ $\{f \in C^*(X,F)$:$Z_{\beta_F X}(f_F^\beta)$ is a neighborhood of $\beta_F X - X$ in the space $\beta_F X\}$.*

Proof. If $f \in C_K(X,F)$ then $f(X)$ is a compact subset of F because $f(X) = f(cl_F(X - Z(f)) \cup Z(f)) = f(cl_F(X - Z(f))) \cup \{0\}$. Consequently $f \in C^*(X,F)$. Thus $C_K(X,F) \subseteq C^*(X,F)$. Now choose $f \in C^*(X,F)$, then $cl_F(X - Z(f))$ is compact from which it follows $cl_{\beta_F X}(X - Z(f)) \subseteq X$......(1). Again from the relation $X = (X - Z(f)) \cup Z(f)$, we set $\beta_F X = cl_{\beta_F X} X = Cl_{\beta_F X}(X - Z(f)) \cup cl_{\beta_F X} Z(f)$, which yields $\beta_F X - cl_{\beta_F X}(X - Z(f)) \subseteq cl_{\beta_F X} Z(f)$......(2). Combining the relations (1),(2), we can write $\beta_F X - X \subseteq \beta_F X - cl_{\beta_F X}(X - Z(f)) \subseteq cl_{\beta_F X} Z(f) \subseteq Z_{\beta_F X}(f_F^\beta)$. This implies that, $cl_{\beta_F X} Z(f)$ and hence $Z_{\beta_F X}(f_F^\beta)$ is a neighborhood of $\beta_F X - X$ in the space $\beta_F X$. Conversely if $f \in C^*(X,F)$ is such that $Z_{\beta_F X}(f_F^\beta)$ is a neighborhood of $\beta_F X - X$ in the space $\beta_F X$, then since $Z_{\beta_F X}(f_F^\beta) \cap X = Z(f)$, it is clear that, no point of $\beta_F X - X$ can be a limiting point of the set $X - Z(f)$ in the space $\beta_F X$. This means that, $cl_{\beta_F X}(X - Z(f)) = cl_X(X - Z(f))$, hence $f \in C_K(X,F)$. The theorem is completely proved. □

As a consequence, we have the following theorem:

Theorem 3.9. *For any CFR space X, $C_K(X,F) = \bigcap_{p \in \beta_F X - X} O_F^p$.*

Proof. If $f \in C_K(X,F)$, then we have observed in the course of proving Theorem 3.8 that, $cl_{\beta_F X} Z(f)$ is a neighborhood of $\beta_F X - X$ in the space $\beta_F X$. The later assertion means that $f \in O_F^p$ for each $p \in \beta_F X - X$. Conversely if $f \in O_F^p$ for each $p \in \beta_F X - X$, then $cl_{\beta_F X} Z(f)$ and hence $Z_{\beta_F X}(f_F^\beta)$ is a neighborhood of $\beta_F X - X$ in the space $\beta_F X$, as we recall that $cl_{\beta_F X} Z(f) \subseteq Z_{\beta_F X}(f_F^\beta)$. Hence from Theorem 3.8, we get $f \in C_K(X,F)$. □

Stucture space of a typical A(X,F)$\in \Sigma(X, F)$

For each $p \in X$, set $M^p_{A,F} = \{f \in A(X, F) : f(p) = 0\}$. It is easy to check that, $\{M^p_{A,F} : p \in X\}$ constitutes the entire family of fixed maximal ideals of $A(X, F)$. The following result can be established by using routine arguments and adapting the proof of Theorem 3.6 of [6].

Theorem 3.10. *The structure space $\mathcal{M}_F(A)$ of the ring $A(X, F)$ (i.e. the set of all maximal ideals of $A(X, F)$ with Hull-Kernel topology) is a compact Hausdorff space. Furthermore the map $\psi_{A,F} \colon X \to \mathcal{M}_F(A)$ defined by the rule $\psi_{A,F}(p) = M^p_{A,F}$ establishes a topological embedding with $\psi_{A,F}(X)$ dense in $\mathcal{M}_F(A)$.*

Thus $\mathcal{M}_F(A)$ or more formally the pair $(\psi_{A,F}, \mathcal{M}_F(A))$ makes a Hausdorff compactification of X. The following result indicates that, the two compact Hausdorff spaces $\mathcal{M}_F(A)$ and $\beta_F X$ are essentially the same.

Theorem 3.11. *$\mathcal{M}_F(A)$ and $\beta_F X$ are homeomorphic for any intermediate ring $A(X, F)$.*

Proof. Define the map $v \colon \mathcal{M}_F(A) \to \beta_F X$ as follows: for each $M \in \mathcal{M}_F(A)$, $v(M)$ is the unique z_F-ultrafilter on X, which contains $\mathcal{Z}_{A,F}[M]$. We have already established (see Remark 2.17) that v is a bijection onto $\beta_F X$. Since any bijection between two compact Hausdorff spaces is a homeomorphism if it is either a continuous map or a closed map, we shall show that, v is a closed map. A typical basic closed set in the space $\mathcal{M}_F(A)$ is a set of the form $\mathcal{M}_F(A)_f = \{M \in \mathcal{M}_F(A) : f \in M\}$, for some $f \in A(X, F)$. It is not hard to check that, $v(\mathcal{M}_F(A)_f) = \bigcap_{E \in \mathcal{Z}_{A,F}(f)} \{\mathfrak{U} \in \beta_F X : E \in \mathfrak{U}\}$ = the intersection of a family of basic closed sets in $\beta_F X$ = a closed set in $\beta_F X$. $\square$

The actual relation between $\mathcal{M}_F(A)$ and $\beta_F X$ is manifested by the following result.

Theorem 3.12. *The two Hausdorff compactifications $(\psi_{A,F}, \mathcal{M}_F(A))$ and $(\eta_{X,F}, \beta_F X)$ of X are topologically equivalent in the sense that, there is a homeomorphism $\upsilon \colon \mathcal{M}_F(A) \to \beta_F X$ which makes the following diagram commutative:*

$$\begin{tikzcd}
X \arrow[r, "\psi_{A,F}"] \arrow[d, "\eta_{X,F}"'] & \mathcal{M}_F(A) \arrow[dl, "\upsilon"] \\
\beta_{F,X} &
\end{tikzcd}$$

Proof. Because of Theorem 3.11 we need to check only the commutativity of the diagram. We recall that, $\psi_{A,F}(p) = M^p_{A,F} \equiv \{f \in A(X,F) : f(p) = 0\}$ and $\eta_{X,F}(p) = A_{F,p} \equiv \{Z \in Z(X,F) : p \in Z\}$ for each point $p \in X$. Next we observe that, if $p \in X$ and $f \in A(X,F)$ are such that, $f(p) = 0$ then $Z(f) \in A_{F,p}$, from which it follows in view of the relation $\cap \mathcal{Z}_{A,F}(f) = Z(f)$ (see Theorem 2.11 (4)) that, $\mathcal{Z}_{A,F}(f) \subseteq A_{F,p}$. Hence we can write $\mathcal{Z}_{A,F}(M^p_{A,F}) \subseteq A_{F,p}$. But recall that $v(M^p_{A,F})$ is the unique z_F-ultrafilter on X, containing $\mathcal{Z}_{A,F}(M^p_{A,F})$. Hence it follows that, $v(M^p_{A,F}) = A_{F,p}$. Since this is true for each $p \in X$, we can write $v \circ \psi_{A,F} = \eta_{X,F}$. The theorem is completely proved. □

Remark 3.13. The structure spaces of any two intermediate rings between $B(X,F)$ and $C(X,F)$ are topologically equivalent, each being equivalent to $\beta_F X$. If $F = \mathbb{R}$, then these structure spaces (of rings lying between $C^*(X)$ and $C(X)$) are the same as βX- a fact established by Plank in 1969 and by Byun and Watson in 1991 by different methods.

Open Question 3.14. Which results amongst Theorems 2.13, 2.15, 2.16 are valid if $B(X,F)$ is replaced by $C^*(X,F)$?

Acknowledgements: The authors express their gratitude to the referee of this paper for making some valuable comments towards the improvement of the original version of this paper.

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