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DENSE SADDLES IN TORUS MAPS

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Abstract. In this paper, we look at a specific class of maps in the torus and explore the consequences of this map having a dense set of periodic saddles. The main result states that under these assumptions, the torus splits into a countable number of invariant cylinders with disjoint interiors and the map is transitive on each cylinder.

1. Introduction

We will focus on a class of maps $F : \mathbb{T}^2 \to \mathbb{T}^2$ which are of the form

$$F : (x, y) = (mx, g(x, y))(\mod 1), \tag{1.1}$$

where $m \in \mathbb{N}$ is > 1 and $g : \mathbb{T}^2 \to S^1$ is C^2. The motivation of this work is to explore the connection between transitivity of a map and the existence of dense periodic saddles in the torus. There are maps on the torus which have dense periodic saddles but are not transitive, as shown at the end of this section. Our main result states that a map on the torus with dense saddles may not be transitive, but there will be a decomposition of the torus into a finite number of cylinders with disjoint interiors with the map transitive on each component.

Our approach will be by using an invariant structure in the tangent bundle called "*invariant, expanding cone system*", explained in Section 2.2. Cone-systems have been studied previously as geometric structures in vector bundles, for example in [4]. The reason we assume that our map has the form (1.1) is because it has an invariant expanding cone system.

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We believe that the results of this paper are applicable to other maps on the torus with an invariant, expanding cone system.

The Jacobian of this map is given by

$$DF(x,y) = \begin{pmatrix} m & 0 \\ \partial_1 g(x,y) & \partial_2 g(x,y) \end{pmatrix} \text{ for } \forall (x,y) \in \mathbb{T}^2. \tag{1.2}$$

Our main result assumes that the map is a *local diffeomorphism*, i.e., its Jacobian is invertible everywhere. This property will be assumed for the rest of the paper. From (1.2) this condition can be stated as

$$\partial_2 g(x,y) \neq 0 \text{ for } \forall (x,y) \in \mathbb{T}^2. \tag{1.3}$$

Let $z_0 = (x_0, y_0) \in \mathbb{T}^2$ be a periodic point of period $p \in \mathbb{N}$. Then one of the eigenvalues of $dF^p(z_0)$ is m^p. Therefore, depending upon whether the other eigenvalue, which equals $|\partial_2(g \circ F^{p-1})(z_0))|$, is lesser than, equal to or greater than 1, z_0 is a saddle, non-hyperbolic or a repellor.

Our main result will carry the additional assumption that the expansion in the X-direction by m dominates any expansion in the Y-direction. This can be stated as

$$|\partial_2 g(x,y)| < m \text{ for } \forall (x,y) \in \mathbb{T}^2. \tag{1.4}$$

We will also find use for a stronger condition on the expansion, which is

$$|\partial_2 g(x,y)| < 0.5m \text{ for } \forall (x,y) \in \mathbb{T}^2. \tag{1.5}$$

Vertical circles. A *vertical circle* is a subset of the torus of the form $\{X = x\}$, which can also be represented as $\{x\} \times S^1$, where $x \in S^1$. It will be denoted as $\boldsymbol{S_x}$. Then the map (1.1) maps vertical circles into vertical circles, that is,

$$\text{For } \forall x \in S^1, \; F(S_x) = S_{F(x)} \tag{1.6}$$

All the vertical circles S_x with x of the form

$$x_0 = \frac{k}{m^n - 1} \pmod 1 \tag{1.7}$$

are invariant under F^n. These will be called the **periodic circles** of the map and will be denoted as $\Gamma_{k,n}$.

$$\Gamma_{k,n} := \{(x,y) \in \mathbb{T}^2 \mid x = \frac{k}{m^n - 1} \pmod 1\}. \tag{1.8}$$

So if $z_0 = (x_0, y_0)$ is a periodic point, then x_0 must be of the form (1.7) and z_0 is a fixed point of the circle map $F^n|S_{x_0}$. Depending upon whether this fixed point is attracting, neutrally stable or repelling for $F^n|S_{x_0}$, z_0 is a saddle, non-hyperbolic periodic point or repellor for F.

A *cylinder* is a set diffeomorphic to $S^1 \times [0,1]$. Recall that a map is *transitive* if it has a dense trajectory or equivalently, for every pair of open sets U and V, there is some $n \in \mathbb{N}$ for which $F^{-n}(U) \cap V \neq \Phi$.

A set X is called *forward-invariant* wrt F if $F(X) \subseteq X$. It is called *strongly forward-inavariant* if $F(X) = X$. *Backward-invariance* and *invariance* is similarly defined.

Theorem 1.1 (Main result)**.** *Let (1.1) be a local diffeomorphism that has dense periodic saddles and satisfies (1.5). Then either the map is transitive or there exists some $p \in \mathbb{N}$ such that the torus is a union of finitely many cylinders with disjoint interiors such that F^p acts transitively on each cylinder and the action of F on the set of cylinders is a permutation whose cycles are of order p.*

The following two corollaries are immediate consequences of the main theorem. They show that the splitting of the torus into invariant cylinders can be refuted by easily satisfiable conditions.

Corollary 1.2. *Let (1.1) be a local diffeomorphism that has dense periodic saddles, satisfies (1.5) and there is a periodic circle $\Gamma_{k,n}$ of the form (1.8) on which F^n is transitive. Then the map F is transitive on $\mathbb{T}^2$.*

Corollary 1.3. *Let (1.1) be a local diffeomorphism that has dense periodic saddles and satisfies (1.5). Moreover, suppose that there are two periodic points whose periods are coprime. Then the map F is transitive on $\mathbb{T}^2$.*

Section 2 has some definitions and properties needed to prove Theorem 1.1. Finally, section 3 presents the proof to Theorem 1.1.

A non-transitive torus map with dense periodic saddles. Consider the cylinder $C = S^1 \times [0,1]$. We will first construct a map on this cylinder which has a dense set of periodic saddles and leaves the boundaries $S^1 \times \{0\}$ and $S^1 \times \{1\}$ invariant. Then the map on the torus can be constructed by gluing corresponding boundaries of each cylinder together. We will continue to use the notation S_x to denote the vertical line segments $\{x\} \times [0,1]$.

The following map on the cylinder is a modification of a torus map studied in [3].

$$F(x,y) = (3x \pmod{()1},\ y + 0.01\sin(2\pi y) + 0.2g(y)\sin^2(\pi x)), \tag{1.9}$$

where $(x,y) \in S^1 \times [0,1]$. Here $g : [0,1] \to [0,1]$ is a smooth map which equals 0 in a small neighborhood of 0 and 1 and equals 1 for $y \in [0.01, 0.99]$. The map has a fixed saddle point z at $(0, 0.5)$. Let R be the set bounded by the circle $S^1 \times [0.4]$ and from the bottom by portions of the unstable manifold of z. Using a bit of arithmetic, it was shown in [3] that $R \subset S^1 \times [0, 0.4]$ and that $R \subset F(R)$. Therefore, every point in R has a preimage in R. Also note that $\frac{\partial F_y}{\partial y} > 1$ in R. From this it follows that the unstable manifold W_u of z is dense in R. This region also has a periodic repellor, so the forward iterates of R covers the interior of the cylinder C. Therefore, W^u is everywhere dense.

Now note that the stable manifold W^s of z contains $S_0 - \{z\}$ and all of its inverse images. The inverse images of S_0 are the vertical lines $\{S_x \mid x = \frac{k}{3^n} \pmod 1),\ k, n \in \mathbb{N}\}$. W^s is dense on each such vertical line and these lines are dense in C. Therefore W^s is dense in C.

Therefore, the intersections of W^u with W^s are transverse homoclinic points and dense in C. Each of them are limit points of periodic points, hence, the set of periodic points is dense in C.

2. Definitions and properties

2.1. Stable and unstable manifolds.

In this section, the definitions of local and global, stable and unstable manifolds are reviewed.

Definition 2.1 (Local stable and unstable manifolds for hyperbolic maps.)**.** Let M be a closed n-manifold, $F : M \to M$ be a C^1 diffeomorphism, $\Lambda \subseteq M$ is a compact hyperbolic set. Then for $\forall x \in \Lambda$, $\forall \epsilon > 0$:
$\boldsymbol{W^s_\epsilon(x)} := \{y \in M \mid \forall n \in \mathbb{N}_0,\ d(f^n(y), f^n(x)) < \epsilon\}$.
$\boldsymbol{W^u_\epsilon(x)} := \{y \in M \mid \forall n \in \mathbb{N}_0,\ d(f^{-n}(y), f^{-n}(x)) < \epsilon\}$.

Definition 2.2 (Global stable and unstable manifolds for hyperbolic maps.)**.** Let M be a closed n-manifold, $F : M \to M$ be a C^1 diffeomorphism, $\Lambda \subseteq M$ is a compact hyperbolic set. Then for $\forall x \in \Lambda$, $\forall \epsilon > 0$:
$\boldsymbol{W^s(x)} := \{y \in M \mid \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\}$.
$\boldsymbol{W^u(x)} := \{y \in M \mid \lim_{n\to\infty} d(f^{-n}(y), f^{-n}(x)) = 0\}$.
It follows from hyperbolic systems' theory and proved in various sources, such as [5], that $\exists \epsilon_0 > 0$ such that for $\forall 0 < \epsilon < \epsilon_0$,
$W^s_\epsilon(x)$ is a manifold and $W^s(x) = \bigcup_{n\in\mathbb{N}_0} F^{-n}(W^s_\epsilon(x))$.
$W^u_\epsilon(x)$ is an embedded manifold and $W^u(x) = \bigcup_{n\in\mathbb{N}_0} F^{n}(W^u_\epsilon(x))$.

Definition 2.3 (Stable and unstable manifolds of hyperbolic periodic points.)**.** Let a point P on a n-dimensional manifold M be a hyperbolic periodic point of period p, of a map F. Then P must be a hyperbolic fixed point of the map F^p. By the Hartman-Grobman theorem [2], there is a neighbourhood W of P in which F^p is C^1 conjugate to the linear map $dF^p(P)$. The local stable and unstable manifolds of P exists in this neighborhood. The global stable and unstable manifolds can then be described as above.

2.2. Invariant cone systems.

Definition 2.4 (Invariant system of cones :)**.** Let M be a manifold, F : $M \to M$ a C^1 map. Let $T(M) = E^u + E^s$ be a splitting of the tangent bundle and for $\forall x \in M$, $\forall \alpha > 0$, the α-unstable cone at x, denoted as $\mathcal{C}^u_\alpha(x)$, is defined to be $\{(v^u, v^s) \in T(x, M) | |v^s| \leq \alpha |v^u|\}$. Then (M, F) is said to have a system of invariant cones wrt the splitting $E^u \oplus E^s$ if $\exists \alpha > 0$ such that for $\forall x \in M$, $\forall v \in \mathcal{C}^u_\alpha(x)$, $v' = dF(x)(v) \in \mathcal{C}^u_\alpha(F(x))$.

Definition 2.5 (Invariant, expanding system of cones:)**.** Let M be a manifold, $F : M \to M$ a C^1 map. Let $T(M) = E^u + E^s$ be a splitting of the tangent bundle. Then (M, F) is said to have a an invariant expanding cone system if $\exists \alpha > 0$, $k > 1$ such that for $\forall x \in M$, $\forall v \in \mathcal{C}^u_\alpha(x)$, $v' = dF(x)(v) \in \mathcal{C}^u_\alpha(F(x))$ and $|v'| > k|v|$.

If the splitting $E^u \oplus E^s$ and constants $\alpha > 0$ and $K > 0$ are clear from the context, then they will be dropped from the notation and the invariant, expanding cone system $\mathcal{C}^u_\alpha$ will be simply denoted as $\{ \boldsymbol{\mathcal{C}(x)} \mid x \in M\}$ or $\{ \boldsymbol{\mathcal{C}_\alpha(x)} \mid x \in M\}$.

We will now describe curves whose tangent bundle is contained in the cone system. Borrowing from a similar idea in physics, we will call such curves *causal*.

Definition 2.6 (Differentiable causal curves)**.** A C^1 curve $\lambda : \mathbb{R} \to M$ is said to be a causal curve if its tangent vector at every point lies inside the cone associated with that point. In other words, for $\forall t \in \mathbb{R}$, $\lambda'(t) \in \mathcal{C}_\alpha(\lambda(t))$.

This definition of causality can be extended from differentiable curves to continuous curves

Definition 2.7 (Causal curves)**.** A C^0 curve $\lambda : \mathbb{R} \to M$ is said to be a causal curve if at every point z_0 on λ and any neighborhood U of z_0, any point z on U can be joined to z_0 by a differentiable, causal curve γ lying inside U.

Causal curves are therefore Lipschitz curves. By Rademacher's theorem (see [1], Theorem 3.1.6), they are differentiable at Lebesgue almost every point. In particular, they are rectifiable and their length can be obtained by integrating their slopes.

Lemma 2.8 (Properties of an expanding system of unstable cones)**.** *Let M be a manifold, $F : M \to M$ a C^1 map with an invariant, expanding cone system wrt the splitting $E^s \bigoplus E^u$ and constants K and α. Let $\lambda : \mathbb{R} \to M$ be a causal curve. Then*

1. *The image $F(\lambda)$ under the map of the causal curve λ is also a causal curve.*
2. *length$(F(\lambda)) > K$ length(λ).*
3. *length$(F^n(\lambda)) > K^n$ length(λ), which $\to \infty$ as $n \to \infty$.*

Proof. (1) For $\forall t \in \mathbb{R}$, $(F \circ \lambda)'(t) = dF(\lambda(t))\lambda'(t)$. Since the cone system is forward invariant, this vector $\in K^u_\alpha(\lambda(t))$, hence $F(\lambda)$ is a causal curve too.

(2) $length(F(\lambda)) = \int_{\mathbb{R}} |(F \circ \lambda)'(t)|dt \leq \int_{\mathbb{R}} k|(\lambda)'(t)|dt = K\ length(\lambda)$.

(3) This follows from (i) and (ii) above.

□

Cone system for (1.1). Let e_x and e_y denote the vector fields along the X and Y directions respectively. Take E^u to be e_x and E^s to be e_y. Then the map (1.1) has an invariant cone system wrt the splitting $e_x \oplus e_y$ if it satisfies (1.4). The cone system will be expanding if the stronger condition (1.5) is satisfied.

In an invariant cone system, the cone $\mathcal{C}(x)$ at a point x is mapped under $DF(x)$ into the cone at $F(x)$. The following quantities a_n track how thin the images $DF^n(x)(\mathcal{C}(x))$ get with the iteration number n.

$$\text{For } \forall n \in \mathbb{N},\ \forall z \in M,\ \alpha_n(z) := \sup\{\frac{\|v\|}{\|u\|} \mid (u, v) \in DF^n(\mathcal{C}_\alpha(z))\}. \tag{2.1}$$

$$\text{For } \forall z \in M,\ \bar{\alpha}(z) := \inf_{n\in\mathbb{N}} \alpha_n(z). \tag{2.2}$$

If $\bar{\alpha}(z) > 0$, then all of the images $DF^n(\mathcal{C}(z))$ will contain the $\bar{\alpha}$-cone wrt the splitting $E^u \oplus E^s$. Note that the $E^u(DF^n(z))$ always lies inside $DF^n(\mathcal{C}(z))$. This can be summarized as follows.

$$\text{For } \forall z \in M,\ \forall n \in \mathbb{N},\ E^u(F^n(z)) \subseteq \mathcal{C}_{\bar{\alpha}}(F^n(z)) \subseteq dF^n(\mathcal{C}_\alpha(z)) \subseteq \mathcal{C}_\alpha(F^n(z)). \tag{2.3}$$

So if $\bar{\alpha}(z) = 0$, E^u must be invariant under DF along the orbit of z. Conversely, if E^u is not an invariant sub-bundle, then $\bar{\alpha} > 0$.

Stable and unstable manifolds. It turns out that in dynamical systems with an expanding, invariant cone system, the stable and unstable manifolds, W^s and W^u, have a nice behavior which have been described in the following two propositions.

Proposition 2.9. *In a 2-manifold M with an invariant, expanding cone system, the unstable manifold of a saddle is an embedded, causal curve.*

Proof. Let z be a saddle, W^u its unstable manifold. By Lemma 4.1, $\mathcal{C}(z) \cap T_z(W^u)$ contains a subspace of dimension 1. Since W^u is 1-dimensional, $T_z(W^u)$ must be contained in the interior of $\mathcal{C}(z)$. By continuity of the tangent space along W^u and of the cone system $\mathcal{C}$, $\exists \epsilon > 0$ such that for $\forall z' \in W^u_\epsilon(z)$, $T_{z'}(W^u) \in \mathcal{C}(z')$. Therefore, since the curve $W^u_\epsilon(z)$ is causal and since $W^u(x) = \bigcup_{n\in\mathbb{N}_0} F^n(W^u_\epsilon(z))$, by Lemma 2.8, it is an embedded, causal curve. □

Proposition 2.10. *In a manifold M with an invariant, expanding cone system, the stable manifold of a saddle is everywhere transverse to the invariant cones.*

Proof. Let W^s be the stable manifold of a saddle z. The proof has two parts. First we will prove that W^s_ϵ is transverse to the cone system for some $\epsilon > 0$. Secondly, we will show this implies that the entire stable manifold is transverse to the cone system.

Since $T_z(W^s)$ is a contracting eigenspace and vectors in $\mathcal{C}(z)$ expand by at least $k > 1$, $T_z(W^s)$ must be disjoint from $\mathcal{C}(z)$. Since both $\mathcal{C}$ and W^s are C^1 structures and $\mathcal{C}(z)$ is a closed set, for sufficiently small $\epsilon > 0$, W^s_ϵ is transverse to the cone system.

Suppose at some $z_0 \in W^s$, $\exists w \in T_{z_0}(W^s) \cap \mathcal{C}(z_0)$. By the definition of the stable manifold, $z_n := F^n(z_0) \to z$, so for every large $n \in \mathbb{N}$, $z_n \in W^s_\epsilon$. By the invariance of the cone structure, $dF^n(w) \in \mathcal{C}(z_n)$, a contradiction of the previous conclusion. So no such z_0 exists and W^s is everywhere transverse to the cone system. □

3. Proof of Theorem 1.1

In this section it will be assumed that F in (1.1) is a local diffeomorphism and its periodic saddles are dense in $\mathbb{T}^2$. If F is transitive, then the theorem is already proved, so we will proceed with the assumption that F is not transitive. So there exists an open subset U of $\mathbb{T}^2$ whose images are not dense in $\mathbb{T}^2$.

By assumption, there exists a periodic point of period $p \in \mathbb{N}$ in U. Hence, $\forall n \in \mathbb{N}$, $F^{pn}(U)$ intersects U. Since U, is connected, so is $F^{pn}(U)$, therefore, for $\forall N \in \mathbb{N}$, $U_n := \bigcup_{0 \le n \le N} F^{pn}(U)$ is a connected set. $U_\infty := \bigcup_{n \in \mathbb{N}_0} F^{pn}(U)$ is open and $K := \overline{U_\infty}$ is closed and hence compact. Both K and U_∞ are forward invariant under F^p.

Given any subset A of a topological space X, $\boldsymbol{Int(A)}$ will denote the interior of the set A.

By Lemma 4.3, $U_\infty = Int(K)$, $\partial(K)$ and K^C are strongly forward and backward invariant under F^p. As a consequence, we can conclude that,

Lemma 3.1. *Unstable manifolds of periodic saddles do not cross ∂K. Moreover, if a saddle lies on ∂K, then its unstable manifold lies in ∂K.*

Claim 1. The connected component of the boundary of K are C^1, causal, closed curves. Thence, we will conclude that K is homeomorphic to a cylinder. This is proved in Section 3.1.

Claim 2. $\mathbb{T}^2$ decomposes into a finite number of such cylinders with disjoint interiors and each cylinder is mapped into and onto another cylinder. This is proved in Section 3.2.

3.1. The proof of Claim 1.

The boundary of K. By making U smaller if necessary, we may assume without loss of generality that U is homeomorphic to an open rectangle and that its boundary has four C^1 components, the top and the bottom boundary are tangent to E^u and its left and right boundaries are tangent to E^s. For $\forall x \in S^1$, $\boldsymbol{K_x}$ is the 1-dimensional set $S_x \cap K$. This is a compact set and hence a union of compact intervals. Therefore, a point $z_0 = (x_0, y_0) \in \partial K$ is either the boundary of a proper component interval of K_{x_0} or a singleton component of K_{x_0}. We will first prove that each connected component of ∂K is an embedded curve. To prove this, we will show that there is a unique C^0 curve embedded in ∂K that passes through z_0. The claim will be proved separately for both the possibilities for z_0 in Lemmas 3.2 and 3.3.

Lemma 3.2. *For $\forall z_0 = (x_0, y_0) \in \partial K$ which are boundary points of proper component intervals of K_{x_0}, $\exists$ a unique C^0 curve embedded in ∂K that passes through z_0.*

Proof. If S_{x_0} is given the usual orientation, then every proper component interval of K_{x_0} has an upper boundary and a lower boundary. Without loss of generality, the point $z_0 = (x_0, y_0)$, which lies on the boundary of a proper component interval of K_{x_0}, is an upper boundary. We will first demonstrate the existence of a continuous curve embedded in ∂K and passing through z_0.

Since $z_0 \in \partial K$, $\exists z_n \in U$ and $k_n \in \mathbb{N} \ni F^{pk_n}(z_n) \to z$ and $F^{pk_n}(z_n) \in S_{x_0}$. Since z is the upper boundary of a component interval, this convergence is from below. Let γ be the upper boundary of U. Let z'_n be the point in γ with the same X-coordinate as z_n. Since F is orientation preserving, $F^{pk_n}(z'_n) \to z$ and $F^{pk_n}(z'_n) \in S_{x_0}$.

Let I be a small open interval in S^1 around x_0. For all $x \in I$, let $y_n(x)$ be the y coordinate of the point where the curve $\gamma_n := F^{pk_n}(\gamma)$ first hits S_x after passing through $F^{pk_n}(z'_n)$. Since γ is a causal curve, so are its images γ_n under F^{pk_n}. Then it follows that $\Gamma(x) := \sup_{n\in\mathbb{N}} y_n(x)$ is a continuous, causal curve passing through z and lying in ∂K.

We will now prove that this embedded curve is unique. Suppose $\exists$ two different curves Γ_1 and Γ_2 in ∂K passing through z_0. Let Q be a periodic saddle close to z_0 such that one of these curves lies above it and the other below it. Then since the unstable manifold of Q is causal, it must intersect one of these curves, which contradicts Lemma 3.1. □

Lemma 3.3. *For $\forall z_0 = (x_0, y_0) \in \partial K$ which are singleton components of K_{x_0}, $\exists$ a unique C^0 curve embedded in ∂K that passes through z_0.*

Proof. Let $z_0 = (x_0, y_0)$ be a singleton component of S_{x_0}. Since it lies on ∂K, it is a limit point of points z_n in the interior of K.

We will first show that these points z_n can be chosen to lie on S_{x_0}. Suppose not, then let $z_n = (x_n, y_n)$. Then without loss of generality, $x_n \to x_0^-$. Let $I(x_n) = [\Gamma_1(x_n), \Gamma_2(x_n)]$ be the component of K_{x_n} that contains z_n. By Lemma 3.2, Γ_1 and Γ_2 can be extended to C^0 curves in a left neighborhood of x_0. Since there are no points in the interior of K in a neighborhood of z_0 in S_{x_0}, Γ_1, Γ_2 intersect S_{x_0} at z_0. Let Q be a periodic saddle close to z_0 such that one of these curves lies above it and the other below it. Then since the unstable manifold of Q is causal, it must intersect one of these curves, which contradicts Lemma 3.1. So the assumption was false and hence, z_0 is a limit of proper component intervals of K_{x_0}.

Let these component intervals be $I_n = [a_n, b_n]$. Without loss of generality, I_n-s converge to z_0 from below. By Lemma 3.1, there exists C^0 curves Γ_n embedded in ∂K and passing through b_n. For $\forall x$ in a neighborhood of x_0, $\Gamma(x) := \varlimsup_{n\to\infty} \Gamma_n(x)$. This Γ lies in ∂K and is C^0 and causal. It is also the unique curve in ∂K passing through z_0. □

Lemma 3.4. *No point $z_0 = (x_0, y_0)$ on a boundary curve of ∂K passing through an upper/lower boundary point can be a singleton component of K_{x_0}.*

Proof. Let Γ_1 be a boundary curve of ∂K passing through an upper boundary point z_1. Without loss of generality, z_0 is the closest point to z_1 lying on Γ_1, so the segment of the curve Γ_1 from z_1 to z_0 must have only upper boundary points and consequently, has an adjacent lower boundary curve Γ_2. Since Γ_1, Γ_2 are C^0 and z_0 is an isolated point of K_{x_0}, they must intersect at z_0. This contradicts the uniqueness of embedded curves in ∂K passing through z_0. □

Lemma 3.5. *A boundary curve of ∂K passing through an upper boundary point, cannot intersect a boundary curve of ∂K passing through a lower boundary point.*

Proof. Let the contrary be true, so there exists a boundary curve Γ_1 of ∂K passing through an upper boundary point z_1 and intersecting a boundary curve Γ_2 of ∂K passing through a lower boundary point z_2. Let the point of intersection be $z_0 = (x_0, y_0)$. Then since Γ_1, Γ_2 are continuous, z is a singleton component of K_{x_0}. However, this is not possible by Lemma 3.4 □

Lemma 3.6. *Let $z \in \partial K$ be a lower boundary point. Then $F^p(z)$ is also a lower boundary point and the connected component of ∂K containing z has only lower boundary points. Analogous statements hold true for upper boundary points.*

Proof. Since F is orientation preserving and by Lemma 3.1, lower boundary points are mapped into lower boundary points. Since $\mathbb{T}^2$ itself is orientable, an embedded curve, which is a co-dimension 1 embedded submanifold, is also orientable and hence, if a boundary has a lower boundary point, then all of its points are lower boundaries. □

Now consider an adjacent upper boundary and lower boundary Γ_1 and Γ_2 respectively. These two curves do not intersect each other. Hence, the region R enclosed by them is either homeomorphic to a cylinder or an infinite tape. If it is a cylinder, then the Claim 1 is proved. So we will demonstrate that it cannot be an infinite tape.

The proof will be by contradiction, so we will assume that R is an infinite tape. Therefore, Γ_1, Γ_2 must be open curves of infinite length. We will first show that none of them can have more than one periodic saddle using the following lemma.

Lemma 3.7. *Let Γ be an causal, open curve in $\mathbb{T}^2$ invariant under F^p. Then at most one periodic point can lie on Γ.*

Proof. Since Γ is causal and is invariant under F^p, it must have infinite length.

The proof will be by contradiction. So let Q_1, Q_2 be two periodic points on Γ with periods p_1, p_2 respectively. Let $N = pp_1p_2$. Then Q_1, Q_2 are fixed points of F^N and Γ is invariant under F^N.

Γ must be the unstable manifold of both the Q_i-s. Let L be the section of the curve joining the Q_i-s. Then for $\forall n \in \mathbb{N}$, $F^{nN}(L)$ is a sub-segment of Γ with the Q_i-s as its endpoints. Since Γ is an open curve and since F is a local diffeomorphism, L is the only such curve-segment, hence $F^N(L) = L$. This contradicts the expansion property of the map F on causal curves. □

However, the next lemma proves that periodic points on the Γ_i-s are dense. This leads to a contradiction and consequently, proves Claim 1.

Lemma 3.8. *Every point z_0 in an upper boundary curve is a limit point of periodic points lying on that curve.*

Proof. Suppose $z_0 = (x_0, y_0)$ is a point on an upper boundary of K. Let Γ_1 be the upper boundary passing through z_0 and let Γ_2 be the adjacent lower boundary. Let I be a small neighborhood of x_0 in S^1.

Then the region $R := \{z = (x, y) \in \mathbb{T}^2 \mid x \in I,\ \Gamma_2(x) \le y \le \Gamma_1(x)\}$ is homeomorphic to a rectangle. Since periodic saddles are dense, $\exists$ a periodic saddle $z_1 = (x_1, y_1)$ in R of period $q \in \mathbb{N}$. Then the circle S_{x_1} must be invariant under $F^p q$ and for a sufficiently large $N \in \mathbb{N}$, all the periodic points on S_{x_1} are fixed under F^{pqN}. Let L be the line segment $S_{x_1} \cap R$. L contains the periodic point Q.

Note that Q is an attracting fixed point for the map $F^{NP}|S_{x_1}$. By Lemma 4.3, the end-points of L must be fixed points. □

3.2. **The proof of Claim 2.** As a result of the lemmas in the previous section, we can conclude that the invariant set K is diffeomorphic to a cylinder $S^1 \times [0, 1]$ and $Int(K)$, K^C and ∂K are invariant under F^p. Now instead of considering the iterated map F^p, we will examine the action of F on K.

Lemma 3.9. *Suppose for some* $m \in \mathbb{N}$, $F^m(K) \cap K \neq \Phi$. *Then* $F^m(K) = K$.

Proof. Let the contrary be assumed, i.e., $F^m(K) \cap K \neq \Phi$ for some $m \in \mathbb{N}$. Since $F^p(K) = K$, it may be assumed without loss of generality that $0 < m < p$. Since F is a local diffeomorphism and $F^{p-m}(F^m(K) = K$, we must have,

$$F^m(\partial K) = \partial(F^m(K)),\ F^m(Int(K)) = Int(F^m(K). \tag{3.1}$$

The boundary of K is composed of two disjoint, closed, causal curves which are the upper and lower boundaries respectively. Since F is orientation preserving, F maps upper(lower) boundaries to upper(lower) boundaries, so $F^m(K)$ is also a cylinder. The only way by which the upper/lower boundary of K intersects the upper/lower boundary of $F^m(K)$ and satisfy (3.1) is if they coincide. Therefore, $F^m(K) = K$. □

Lemma 3.10. *The images of* K *under* F *form a disjoint collection of cylinders.*

Proof. Suppose for some $0 \le m < n < p$, $F^m(K) \cap F^n(K) \neq \Phi$. Then $F^{p-n}(F^m(K) \cap F^n(K)) \neq \Phi$. But $F^{p-n}(F^m(K) \cap F^n(K)) \subseteq F^{p-n+m}(K) \cap F^p(K) = F^{p-n+m}(K) \cap K$.
Therefore, $\exists p' := p - (n - m)$ which is less than p and for which $F^{p'}(K) \cap K \neq \Phi$. Without loss of generality, p' is the minimum such integer > 0. Then by Lemma 3.9, this implies that $F^{p'}(K) = K$. From this it follows that the images $K = F^0(K), \ldots, F^{p'-1}(K)$ are all distinct cylinders. □

Lemma 3.11. *The number of periodic cylinders is finite.*

Proof. Henceforth, K and its images $F^1(K), F^2(K), \ldots$ will be called *periodic cylinders.* Since by assumption, F is not transitive, it does not have dense trajectories. Therefore, every point is in a periodic cylinder. We will show that there are only a finite number of periodic cylinders, whence, $\mathbb{T}^2$ can be decomposed into a finite "stack" of cylinders with disjoint interiors.

Let K be a periodic cylinder. For $\forall k, n \in \mathbb{N}$, the intersection $\Gamma_{k,n} \cap \partial K$ has a periodic point, where $\Gamma_{k,n}$ is described in (1.8). Since (1.1) is C^1, each $\Gamma_{k,n}$ can have a finite number of periodic points on it wrt the map F^n. Therefore, the set of such periodic cylinders K must be finite in number. □

Lemma 3.12. *All the periodic cylinders have the same period.*

Proof. Let p be the minimum period of a periodic cylinder K_1. Therefore, if Γ is its upper boundary, then $F^p(\Gamma) = \Gamma$. But Γ is the lower boundary of the cylinder K_2 stacked above K_1. Therefore, the period of K_2 must be a divisor of p and because of the minimality of p, must be p itself. A repetition of this argument a finite number of times establishes that all the cylinders have the same period p. □

Therefore, we have proved our main result Theorem 1.1.

4. Appendix : Some lemmas

Lemma 4.1. *Let z be a saddle and W^u its unstable manifold. If $dim(W^u) = 1$, then W^u is an embedded causal curve.*

Proof. Suppose that M is an n-manifold. Let S^{n-1} be the unit sphere in $T_z(M)$. Then the intersection $Q := \mathcal{C}(z) \cap S^{n-1}$ is compact. If the dimension of E^u is k for some $0 < k < n$, and $\alpha = tan(\theta)$ for some $\theta \in (0, \frac{\pi}{2}$, then Q is diffeomorphic to $S^{k-1} \times D^{n-k-1} \times [-\theta, \theta]$ via the map $\phi : (u, v, t) \mapsto \cos(t)u + \sin(t)v$.

If $k = 1$, then $Q \cong S^0 \times D^{n-1}$. Now consider the map $G : S^{n-1} \to S^{n-1}$ defined as $G(w) = \frac{dF(z)(w)}{\|w\|}$. This map is well defined and smooth because $dF(z)$ is invertible and linear. Since $\mathcal{C}(z)$ is invariant under $dF(z)$, $G : K \to K$. Therefore, by the Brower fixed point theorem, G has a fixed point w in K. But w is a fixed point of G iff $\exists \lambda > 0$ such that $dF(z)(w) = \lambda w$. Since dF is an expanding map on $\mathcal{C}$, λ must be > 1.

Since $dF(z)$ is hyperbolic, all subspaces of $T_z(M)$ invariant under $dF(z)$ must be subspaces of either $T_z(W^u)$ or $T_z(W^s)$. In particular, the eigenvector w must be in one of these subspaces. Since its eigenvalue λ is > 1, w must $\in T_z(W^u)$. Then the span of w is the 1-dimensional subspace contained in the intersection $\mathcal{C}(z) \cap T_z(W^u)$. □

Lemma 4.2. *Let $F : M \to M$ be a local diffeomorphism on a compact manifold M. Let the periodic points of F be dense in M and let $U \subset M$ be open and forward-invariant under F. Suppose that $K := \bar{U}$ is a proper subset of M. Then $F^p(K) = K$, $F(\partial K) = \partial K$ and $F(K^C) = K^C$.*

Proof. Since K is forward invariant under F, $F(K) \subseteq K$. Suppose it is a strict subset, i.e., $F(K) \subset K$. Since K is compact, $F(K)$ is compact and hence closed. Therefore, $K - F(K)$ has non-empty interior V. Let $Q \in V$ be a periodic point of period q. Then $F^q(Q) = Q$. However, $Q = F^q(Q) \in F^q(K)$ which is disjoint from V which contains Q, leading to a contradiction. Hence the assumption was wrong and $F(K) = K$.

We will first prove that $F(\partial K) \subseteq \partial K$. Let the contrary be assumed, hence $\exists x \in \partial K \ni F(x) \in Int(K)$. Since F is a local diffeomorphism, it is an open mapping too. Hence, $\exists$ a neighborhood V of x such that $F(V)$ is an open set contained in the interior $Int(K)$ of K. Since x is a boundary point, V contains an open set in the exterior of K. Let Q be a periodic point of period q lying in $V - K$. Then $F^q(Q) = Q$. But $F(Q) \in F(V) \subset K$, and by the forward invariance of K under F, $F^n(Q)$ never exits K and hence is never equal to Q which lies outside K, leading to a contradiction.

We will next prove that in fact, strict equality holds. Let the contrary be assumed, i.e., $F(\partial K) \subset \partial K$. Then $\exists x \in \partial K \ni F(x)$ is disjoint from ∂K. However, since $F(K) = K$, x must have an inverse image y in $Int(K)$. Take a neighborhood V of y in K. Then $F(V)$ is a neighborhood of x. Since x is a boundary point, $F(V)$ intersects K^C. This contradicts the forward invariance of K. Hence the initial assumption was untrue and $F(\partial K)$ must equal ∂K.

The last equality follows from the previous two. □

Lemma 4.3. *Let F be a a C^1 map on S^1 with a non-zero derivative. Let $J \subset S^1$ be a compact set such that both J and ∂J are forward invariant. Let a component interval L of J contain an attractor. Then the endpoints of L are periodic points.*

Proof. For $N \in \mathbb{N}$ sufficiently large, all the periodic points of F^N are fixed points. J and ∂J remain invariant under F^N. Consider an endpoint A of L. The proof will be by contradiction, so suppose that A is not a fixed point of F^N.

Let Q be the fixed point on L closest to A. By assumption, $Q \neq A$. Q must be an attractor or repellor. We will prove that both cases lead to contradictions and hence, the assumption about A not being a fixed point will be proved false.

If Q is an attractor, then A is in the basin of attraction of Q and F^N maps A closer to Q. In other words, $F^N(A) \in Int(L) \subseteq Int(J)$, violating the invariance of ∂J.

If Q is a repellor, then A lies in the basin of repulsion of Q and hence, A has an inverse image under F^N in the interior of the line segment QA. Since F has non-zero derivative, $F^N(QA)$ must contain a neighborhood of A. Since $A \in \partial J$, $F^N(QA)$ intersects the exterior of J. This contradicts the invariance of J. □

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