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by

Paloma Hernández and Héctor Méndez

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
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ENTROPY OF INDUCED DENDRITE HOMEOMORPHISMS

PALOMA HERNÁNDEZ AND HÉCTOR MÉNDEZ

Abstract. Let $f : D \to D$ be a dendrite homeomorphism. Let 2^D denote the hyperspace of all nonempty compact subsets of D endowed with the Hausdorff metric. Let $2^f : 2^D \to 2^D$ be the induced homeomorphism. We show in this note that the topological entropy of 2^f has only two possible values: 0 or ∞. This claim generalizes a result due to M. Lampart and P. Raith.

1. Introduction and some definitions

A *continuum* is a nonempty compact and connected metric space.

Let $X = (X, d)$ be a continuum. Let 2^X be the collection of all nonempty compact subsets of X endowed with the Hausdorff metric H_d induced by metric d. If Y is a continuum and $Y \subset X$ then Y is a *subcontinuum* of X.

It is said that X is

- an *arc* provided that it is homeomorphic to the unit interval $[0, 1]$,
- a *simple closed curve* provided that it is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
- a *dendrite* provided that it is a locally connected and contains no simple closed curves.

Let $\mathbb{N}$ denote the set of all positive integers. A *mapping* is a continuous function. Let $f : X \to X$ be a mapping.

Let $2^f : 2^X \to 2^X$ be the mapping induced in 2^X by f. For each $n \in \mathbb{N}$ and for each $A \in 2^X$, $\left(2^f\right)^n (A) = f^n(A)$.

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In section 5 we recall the definition of topological entropy of a mapping $f : X \to X$ and some of its basic properties.

In year 2010 M. Lampart and P. Raith, [6], proved the following result.

Theorem 1.1. *Let $I = [0, 1]$ be the unit interval and $f : I \to I$ be a homeomorphism. Then*

- *topological entropy of $2^f : 2^I \to 2^I$ has only two possible values: 0 or ∞;*
- *topological entropy of $2^f : 2^I \to 2^I$ is ∞ if and only if $f \circ f$ is not the identity map.*

Our main result is the following: Let D be a dendrite and $f : D \to D$ be a homeomorphism. Then

- topological entropy of $2^f : 2^D \to 2^D$ has only two possible values: 0 or ∞;
- topological entropy of $2^f : 2^D \to 2^D$ is ∞ if and only if the set of recurrent points of f is distinct from D.

2. Preliminary results

Let $X = (X, d)$ be a compact metric space. From now on we assume that X is nondegenerate (it contains more than one point). Let $f : X \to X$ be a mapping. Given a point x in X, the *orbit of x under f* is the sequence

$$o(x, f) = \{f^n(x) : n \geq 0\},$$

where f^0 denotes the identity map in X, $f^1 = f$, and for each $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$. If there exists $n \in \mathbb{N}$ with $f^n(x) = x$, then x is a *periodic point* of f. If $f(x) = x$, then x is a *fixed point* of f. Let $Per(f)$ and $Fix(f)$ denote the set of all periodic points and of all fixed points of f, respectively. If $x \in Per(f)$, then $n_0 = \min\{n \in \mathbb{N} : f^n(x) = x\}$ is the *period* of x.

The *omega limit set of x under f* is the set

$$\omega(x, f) = \left\{y \in X : \exists \{n_1 < n_2 < \cdots\} \text{ with } \lim_{i\to\infty} f^{n_i}(x) = y\right\}.$$

If $x \in \omega(x, f)$, then it is said that x is a *recurrent point* of f. Let $R(f)$ denote the set of all recurrent points of f. It is known that for each $N \in \mathbb{N}$, $R(f^N) = R(f)$. See [2].

Let $\Lambda(f) = \cup\{\omega(x, f) : x \in X\}$. Note that

$$Fix(f) \subset Per(f) \subset R(f) \subset \Lambda(f).$$

If f is a homeomorphism, the *alpha limit set of x under f* is the set

$$\alpha(x, f) = \omega(x, f^{-1}).$$

If $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball around $x \in X$ with radius ε. If $A \subset X$, then the symbols $cl(A)$, $int(A)$ and $bd(A)$ stand for the closure, the interior and the boundary of A in X. Furthermore, if $A \neq \emptyset$,

$$N(A, \varepsilon) = \{y \in X : \text{ there is } x \in A,\ d(y, x) < \varepsilon\} = \cup \{B(x, \varepsilon) : x \in A\},$$

and $diam(A) = \sup \{d(x, y) : x \text{ and } y \text{ in } A\}$. Symbol $|A|$ stands for the cardinality of A.

A nonempty subset $A \subset X$ is *invariant* under $f : X \to X$ if $f(A) \subset A$; it is *strongly invariant* provided that $f(A) = A$. It is said that A is a *minimal set* of f if it is closed, invariant and for any closed subset $B \subset A$, $B \neq \emptyset$, that is invariant under f, we have that $B = A$.

Proposition 2.1 contains some basic properties of $\omega(x, f)$. See [2].

Proposition 2.1. *Let x in X.*

- *$\omega(x, f)$ is closed and nonempty.*
- *$\omega(x, f)$ is strongly invariant.*
- *For each $m \in \mathbb{N}$, $f(\omega(x, f^m)) = \omega(f(x), f^m)$.*
- *For each $m \in \mathbb{N}$,*

  $$\omega(x, f) = \omega(x, f^m) \cup \omega(f(x), f^m) \cup \cdots \cup \omega(f^{m-1}(x), f^m).$$

  Thus, $\omega(x, f)$ is finite if and only if for some m, $\omega(x, f^m)$ is finite.
- *Let $y \in X$. If $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$, then $\omega(x, f) = \omega(y, f)$.*
- *If cardinality of $\omega(x, f)$ is finite, say N, then there exists $y \in Per(f)$ of period N with*

  $$\omega(x, f) = \left\{y, f(y), f^2(y), \ldots, f^{N-1}(y)\right\}.$$

  Therefore, if $\omega(x, f)$ is finite, then it is a minimal set of f.

Proof of Lemma 2.2 is a direct consequence of Proposition 2.1.

Lemma 2.2. *Let $x, y \in X$. If for some $N \in \mathbb{N}$, $\omega(x, f^N) = \omega(y, f^N)$, then $\omega(x, f) = \omega(y, f)$.*

Let X be a continuum. Let A and B be two elements of 2^X. Then

$$H_d(A, B) = \inf \{\varepsilon > 0 : A \subset N(B, \varepsilon) \quad \text{and} \quad B \subset N(A, \varepsilon)\}$$

defines a metric in 2^X, the *Hausdorff metric*. See [4] and [8].

Let $\{A_n\}$ be a sequence in 2^X and $A \in 2^X$. If $\lim_{n\to\infty} H_d(A_n, A) = 0$, then we write $\lim A_n = A$.

3. Dendrites

In this section we recall some basic properties of dendrites and of maps defined on dendrites. Let D denote a nondegenerate dendrite. Proofs of Theorems 3.1 and 3.2 can be found in [8].

Theorem 3.1. *The following conditions hold:*

- *Every connected subset of D is arcwise connected.*
- *Each subcontinuum of D is a dendrite.*
- *The intersection of any two connected subsets of D is connected.*
- *For each $\varepsilon > 0$, there are finitely many dendrites $D_1, \dots, D_n$ contained in D such that $diam(D_i) < \varepsilon$ and $D = \cup_{i=1}^{n} D_i$.*
- *For every dendrite mapping $f : D \to D$, there is a point $x \in D$ such that $f(x) = x$.*

Let $x \in D$. It is said that x is an *end point* of D provided that $D \setminus \{x\}$ is connected; x is a *cut point* of D if $D \setminus \{x\}$ is not connected. The *order* of x, $ord(x)$, is the cardinality of the set of all components of $D \setminus \{x\}$. Each point of D is of order $\leq \aleph_0$ (see [8]). If $ord(x) \geq 3$, it is said that x is a *branch point* of D.

Theorem 3.2. *The following conditions hold:*

- *Each nondegenerate subcontinuum of D contains uncountably many cut points.*
- *The set of all branch points of D is countable.*

Corollary 3.3. *Each nondegenerate subcontinuum of D contains cut points of order* 2.

Proof. The result follows immediately from Theorem 3.2. □

Propositions 3.4 and 3.5 are proved in [7].

Proposition 3.4. *Let $\{A_n\}$ be a sequence of nonempty connected subsets of D such that for each pair $n \neq m$, $A_n \cap A_m = \emptyset$. Then*

$$\lim_{n\to\infty} diam\,(A_n) = 0.$$

Given two distinct points a and b in D, there is only one arc from a to b contained in D. We denote such an arc with $[a, b]$. Also we use the following notation: $(a, b] = [a, b] \setminus \{a\}$, $[a, b) = [a, b] \setminus \{b\}$, and $(a, b) = [a, b] \setminus \{a, b\}$.

Proposition 3.5. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any pair of points $a, b \in D$, $d(a, b) < \delta$ implies $diam([a, b]) < \varepsilon$.*

Corollary 3.6. *Let $\{a_n\}$ be a sequence in $D \setminus \{a\}$ such that* $\lim_{n\to\infty} a_n = a$. *Then*

$$\lim_{n\to\infty} diam\,([a_n, a]) = 0.$$

Proof. The result follows immediately from Proposition 3.5. □

Proposition 3.7. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any collection of points $a, b, u, v \in D$, if $d(a, u) < \delta$ and $d(b, v) < \delta$, then*

$$H_d([a, b], [u, v]) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Let $a, b, u, v \in D$. By Proposition 3.5, there exists $\delta > 0$ such that if $d(a, u) < \delta$ and $d(b, v) < \delta$, then $diam([a, u]) < \varepsilon$ and $diam([b, v]) < \varepsilon$.

Consider the following sets:

$$J = [u, a] \cup [a, b] \cup [b, v] \quad \text{and} \quad K = [u, a] \cup [u, v] \cup [b, v].$$

Note that J and K are both connected. It follows that

$$[u, v] \subset J \quad \text{and} \quad [a, b] \subset K.$$

Hence for each $x \in [u, v]$ there exists $t \in [a, b]$, $d(x, t) < \varepsilon$, and for each $y \in [a, b]$ there exists $s \in [u, v]$, $d(y, s) < \varepsilon$. Thus $H_d([a, b], [u, v]) < \varepsilon$. □

Corollary 3.8. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of points in D. Let $a, b \in D$ be two distinct points such that $lim_{n\to\infty} a_n = a$ and $lim_{n\to\infty} b_n = b$.*

(1) *Then* $\lim[a_n, b_n] = [a, b]$.
(2) *For each arc $[s, t] \subset [a, b]$, $\{s, t\} \cap \{a, b\} = \emptyset$, there exists $\delta > 0$ such that for each pair of points $u, v \in D$ with $d(a, u) < \delta$ and $d(b, v) < \delta$, $[s, t] \subset [u, v]$.*
(3) *For each arc $[s, t] \subset [a, b]$, $\{s, t\} \cap \{a, b\} = \emptyset$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $[s, t] \subset [a_n, b_n]$.*
(4) *For each point $s \in (a, b)$, there exists $n_0 \in \mathbb{N}$ such that $[a, s] \subset [a, b_n]$ provided that $n \geq n_0$.*

Proof. The first claim is an immediate consequence of Proposition 3.7.

To prove claim (2), let $[s, t]$ and $[a, b]$ be two arcs in D such that $[s, t]$ is contained in $[a, b]$ and $\{s, t\} \cap \{a, b\} = \emptyset$.

Let $\varepsilon > 0$ such that

$$\varepsilon < \min\{\min\{d(a, x) : x \in [s, t]\}, \ \min\{d(b, x) : x \in [s, t]\}\}.$$

By Proposition 3.5, there exists $\delta > 0$ with the property that for any pair of points $u, v \in D$, $d(u, a) < \delta$ and $d(v, b) < \delta$ imply $diam([a, u]) < \varepsilon$ and $diam([b, v]) < \varepsilon$.

Notice that

$$[a, b] \subset [a, u] \cup [u, v] \cup [v, b].$$

Since $[s, t] \cap ([a, u] \cup [v, b]) = \emptyset$, then $[s, t] \subset [u, v]$.

Claim (3) is a consequence of claim (2).

Proof of claim (4) is similar to the one given above to claim (2) (see also reference [7]). □

Lemma 3.9. *Let $U \subset D$, $U \neq D$, be a nonempty open connected set. Let $x_0 \in bd(U)$. Then for each $u \in U$,*

$$[u, x_0] \cap U = [u, x_0).$$

Proof. Let us consider the arc $[u, x_0]$. There exists a sequence $\{x_n\}$, contained in U, such that $\lim_{n\to\infty} x_n = x_0$.

Let $v \in [u, x_0)$. By Corollary 3.8, there exists $n \in \mathbb{N}$ such that $[u, v] \subset [u, x_n] \subset U$. Thus $v \in U \cap [u, x_0) \subset U \cap [u, x_0]$.

Clearly $[u, x_0] \cap U \subset [u, x_0)$. Hence $[u, x_0] \cap U = [u, x_0)$. □

4. Dynamics of dendrite homeomorphisms

We collect in this section some basic properties of dendrite homeomorphisms. Recall D represents a nondegenerate dendrite.

Proof of Proposition 4.1 can be found in [9].

Proposition 4.1. *Let $f : D \to D$ be a homeomorphism. Then for each arc $[a, b]$ contained in D, $f([a, b]) = [f(a), f(b)]$.*

Consider the unit interval $[0, 1]$. If $f : [0, 1] \to [0, 1]$ is an increasing homeomorphism, then for each $x \in [0, 1]$, $o(x, f)$ is a monotone sequence. Therefore, $|\alpha(x, f)| = 1$ and $|\omega(x, f)| = 1$. Furthermore, if $Fix(f) \cap (0, 1) = \emptyset$, then one of the following two conditions holds:

- For every $x \in (0, 1)$, $\alpha(x, f) = \{0\}$ and $\omega(x, f) = \{1\}$, or
- for every $x \in (0, 1)$, $\alpha(x, f) = \{1\}$ and $\omega(x, f) = \{0\}$.

Proposition 4.2. *Let $f : D \to D$ be a homeomorphism. Let $u, v \in D$ be two points such that $f(u) = u$ and $f(v) = v$. Then for each point x in the arc $[u, v]$, $|\omega(x, f)| = 1$ and $|\alpha(x, f)| = 1$. Furthermore, if $Fix(f) \cap (u, v) = \emptyset$, then one of the following two conditions holds:*

1. *For every $x \in (u, v)$, $\alpha(x, f) = \{u\}$ and $\omega(x, f) = \{v\}$, or*
2. *for every $x \in (u, v)$, $\alpha(x, f) = \{v\}$ and $\omega(x, f) = \{u\}$.*

Proof. Let $h : [0, 1] \to [u, v]$ be a homeomorphism with $h(0) = u$ and $h(1) = v$. Then $g : [0, 1] \to [0, 1]$ given by $g = h^{-1} \circ f \circ h$ is an increasing homeomorphism.

Notice that for each point $x \in [u, v]$, $h^{-1}(f(x)) = g(h^{-1}(x))$. Hence for each $x \in [u, v]$, $|\alpha(x, f)| = 1$ and $|\omega(x, f)| = 1$.

If $Fix(f) \cap (u, v) = \emptyset$, then $Fix(g) \cap (0, 1) = \emptyset$. Now the proof of the second part follows immediately. □

Corollary 4.3. *Let $f : D \to D$ be a homeomorphism. Let $u, v \in D$ and $N \in \mathbb{N}$. If $f^N([u, v]) = [u, v]$, then for each $x \in [u, v]$, $|\alpha(x, f)| \le 2N$ and $|\omega(x, f)| \le 2N$.*

Proof. The result is an immediate consequence of Proposition 4.2. □

Theorem 4.4 is one of the main results in [1].

Theorem 4.4. *Let $f : D \to D$ be a homeomorphism and $x \in D$. Then $\omega(x, f)$ is either a periodic orbit or a Cantor set. Moreover, if $\omega(x, f)$ is a Cantor set, then f restricted to $\omega(x, f)$ is an adding machine.*

Note that in both cases considered in Theorem 4.4, the limit set $\omega(x, f)$ is a minimal set of f.

Proposition 4.5. *Let $f : D \to D$ be a homeomorphism and $x \in D$. Then*

- $f(\alpha(x, f)) = \alpha(x, f)$.
- $\alpha(x, f)$ *is a minimal set of* f.

Proof. The result is an immediate consequence of $\alpha(x, f) = \omega(x, f^{-1})$ and of Theorem 4.4. □

Corollary 4.6. *Let $f : D \to D$ be a homeomorphism, and $x \in D$. Then $x \in \omega(x, f)$ if and only if $x \in \alpha(x, f)$. Furthermore, if $x \notin \omega(x, f)$, then for every $y \in D$, $x \notin (\omega(y, f) \cup \alpha(y, f))$.*

Proof. The result is an immediate consequence of Proposition 4.5. □

Notice that corollary 4.6 says that for each dendrite homeomorphism, $\Lambda(f) \subset R(f)$. Therefore in this setting $R(f) = \Lambda(f)$.

In [9] the author proved the following two interesting and useful results (Propositions 4.7 and 4.8).

Proposition 4.7. *Let $f : D \to D$ be a homeomorphism. Then*

$$R(f) = \Lambda(f) = cl(Per(f)).$$

Proposition 4.8. *Let $f : D \to D$ be a homeomorphism. If $R(f) = D$, then every cut point of D is a periodic point of f.*

Proof of Lemma 4.9 can be found in [1].

Lemma 4.9. *Let $f : D \to D$ be a homeomorphism. If x_0 is an end point of D such that $f(x_0) = x_0$, then $|Fix(f)| \geq 2$.*

Lemma 4.10. *Let $f : D \to D$ be a homeomorphism. Let a, b, c be three distinct end points of D. If $\{a, b, c\} \subset Fix(f)$, then there exists a cut point of D, say u, such that $u \in Fix(f)$.*

Proof. Consider the arcs $[a, b]$, $[a, c]$ and $[b, c]$. Since a, b, c are three distinct end points of D, there exists $u \in (a, b)$ such that $[a, b] \cap [a, c] = [a, u]$.

Therefore

$$\{u\} = [a, b] \cap [a, c] \cap [b, c].$$

Since $f([a, b]) = [a, b]$, $f([a, c]) = [a, c]$ and $f([b, c]) = [b, c]$, it follows that $f(u) = u$. □

Lemma 4.11. *Let $f : D \to D$ be a homeomorphism. Let $a, b \in Fix(f)$, $a \neq b$. If a and b are end points of D and $|Fix(f)| = 2$, then one of the following two conditions holds:*

(1) *For every $x \in (D \setminus \{a, b\})$, $\alpha(x, f) = \{a\}$ and $\omega(x, f) = \{b\}$.*
(2) *For every $x \in (D \setminus \{a, b\})$, $\alpha(x, f) = \{b\}$ and $\omega(x, f) = \{a\}$.*

Proof. The arc $[a, b]$ is strongly invariant under f. If $x \in (a, b)$, then $f(x) \neq x$. Hence for every $x \in (a, b)$, $\alpha(x, f) = \{a\}$ and $\omega(x, f) = \{b\}$, or $\alpha(x, f) = \{b\}$ and $\omega(x, f) = \{a\}$.

Let us assume the first option: For every $x \in (a, b)$, $\alpha(x, f) = \{a\}$ and $\omega(x, f) = \{b\}$ (the other case is similar). Let $u \in D$, $u \notin [a, b]$. Since a and b are end points of D, there exists $x \in (a, b)$ such that $[u, x] \cap [a, b] = \{x\}$.

Continuum D has no simple closed curves hence, for each $n \in \mathbb{N}$, the arc $f^n([u, x]) = [f^n(u), f^n(x)]$ is disjoint from the arc $[u, x]$. It implies that for every pair $n, m \in \mathbb{Z}$, with $n \neq m$, $f^n([u, x]) \cap f^m([u, x]) = \emptyset$.

Therefore
$$\lim_{n\to\infty} diam(f^n([u, x])) = 0 \quad \text{and} \quad \lim_{n\to-\infty} diam(f^n([u, x])) = 0.$$

Thus $\alpha(u, f) = \alpha(x, f) = \{a\}$ and $\omega(u, f) = \omega(x, f) = \{b\}$. $\square$

5. Topological entropy

In this section we recall the definition of topological entropy and some of its basic properties. Let $X = (X, d)$ denote a nondegenerate compact metric space. Let $f : X \to X$ be a mapping.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. A subset $A \subset X$ is said to (n, ε)-*span* X if for any $x \in X$ there exists $a \in A$ with
$$d(f^i(x), f^i(a)) < \varepsilon, \quad \text{for} \quad 0 \leq i \leq n - 1.$$

Let $r(n, \varepsilon)$ denote the smallest cardinality of any (n, ε)-spanning set for X. Let
$$r(\varepsilon, f) = \limsup_{n\to\infty} \left(\frac{1}{n}\right) \log(r(n, \varepsilon)).$$

The *topological entropy* of f is given by
$$ent(f) = \lim_{\varepsilon\to 0} r(\varepsilon, f).$$

Proofs of Propositions 5.1, 5.2 and 5.3 can be found in references [2] and [10]. Proposition 5.4 is proved in [3].

Proposition 5.1. *Let $n \in \mathbb{N}$. Then $ent(f^n) = n \cdot ent(f)$. Furthermore, if $f : X \to X$ is a homeomorphism, then $ent(f^{-1}) = ent(f)$.*

Proposition 5.2. *Let $A \subset X$ be a closed and invariant set of $f : X \to X$. Then $ent(f) \geq ent(f|_A)$.*

Proposition 5.3. *Let X and Y be compact metric spaces, $f : X \to X$ and $g : Y \to Y$ be two mappings, and $h : X \to Y$ be a surjective mapping. If for every $x \in X$, $h(f(x)) = g(h(x))$, then $ent(f) \geq ent(g)$. If h is a homeomorphism, then $ent(f) = ent(g)$.*

Consider the space of 2 symbols,

$$\Sigma_2 = \{\mathbf{t} = (\ldots, t_{-2}, t_{-1} \cdot t_0, t_1, t_2, \ldots) : t_n \in \{0,1\}, n \in \mathbb{Z}\},$$

and the shift homeomorphism $\sigma_2 : \Sigma_2 \to \Sigma_2$, given by

$$\sigma_2(\ldots, t_{-2}, t_{-1} \cdot t_0, t_1, t_2, \ldots) = (\ldots, t_{-2}, t_{-1}, t_0 \cdot t_1, t_2, \ldots).$$

Proposition 5.4. *Then $ent(\sigma_2) = \log(2)$.*

Theorem 5.5 is proved in [1]. It is also a direct consequence of Theorem 4.4.

Theorem 5.5. *Let $f : D \to D$ be a dendrite homeomorphism. Then $ent(f) = 0$.*

The following result is an immediate consequence of Theorem 17 in [5].

Theorem 5.6. *Let $f : X \to X$ be a homeomorphism. If for some point $x \in X$,*

$$x \notin (\alpha(x, f) \cup \omega(x, f)),$$

then $ent(2^f) \geq \log(2)$.

The proof we present to Theorem 5.7 follows, with slight changes, the proof given to Theorem 2 in [6]. For the sake of completeness we provide it here.

Theorem 5.7. *Let $f : X \to X$ be a homeomorphism. Let $K, L \in 2^X$. If there exists an infinite countable set $A = \{a_0, a_1, a_2, \ldots\} \subset X$ such that*

- *for every $i \geq 0$, $\alpha(a_i, f) = K$ and $\omega(a_i, f) = L$,*
- *for every $i \geq 0$, $a_i \notin (K \cup L)$, and*
- *for every pair $i \neq j$, $i \geq 0$, $j \geq 0$,*

$$\left\{f^k(a_i) : k \in \mathbb{Z}\right\} \cap \left\{f^k(a_j) : k \in \mathbb{Z}\right\} = \emptyset,$$

then $ent(2^f) = \infty$.

Proof. Let $N \in \mathbb{N}$. We claim that $ent(2^f) \geq N \cdot \log(2)$.

Step 1. Let $\Gamma_N = \prod_{k \in \mathbb{Z}} \{0,1\}^N$.

We use the following notation. Let $\mathbf{u} \in \Gamma_N$,

$$\mathbf{u} = (\ldots, (u_{-10}, \ldots, u_{-1N-1}) \cdot (u_{00}, \ldots, u_{0N-1}), (u_{10}, \ldots, u_{1N-1}), \ldots).$$

Note that Γ_1 is homeomorphic to Σ_2.

Let $\varphi : \Gamma_N \to \Gamma_N$ be the mapping given by $\mathbf{v} = \varphi(\mathbf{u})$ where

$$\mathbf{v} = (\ldots, (u_{-10}, \ldots, u_{-1N-1}), (u_{00}, \ldots, u_{0N-1}) \cdot (u_{10}, \ldots, u_{1N-1}), \ldots).$$

The map φ is a homeomorphism and, in a way we are going to see, it is connected with the shift map σ_2.

Let $h : \Gamma_N \to \Sigma_2$ be a homeomorphism given in the following way: Let $i \in \mathbb{Z}$. There exists a unique pair of numbers $j \in \mathbb{Z}$ and $0 \leq k \leq N-1$ such that $i = jN + k$. Then the coordinate t_i of $\mathbf{t} = h(\mathbf{u})$ satisfy $t_i = u_{jk}$:

$$\mathbf{t} = (\ldots, u_{-10}, \ldots, u_{-1N-1} \cdot u_{00}, \ldots, u_{0N-1}, u_{10}, \ldots, u_{1N-1}, \ldots) .$$

That is, h simply *erase* most of the parenthesis in $\mathbf{u}$.

It is easy to see that for each $\mathbf{u} \in \Gamma_N$, $(\sigma_2)^N(h(\mathbf{u})) = h(\varphi(\mathbf{u}))$.

Thus $ent(\varphi) = ent((\sigma_2)^N) = N \cdot \log(2)$.

Step 2. Consider the first N elements of A, $\{a_0, a_1, a_2, \ldots, a_{N-1}\}$.

Let $M \subset X$,

$$M = \{f^k(a_0) : k \in \mathbb{Z}\} \cup \{f^k(a_1) : k \in \mathbb{Z}\} \cup \cdots \cup \{f^k(a_{N-1}) : k \in \mathbb{Z}\},$$

and $\mathcal{F} \subset 2^X$,

$$\mathcal{F} = \left\{B \in 2^X : (K \cup L) \subset B \subset (K \cup L \cup M)\right\} .$$

It is not difficult to prove that collection $\mathcal{F}$ is a compact subset of 2^X and it is strongly invariant under the induced homeomorphism $2^f : 2^X \to 2^X$.

Now let $g : \mathcal{F} \to \Gamma_N$ be the mapping given in this way: Take $B \in \mathcal{F}$. Then the coordinates of $\mathbf{u} = g(B)$, u_{jk}, $j \in \mathbb{Z}$, $0 \leq k \leq N-1$, are given by

$$u_{jk} = \begin{cases} 1, & \text{if } f^j(a_k) \in B, \\ \\ 0, & \text{if } f^j(a_k) \notin B. \end{cases}$$

Notice that $g : \mathcal{F} \to \Gamma_N$ is a homeomorphism. Furthermore, for each element $B \in \mathcal{F}$, $\varphi(g(B)) = g((2^f)^{-1}(B))$.

Hence,

$$ent(2^f|_{\mathcal{F}}) = ent((2^f)^{-1}|_{\mathcal{F}}) = N \cdot \log(2).$$

It follows that for each $N \in \mathbb{N}$, $ent(2^f) \geq N \cdot \log(2)$.

Thus $ent(2^f) = \infty$. □

6. Entropy of induced homeomorphism $2^f : 2^D \to 2^D$

Let D denote a nondegenerate dendrite. Let $f : D \to D$ be a homeomorphism. In order to show that the topological entropy of the induced homeomorphism $2^f : 2^D \to 2^D$ has only two possible values, 0 or ∞, we consider two cases:

- The set of recurrent points of f is a proper subset of D, $R(f) \neq D$.
- Every point of D is a recurrent point of f, $R(f) = D$.

With Proposition 6.1 and Theorem 6.3 we solve the first case. We consider the second case in Propositions 6.4 and 6.5 and in Theorem 6.6.

Proposition 6.1. *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Let $x_0 \in D \setminus R(f)$ and U be the component of $D \setminus R(f)$ that contains x_0. Then for each $x \in U$,*

$$\alpha(x, f) = \alpha(x_0, f) \quad \textit{and} \quad \omega(x, f) = \omega(x_0, f).$$

Proof. Let $x_0 \in D \setminus R(f)$ and U be the component of $D \setminus R(f)$ that contains the point x_0. Note that the sets $R(f) = cl(Per(f))$ and $D \setminus R(f)$ are strongly invariant under f. Since $f : D \to D$ is a homeomorphism, for each $n \in \mathbb{Z}$, $f^n(U)$ is a component of $D \setminus R(f)$.

We consider two cases.

Case 1. There exists $n \in \mathbb{N}$ such that $f^n(U) = U$.

Let $N = \min\{n \in \mathbb{N} : f^n(U) = U\}$. Let $g : D \to D$ be the homeomorphism given by $g = f^N$ and let $W = cl(U)$. Since $g(U) = U$, $g(W) = W$.

There exists a fixed point of g, say u_0, in the dendrite W. Note that the point u_0 is not in U. Hence u_0 is an end point of W.

By Lemma 4.9, there exists another fixed point of g in W.

Let $u_1 \in Fix(g) \cap W$, $u_1 \neq u_0$. Note u_1 is an end point of W as well.

Notice that g cannot have a third fixed point in W. For, by Lemma 4.10, it implies that there exists a cut point u of W with $g(u) = u$, a contradiction.

Therefore g has exactly two fixed points in dendrite W. Both of them are end points of W.

By Lemma 4.11, for every point $x \in (W \setminus \{u_0, u_1\})$,

$$\alpha(x, f^N) = \alpha(x_0, f^N) \quad \text{and} \quad \omega(x, f^N) = \omega(x_0, f^N).$$

It implies, by Lemma 2.2, that for each $x \in (W \setminus \{u_0, u_1\})$,

$$\alpha(x, f) = \alpha(x_0, f) \quad \text{and} \quad \omega(x, f) = \omega(x_0, f).$$

Case 2. For every $n \in \mathbb{N}$, $f^n(U) \cap U = \emptyset$.

Hence for each pair $n, m \in \mathbb{Z}$, with $n \neq m$, $f^n(U) \cap f^m(U) = \emptyset$.

By Proposition 3.4,

$$\lim_{n\to\infty} diam(f^n(U)) = 0 \quad \text{and} \quad \lim_{n\to-\infty} diam(f^n(U)) = 0. \tag{6.1}$$

Thus, for every $x \in U$, $\alpha(x, f) = \alpha(x_0, f)$ and $\omega(x, f) = \omega(x_0, f)$. □

Corollary 6.2. *Let $f : D \to D$ be a homeomorphism and $x \in X$. Then $\alpha(x, f)$ is finite if and only if $\omega(x, f)$ is finite.*

Proof. Let $x \in D$. If $x \in R(f)$, then $\alpha(x, f) = \omega(x, f)$ and the conclusion readily follows.

Let us assume that $x \in D \setminus R(f)$. Let U be the open component of $D \setminus R(f)$ that contains x. According to Proposition 6.1, there are two cases: In the first one, when $f^N(U) = U$ for some $N \in \mathbb{N}$, it easy to see that for every point $x \in U$, $|\alpha(x, f)| \leq N$ and $|\omega(x, f)| \leq N$. In fact, the cardinality of any of these two limit sets is a factor of N.

In the second case our claim is this: for every $x \in U$, $\alpha(x, f) = \omega(x, f)$. Take $y \in bd(U)$. By limits in (6.1), $\alpha(x, f) = \alpha(y, f)$ and $\omega(x, f) = \omega(y, f)$. Since $y \in R(f)$, $\alpha(y, f) = \omega(y, f)$. □

Theorem 6.3. *Let $f : D \to D$ be a homeomorphism such that $R(f) \neq D$. Then $ent(2^f) = \infty$.*

Proof. Let U be a nonempty component of $D \setminus R(f)$. Note that U is an infinite uncountable set. Hence, it is possible to define an infinite countable set $A = \{a_0, a_1, a_2, \ldots\} \subset U$ such that for each pair $i \neq j$,

$$\left\{f^k(a_i) : k \in \mathbb{Z}\right\} \cap \left\{f^k(a_j) : k \in \mathbb{Z}\right\} = \emptyset.$$

By proposition 6.1, for each $i \geq 0$,

$$\alpha(a_i, f) = \alpha(a_0, f) \quad \text{and} \quad \omega(a_i, f) = \omega(a_0, f).$$

Note that for each $i \geq 0$, $a_i \notin (\alpha(a_0, f) \cup \omega(a_0, f))$. Therefore, by Theorem 5.7, $ent(2^f) = \infty$. □

Proposition 6.4. *Let D be a dendrite. Let $\varepsilon > 0$. There exists a finite set $E \subset D$ of cut points of order 2 such that each component U of $D \setminus E$ has diameter $< \varepsilon$.*

Proof. Let $\varepsilon > 0$. There are finitely many dendrites $D_1, \ldots, D_n$ contained in D such that

- $diam(D_i) < \frac{\varepsilon}{4}$;
- $D = \cup_{i=1}^{n} D_i$; and
- for each i, $D_i \setminus (\cup_{j \neq i} D_j) \neq \emptyset$.

For each i, fix $x_i \in D_i \setminus (\cup_{j \neq i} D_j)$. Now for each set $\{i, j\}$ with $D_i \cap D_j = \emptyset$ take x_{ij} in the arc $[x_i, x_j]$ such that $x_{ij} \notin D_i \cup D_j$ and $ord(x_{ij}) = 2$ (see Corollary 3.3). Let E be the set whose elements are all such points x_{ij}.

Note the following: If $B \subset D$ is a connected set with

$$B \cap D_i \neq \emptyset, \quad B \cap D_j \neq \emptyset \quad \text{and} \quad D_i \cap D_j = \emptyset,$$

then $x_{ij} \in B$.

Let U be a component of $D \setminus E$. Let k be such that $U \cap D_k \neq \emptyset$. Then for each l with $U \cap D_l \neq \emptyset$, $D_k \cap D_l \neq \emptyset$. Hence $diam(U) < \varepsilon$. □

Proposition 6.5. *Let D be a dendrite. Let $f : D \to D$ be a homeomorphism such that $R(f) = D$. Then for each $\varepsilon > 0$, there exists a finite collection of dendrites, $\{D_1, D_2, \ldots, D_m\}$, $D_i \subset D$, with the following properties:*

- *For each i, $1 \leq i \leq m$, $diam(D_i) < \varepsilon$.*
- *For each i there exists j, $1 \leq i, j \leq m$, $f(D_i) = D_j$.*

- *If $i \neq j$, then $D_i \cap D_j = \emptyset$ or $|D_i \cap D_j| = 1$.*
- *If $D_i \cap D_j = \{x_0\}$, then $x_0 \in Per(f)$.*
- *If $i \neq j$, $i \neq k$ and $j \neq k$, then $D_i \cap D_j \cap D_k = \emptyset$.*

Proof. Let $\varepsilon > 0$. According to Proposition 6.4 there exists a finite set $E \subset D$ of cut points of order 2 such that each component U of $D \setminus E$ has diameter $< \varepsilon$.

Notice that the boundary of each component U of $D \setminus E$ intersects E in a nonempty set. Each point of E is in the boundary of exactly two components of $D \setminus E$. Hence the cardinality of the collection of all components of $D \setminus E$ is finite.

Let $x \in E$. By Proposition 4.8, x is a periodic point of f. Since f is a homeomorphism, every $y \in o(x, f)$ is a cut point of D.

Let

$$F = \{y \in D : y \in o(x, f),\, x \in E\}.$$

Note that F is a finite set and each point of F is of order 2. It follows that the cardinality of the collection of all components of $D \setminus F$ is finite.

Since $E \subset F$, for each component W of $D \setminus F$ there exists some component U of $D \setminus E$ such that $W \subset U$. Hence every component of $D \setminus F$ has diameter $< \varepsilon$.

Let $\{W_1, W_2, \ldots, W_m\}$ be the components of $D \setminus F$. For each $1 \leq i \leq m$, let us define $D_i = cl(W_i)$. Notice that each dendrite D_i has diameter $< \varepsilon$.

It is immediate that F and $D \setminus F$ are strongly invariant sets of the homeomorphism $f : D \to D$. The image under f of a component of $D \setminus F$ is a component of $D \setminus F$. Hence for each i there exists j, $1 \leq i, j \leq m$, such that $f(D_i) = D_j$.

Since D has no simple closed curves, for each $i \neq j$, $D_i \cap D_j = \emptyset$ or $|D_i \cap D_j| = 1$. If $i \neq j$ and $x \in D_i \cap D_j$, then $x \in F$ and $x \in Per(f)$.

Since each point of F is a cut point of order 2, $D_i \cap D_j \cap D_k = \emptyset$ provided that $i \neq j$, $i \neq k$ and $j \neq k$. □

Theorem 6.6. *Let D be a dendrite. Let $f : D \to D$ be a homeomorphism such that $R(f) = D$. Then $ent(2^f) = 0$.*

Proof. Let $\varepsilon > 0$. Let $\{D_1, D_2, \ldots, D_m\}$ be a finite collection of dendrites in D that satisfy the conditions of Proposition 6.5. Let

$$F = \{x \in D : \text{there exist } i \neq j,\ x \in D_i \cap D_j\}.$$

Let $G = \left\{A \in 2^D : A \subset F\right\}$. Since F is a finite set, G is a finite collection of points of 2^D. Let $k = |G|$.

Let $B \in 2^D$. Consider an element $A \in G$ with the following property: For each i, $1 \leq i \leq m$,

$$B \cap D_i \neq \emptyset \quad \text{if and only if} \quad A \cap D_i \neq \emptyset.$$

Note that for each i, $1 \leq i \leq m$, and for each each $n \geq 0$, there exists $1 \leq j \leq m$ such that

$$(f^n(B \cap D_i) \cup f^n(A \cap D_i)) \subset D_j.$$

Therefore, for each i and n, $H_d(f^n(B \cap D_i), f^n(A \cap D_i)) < \varepsilon$.

Since

$$f^n(B) = \cup_{i=1}^m f^n(B \cap D_i) \quad \text{and} \quad f^n(A) = \cup_{i=1}^m f^n(A \cap D_i),$$

then

$$H_d(f^n(B), f^n(A)) < \varepsilon.$$

It follows that G is an (n, ε)-spanning set for 2^D and mapping 2^f.

We have that for each $n \geq 0$, $r(n, \varepsilon) \leq |G| = k$. Then

$$r(\varepsilon, 2^f) = \limsup_{n \to \infty} \left(\frac{1}{n}\right) \log(r(n, \varepsilon)) = 0.$$

Thus, $ent(2^f) = 0$. □

Corollary 6.7. *Let D be a dendrite. Let $f : D \to D$ be a homeomorphism. Then $ent(2^f)$ has only two possible values: 0 or ∞. Furthermore, $ent(2^f) = \infty$ if and only if $R(f) \neq D$.*

Proof. The result is an immediate consequence of Theorems 6.3 and 6.6. □

7. Final part

The next result is easy to prove.

Proposition 7.1. *Let $f : [0,1] \to [0,1]$ be a homeomorphism. Then $R(f) = [0,1]$ if and only if f^2 is the identity map.*

Theorem 7.2 is due to M. Lampart and P. Raith, [6].

Theorem 7.2. *Let $f : [0,1] \to [0,1]$ be a homeomorphism. Then $ent(2^f)$ has only two possible values: 0 or ∞. Furthermore, $ent(2^f) = \infty$ if and only if $f^2 \neq id$.*

Proof. The result immediately follows from Corollary 6.7 and Proposition 7.1. □

Conjecture 7.3 and Question 7.4 propose some interesting paths to follow. Both of them are due to M. Lampart and P. Raith as well, [6].

Conjecture 7.3. Let X be a continuum and $f : X \to X$ be a homeomorphism. Then entropy of induced map $2^f : 2^X \to 2^X$ has only two possible values: 0 or ∞.

Question 7.4. Which topological spaces X satisfy that $ent(2^f) \in \{0, \infty\}$ for all continuous maps f (for all homeomorphisms f)?

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(Hernández and Méndez) Departamento de Matemáticas, Facultad de Ciencias, UNAM, Ciudad Universitaria, C.P. 04510, D. F. MEXICO.

E-mail address, Hernández: paloma_hz@yahoo.com.mx

E-mail address, Méndez: hml@ciencias.unam.mx