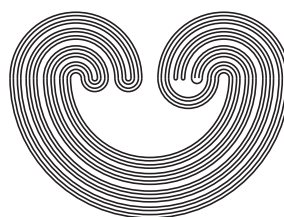


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EXTENSIONS OF ULTRAMETRIC SPACES

COLLINS AMBURO AGYINGI

ABSTRACT. The concept of the tight span of a metric space was introduced and studied by Dress. It is known that his (Dress) theory is equivalent to the theory of the injective hull of a metric space independently discussed by Isbell some years earlier. Dress showed in particular that for a metric space X the tight extension T_X is maximal among the tight extensions of X . In a paper by Bayod et al., it was shown that Isbell's approach can be modified to work similarly for ultrametric spaces. They went ahead and constructed the tight extension for an arbitrary ultrametric space X , which in this article we shall call the ultrametric tight (*um-tight*) extension of X and is denoted uT_X . Continuing that work we show in the present paper that large parts of the theory developed by Dress do not use the triangle inequality of the metric and when appropriately modified will hold unchanged for ultrametric spaces. In particular we shall show that for an ultrametric space X , uT_X is a maximal (among the *um-tight*) extensions of X .

1. INTRODUCTION

We say that a metric space Y is “injective” if every mapping which increases no distance from a subspace of any metric space X to Y can be extended, increasing no distance, over X . These spaces were introduced in [2] by Aronszajn and Panitchpakdi, and they called them “hyperconvex.”

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Isbell [6] proved that every metric space X has an “injective hull,” i.e., an isometric embedding $e : X \rightarrow E$ such that E is injective and no injective proper subspace of E contains $e(X)$. Later on Dress [14] independently introduced the concept of “tight extension” of a metric space X and proved that a “tight span” (i.e., a maximal tight extension) of X is the same as an injective envelope of X .

In this paper we are going to consider the case of ultrametric spaces. It should be noted at this point that ultrametric spaces have applications in pure mathematics as well as in Physics (check for instance the excellent survey [3] and the references given there, in order to get a feeling of the way and the depth in which ultrametric concepts play a role in some parts of modern physics).

It can be shown that no ultrametric space with more than one point is injective (use [4, pp. 46-48]). Thus we shall restrict ourselves to the following weaker definition: An ultrametric space Y is said to be ultrametrically injective if every contractive mapping from a subspace of any ultrametric space X to Y can be extended to a contraction over X .

In [10] the concept of tight extension was studied for ultrametric spaces. In particular such an extension was constructed and it was shown that a compact ultrametric space is ultrametrically hyperconvex if and only if it is spherically complete. The last statement implies that every compact ultrametric space will be equal to its corresponding hyperconvex hull. Recall [8, Definition 4.1] that a (ultra)metric space (X, d) is said to be hyperconvex if for any indexed class of closed balls $B(x_i, r_i)$, $i \in I$, of X which satisfy

$$d(x_i, x_j) \leq r_i + r_j, \quad i, j \in I,$$

it is necessarily the case that

$$\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset.$$

We will show in this article that every ultrametric space X has a *um*-tight extension uT_X which is maximal among the *um*-tight extensions of X .

2. PRELIMINARIES

In this section we start by recalling some basic concepts from the theory of ultrametric spaces that will be useful in the development of this article.

In Section 3 we present a summary of the construction of an extension of an ultrametric space according to [10] and which is tight (in the sense of Dress [14]).

We remark here that ultrametric spaces are also known in the literature as non Archimedean metric spaces.

Example 2.1. ([9, Page 7]) Let $X = [0, \infty)$. Define $n(x, y) = \max\{x, y\}$ if $x \neq y$ and $n(x, y) = 0$ if $x = y$. Then it is easy to see that (X, n) is an ultrametric space.

In some cases we will replace $[0, \infty)$ with $[0, \infty]$ and in this case we shall speak of an extended ultrametric.

Lemma 2.2. (compare [9, Proposition 2.1]) Let $\alpha, \beta, \gamma \in [0, \infty)$. Then the following are equivalent:

- (a) $n(\alpha, \beta) \leq \gamma$
- (b) $\alpha \leq \max\{\beta, \gamma\}$.

Proof. (a) \Rightarrow (b)

To reach a contradiction, suppose that $\alpha > \max\{\beta, \gamma\}$. Since $\alpha > \beta$, we have $n(\alpha, \beta) = \alpha \leq \gamma$ by part (a) and the way n was defined. Thus we have that $\alpha \leq \max\{\beta, \gamma\} < \alpha$ and this is a contradiction.

(b) \Rightarrow (a)

Suppose on the contrary that $n(\alpha, \beta) > \gamma$. Then $n(\alpha, \beta) = \alpha$ and $\alpha > \beta$ and hence $\alpha > \gamma$ which implies that $\alpha > \max\{\beta, \gamma\}$. We have by (b) that $\alpha \leq \max\{\beta, \gamma\}$ which is a contradiction. \square

We have the following corollaries.

Corollary 2.3. Let (X, d) be an ultrametric space. Consider a map $f : X \rightarrow [0, \infty)$ and let $x, y \in X$. Then the following are equivalent:

- (a) $n(f(x), f(y)) \leq d(x, y)$
- (b) $f(x) \leq \max\{f(y), d(x, y)\}$.

Definition 2.4. A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two (ultra-) metric spaces (X, d_X) and (Y, d_Y) is called nonexpansive provided that $d_Y(f(x), f(y)) \leq d_X(x, y)$ whenever $x, y \in X$.

Corollary 2.5. Let (X, d) be an ultrametric space. Then the map $f : (X, d) \rightarrow ([0, \infty), n)$ is a nonexpansive map if and only if $f(x) \leq \max\{f(y), d(x, y)\}$ whenever $x, y \in X$.

Definition 2.6. A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two (ultra-) metric spaces (X, d_X) and (Y, d_Y) is said to be an isometry provided that $d_Y(f(x), f(y)) = d_X(x, y)$ whenever $x, y \in X$. Two (ultra-) metric spaces (X, d_X) and (Y, d_Y) are said to be isometric provided that there exists a bijective isometry between them.

3. ULTRAMETRICALLY INJECTIVE HULLS OF ULTRAMETRIC SPACES

In this section, we shall recall some results from the theory of hyperconvex hulls of ultrametric spaces due to [10].

We shall proceed by recalling the construction, by Bayod et al. ([10]), of an ultrametrically injective hull for any arbitrary ultrametric space X .

Definition 3.1. Let (X, d) be an ultrametric space and let $\mathcal{F}(X, d)$ be the set of all functions f on (X, d) where $f : X \rightarrow [0, \infty)$.

For any such functions f and g set $N(f, f) = 0$ and

$$N(f, g) = \inf\{\max\{f(x), g(x)\} : x \in X\}.$$

Then one sees immediately that N is an extended ultrametric on $\mathcal{F}(X, d)$.

Let (X, d) be an ultrametric space. We shall say that a function $f \in \mathcal{F}(X, d)$ is ultra-ample if for all $x, y \in X$, we have $d(x, y) \leq \max\{f(x), f(y)\}$.

Let us denote by UP_X the set of all ultra-ample functions on an ultrametric space (X, d) . The proof of the following lemma is obvious.

Lemma 3.2. Let (X, d) be an ultrametric space. For each $a \in X$, $f_a(x) := d(x, a)$ whenever $x \in X$, is an ultra-ample function belonging to UP_X .

Let (X, d) be an ultrametric space. We say that a function f is minimal among the ultra-ample functions on (X, d) if it is an ultra-ample function and if g is ultra-ample on (X, d) and for each $x \in X$, $g(x) \leq f(x)$ then $f = g$. By UT_X we shall denote the set of all minimal functions on (X, d) equipped with the restriction of N to UT_X , which we shall still denote by N . Note that the restriction of N to UT_X is indeed an ultrametric on UT_X (check part (a) of Theorem 3.10 below). In the following we shall call (UT_X, N) the ultra-metrically injective hull of (X, d) .

Lemma 3.3. ([10, Lemma 3]) Let (X, d) be an ultrametric space. Then we have the following:

(a) For each $z \in X$, the map $f_z : X \rightarrow \mathbb{R}$ defined by $f_z(x) = d(x, z)$ belongs to UT_X .

(b) If f is ultra-ample on (X, d) and $x, y \in X$, then the map $h : X \rightarrow [0, \infty)$ defined by $h(z) = f(z)$ when $x \neq z$ and $h(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f(y)\}$ is also ultra-ample on (X, d) .

Lemma 3.4. ([10, Lemma 4]) Let (X, d) be an ultrametric space. Then we have that for $f \in UT_X$ and $x, y \in X$ one has $f(x) < \max\{d(x, y), f(y)\}$ and therefore one has either $f(x) = f(y) > d(x, y)$ or $f(x) = d(x, y) > f(y)$ or $f(y) = d(x, y) > f(x)$.

Lemma 3.5. Let (X, d) be an ultrametric space and suppose that f is a minimal ultra-ample function on (X, d) . Then

$$\begin{aligned} f(x) &= \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f(y)\} \\ &= \sup\{f_x(y) : y \in X \text{ and } f_x(y) > f(y)\} \end{aligned}$$

whenever $x \in X$.

Proof. Let $x \in X$. We see that $\sup\{d(x, y) : y \in X \text{ and } d(x, y) > f(y)\} \leq f(x)$, since $d(x, y) \leq \max\{f(y), f(x)\}$ by ultra-ampleness of f .

Suppose now that there is $x_0 \in X$ such that $\sup\{d(x_0, y) : y \in X \text{ and } d(x_0, y) > f(y)\} < f(x_0)$. Set $h(x) = f(x)$ if $x \in X$ and $x \neq x_0$, and $h(x_0) = \sup\{d(x_0, y) : y \in X \text{ and } d(x_0, y) > f(y)\}$. Then by part (c) of Lemma 3.3, we have that h is ultra-ample. Moreover one can see also that $h < f$. This however contradicts the fact that f is minimal ultra-ample. Thus we conclude that $f(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f(y)\}$ whenever $x \in X$. \square

Thus we have that

$$UP_X = \{f : X \rightarrow [0, \infty) : \forall x, y \in X, d(x, y) \leq \max\{f(x), f(y)\},$$

$$UT_X = \{f \in UP_X : \forall x \in X, f(x) = \sup\{d(x, y) : f(y) < d(x, y)\}\},$$

where the last supremum is understood to be zero in case $f(y) \geq d(x, y)$.

Proposition 3.6. *Let f be an ultra-ample function on an ultrametric space (X, d) such that*

$$f(x) \leq \max\{f(y), d(y, x)\}$$

whenever $x, y \in X$. Furthermore let us suppose that there is a sequence $(a_n)_{n \in \mathbb{N}}$ in X with $\lim_{n \rightarrow \infty} f(a_n) = 0$. Then the function f is minimal ultra-ample.

Proof. Suppose that this is not the case. This means that there is an ultra-ample function h such that $h < f$. Let us assume without loss of generality that there is $x_0 \in X$ such that $h(x_0) < f(x_0)$. Therefore we have that $0 < f(x_0) \leq \max\{f(a_n), d(x_0, a_n)\}$ whenever $n \in \mathbb{N}$ (by our assumption above). Since h is ultra-ample and $h < f$, we will also have that $d(x_0, a_n) \leq \max\{h(x_0), f(a_n)\}$ whenever $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} f(a_n) = 0$, we conclude that $f(x_0) \leq h(x_0)$ - which is a contradiction. Therefore we deduce that the function f is minimal ultra-ample. \square

Proposition 3.7. *(Compare [14, Section 1.3]) Let (X, d) be an ultrametric space. Then UT_X consists of all functions which are "minimal" in UP_X .*

Proof. To prove this Proposition, we prove that there is no $g \in UP_X$ with $g < f$ but $g \neq f$. This is so since on the one hand, $g \leq f \in UT_X$ and $g \in UP_X$ implies

$$\begin{aligned} f(x) &= \sup\{d(x, y) : y \in X \text{ and } d(y, x) > f(y) \geq g(y)\} \\ &= \sup\{d(y, x) : y \in X \text{ and } d(y, x) \geq g(y)\} \\ &\leq g(x). \end{aligned}$$

Thus

$$(3.1) \quad f(x) \leq g(x).$$

Using 3.1 and the condition that $g \leq f$, we have thus shown that $f = g$.

On the other hand, suppose that for some $x \in X$ and $f \in UP_X$, we have that $f(x) > \sup\{d(y, x) : y \in X \text{ and } d(y, x) > f(y)\}$.

For each $x \in X$ and $f \in UP_X$ set $(p_x(f))(z) = f(z)$ if $z \in X \setminus \{x\}$ and $(p_x(f))(x) = \sup\{d(x, y) : y \in X \text{ and } d(x, y) > f(y)\}$.

To show that $p_x(f)$ is ultra-ample, we shall consider the following cases:

Case 1: If $z = x$ and $y = x$, then the result holds since $d(x, x) = 0$.

Case 2: If $z \neq x$ and $y \neq x$, then $(p_x(f))(z) = f(z)$ so that

$$\max\{(p_x(f))(z), (p_x(f))(y)\} = \max\{f(z), f(y)\} \geq d(z, y).$$

Case 3: $z = x$ and $y \neq x$. In this case $(p_x(f))(y) = f(y)$ and $(p_x(f))(z) = \sup\{d(z, y) : y \in X \text{ and } d(z, y) > f(y)\}$ so that

$$\begin{aligned} \max\{(p_x(f))(z), ((p_x(f))(y))\} &= \max \left\{ \sup_{y \in X} \{d(z, y) : d(z, y) > f(y)\}, f(y) \right\} \\ &\geq d(z, y). \end{aligned}$$

Case 4: In a manner similar to case 3, the result can be shown.

Thus $p_x(f)$ is ultra-ample and also satisfies $p_x(f) \leq f$ by the way it was constructed.

Thus by taking $g = p_x(f)$, we can conclude that for any $f \in UP_X$, $g \leq f$. \square

Lemma 3.8. *Let (X, d) be an ultrametric space. If for each $x \in X$ and $f_1, f_2, g_1, g_2 \in UP_X$ we have that $f_1(x) \leq f_2(x)$ and $g_1(x) \leq g_2(x)$, then $N(f_1, f_2) \leq N(g_1, g_2)$.*

The proof of the next theorem follows from Lemma 3.8.

Theorem 3.9. *Let (X, d) be an ultrametric space. If for each $x \in X$ and $f, g \in UP_X$ we define $p_x(f)$ as in the proof of Proposition 3.7, then we have that $N(p_x(f), p_x(g)) \leq N(f, g)$.*

Theorem 3.10. ([10, Theorem 5]) *Let (X, d) be an ultrametric space. Then we have the following.*

- (a) N is an ultrametric on UT_X .
- (b) For $f \in UT_X$ and $z \in X$, $N(f, f_z) = f(z)$.
- (c) The map $\varphi : X \rightarrow UT_X$ defined by $\varphi(z) = f_z$ is an isometric embedding of X into UT_X .

Theorem 3.11. ([10, Theorem 6]) *For any ultrametric space (X, d) , the space (UT_X, N) is an ultramerically injective hull of X .*

4. ULTRAMETRIC TIGHT EXTENSIONS

In this section we generalize some crucial results about tight extensions of metric spaces from [14] to ultrametric spaces.

Lemma 4.1. (compare [14, Theorem 1]) *Let (X, d) be an ultrametric space. Then for any $f, g \in UT_X$, we have that*

$$N(f, g) = \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > f(x_1) \text{ and } d(x_1, x_2) > g(x_2)\}.$$

Proof. Assume on the one hand that for some $f, g \in UT_X$ we have that $N(f, g) > 0$. Then we have by the definition of N that

$$\begin{aligned} N(f, g) &= \inf\{\max\{f(x), g(x)\} : x \in X\} \\ &\leq \inf\{\max\{f(x), g(x)\} : x \in X \text{ and } g(x) > f(x)\} \\ &= \inf\{g(x) : x \in X \text{ and } g(x) > f(x)\} \\ &\leq \sup\{g(x) : x \in X \text{ and } g(x) > f(x)\}. \end{aligned}$$

Thus we have that

$$N(f, g) \leq \sup\{g(x) : x \in X \text{ and } g(x) > f(x)\}.$$

Notice that $\{g(x) : x \in X \text{ and } g(x) > f(x)\} \neq \emptyset$. Indeed, if that was the case, then we will have that $g(x) \leq f(x)$ for every $x \in X$ and then by minimality $g = f$, so $N(f, g) = 0$.

Let $\alpha = \sup\{g(x) : x \in X \text{ and } g(x) > f(x)\}$. This means that for any $\epsilon > 0$ we can find $x_1 \in X$ such that $g(x_1) > f(x_1)$ and $\alpha - \epsilon < g(x_1)$ (by the supremum characterization of α). Define $\epsilon_1 = \min\{g(x_1) - f(x_1), \epsilon\} > 0$. Since $g(x_1) > 0$ (this is so by our assumption above that $g(x_1) > f(x_1) > 0$), we have by Lemma 3.5 that there is $x_2 \in X$ with $d(x_1, x_2) > g(x_2)$ and $g(x_1) - \epsilon_1 < d(x_1, x_2)$. Notice that $\epsilon_1 \leq g(x_1) - f(x_1)$. Therefore we get that $f(x_1) \leq g(x_1) - \epsilon_1 < d(x_1, x_2)$. Thus $\alpha - \epsilon - \epsilon_1 < d(x_1, x_2)$ with $d(x_1, x_2) > g(x_2)$ and $d(x_1, x_2) > f(x_1)$.

Since ϵ was arbitrary, we have shown that

$$\alpha \leq \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > f(x_1) \text{ and } d(x_1, x_2) > g(x_2)\}.$$

Hence we have that

$$N(f, g) \leq \sup_{x_1, x_2 \in X} \{d(x_1, x_2) : d(x_1, x_2) > f(x_1) \text{ and } d(x_1, x_2) > g(x_2)\}$$

which also holds in the remaining case where for $f, g \in UT_X$ we have $N(f, g) = 0$.

We have by the triangle inequality that

$$N(f_{x_1}, f_{x_2}) \leq \max\{N(f_{x_1}, f), N(f, g), N(g, f_{x_2})\} = \max\{f(x_1), N(f, g), g(x_2)\}$$

whenever $f, g \in UT_X$ and $x_1, x_2 \in X$; thus $N(f_{x_1}, f_{x_2}) \leq N(f, g)$ whenever $N(f_{x_1}, f_{x_2}) > f(x_1)$ and $N(f_{x_1}, f_{x_2}) > g(x_2)$. We have therefore shown that for any $f, g \in UT_X$

$$\sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > f(x_1) \text{ and } d(x_1, x_2) > g(x_2)\} \leq N(f, g).$$

This establishes the equality

$$N(f, g) = \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > f(x_1) \text{ and } d(x_1, x_2) > g(x_2)\}$$

whenever $f, g \in UT_X$. \square

Proposition 4.2. (Compare [14, Section 1.9]) *Let (X, d) be an ultrametric space. There exists a retraction map $p : UP_X \rightarrow UT_X$, i.e., a map that satisfies the following conditions*

- (a) $N(p(f), p(g)) \leq N(f, g)$ whenever $f, g \in UP_X$.
- (b) $p(f) \leq f$ whenever $f \in UP_X$.
- (In particular we have that $p(f) = f$ whenever $f \in UT_X$.)

Proof. We will proceed by the use of Zorn's Lemma.

Indeed, let (X, d) be an ultrametric space and let \mathcal{P} be the set of all maps from UP_X to UP_X satisfying conditions (a) and (b) in Proposition 4.2.

Order \mathcal{P} by

$$p \preceq q \Leftrightarrow p(f) \leq q(f) \text{ and } N(p(f), p(g)) \leq N(q(f), q(g))$$

for all $f, g \in UP_X$ and $p, q \in \mathcal{P}$. Then $\mathcal{P} \neq \emptyset$ since the identity map belongs to \mathcal{P} .

We have to check now that \preceq is actually a partial order.

Reflexivity is obvious since every map is equal to itself.

Let now $p, q \in \mathcal{P}$ such that $p \preceq q$ and $q \preceq p$.

$$p \preceq q \Rightarrow p(f) \leq q(f) \text{ and } N(p(f), p(g)) \leq N(q(f), q(g)), \text{ for } f, g \in UP_X$$

$$q \preceq p \Rightarrow q(f) \leq p(f) \text{ and } N(q(f), q(g)) \leq N(p(f), p(g)), \text{ for } f, g \in UP_X$$

$p(f) \leq q(f)$ and $q(f) \leq p(f)$ implies that $p(f) = q(f)$ so that we can conclude that $p = q$.

Also $N(p(f), p(g)) \leq N(q(f), q(g))$ and $N(q(f), q(g)) \leq N(p(f), p(g))$ implies that $p = q$. This shows that \preceq is anti-symmetric.

Suppose now that $p, q, s \in \mathcal{P}$ such that $p \preceq q$ and $q \preceq s$.

$p \preceq q \Rightarrow p(f) \leq q(f)$ and $N(p(f), p(g)) \leq N(q(f), q(g))$, for $f, g \in UP_X$

$q \preceq s \Rightarrow q(f) \leq s(f)$ and $N(q(f), q(g)) \leq N(s(f), s(g))$, for $f, g \in UP_X$

$p(f) \leq q(f)$ and $q(f) \leq s(f)$ implies that $p(f) \leq s(f)$ by transitivity of $[0, \infty)$ as a subset of \mathbb{R} with the usual ordering \leq .

Also $N(p(f), p(g)) \leq N(q(f), q(g))$ and $N(q(f), q(g)) \leq N(s(f), s(g))$ implies that $N(p(f), p(g)) \leq N(s(f), s(g))$. Thus $p \preceq s$. This shows that \preceq is transitive. Therefore (\mathcal{P}, \preceq) is a partially ordered set.

To complete the proof, we have to show that every chain in \mathcal{P} has a lower bound.

Let $\emptyset \neq \mathcal{K} \subset \mathcal{P}$ be a chain and define $s : UP_X \rightarrow UP_X$ by

$$s(f)(x) := \inf_{k \in \mathcal{K}} (k(f))(x)$$

whenever $x \in X$. Since $k(f) \in \mathcal{P}$, we have that $s(f) \in \mathcal{P}$.

Indeed observe that $s(f) \leq k(f) \leq f$, $\forall f \in UP_X$. Thus $s(f) \leq f$ and condition (b) is satisfied.

To check condition (a), we check that $N(s(f), s(g)) \leq N(k(f), k(g)) \leq N(f, g)$. But this follows from Lemma 3.8 since $s(f) \leq k(f)$ and $k(f) \leq f$.

Thus we have that condition (a) is satisfied and since s is a map from UP_X to UP_X , we conclude that $s \in \mathcal{P}$ and s is a lower bound of the chain \mathcal{K} by construction. We therefore appeal to Zorn's lemma to conclude that \mathcal{P} has a minimal element, say m , with respect to the partial order \preceq .

To complete the proof, it suffices to show that $m(f) \in UT_X$ whenever $f \in UP_X$.

For each $x \in X$, we obviously have that $p_x \circ m \in \mathcal{P}$ and $p_x \circ m \preceq m$ (where p_x is as defined in Proposition 3.7). Hence by minimality of m , we have $p_x \circ m = m$. It therefore follows that for each $x \in X$, $p_x(m(f)) = m(f)$ whenever $f \in UP_X$. Thus by the definition of elements in UT_X , we conclude that $m(f) \in UT_X$ whenever $f \in UP_X$. \square

Proposition 4.3. (compare [1, Proposition 3]) *Let (Y, d) be an ultrametric space and $\emptyset \neq X$ be a subspace of (Y, d) . Then there exists an isometric embedding $\tau : UT_X \rightarrow UT_Y$ such that $\tau(f)|_X = f$ whenever $f \in UT_X$.*

Proof. Let us fix $x_0 \in X$ and choose a retraction $p : UP_Y \rightarrow UT_Y$ satisfying the conditions of Proposition 4.2. Also let $s : UT_X \rightarrow UP_Y$ be defined as $s(f) = f'$ where $f'(y) = f(y)$ whenever $y \in X$, and $f'(y) = \max\{f(x_0), d(x_0, y)\}$ whenever $y \in Y \setminus X$.

We shall consider the following cases to prove that f' belongs to UP_Y .

Case 1: $x \in X$ and $y \in X$.

Then $\max\{f'(x), f'(y)\} = \max\{f(x), f(y)\} \geq d(x, y)$.

Case 2: $x \in Y \setminus X$ and $y \in Y \setminus X$.

Then $\max\{f'(x), f'(y)\} = \max\{f(x_0), f(y_0), d(x, x_0), d(x_0, y)\} \geq$
 $\geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Case 3: $x \in X$ and $y \in Y \setminus X$.

Then $\max\{f'(x), f'(y)\} = \max\{f(x), f(x_0), d(x_0, y)\} \geq$
 $\geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Case 4: $x \in Y \setminus X$ and $y \in X$.

Then $\max\{f'(x), f'(y)\} = \max\{f(x_0), f(y), d(x, x_0)\} \geq$
 $\geq \max\{d(x, x_0), d(x_0, y)\} \geq d(x, y)$.

Thus $f' \in UP_Y$.

Define the map $\tau = p \circ s$. Then $\tau(f)|_X = p(f')|_X = f$ whenever $f \in UT_X$ since $p(f') \leq f'$. Thus $p(f')|_X \leq f'|_X = f$, and f is minimal on X .

Moreover for any $f, g \in UT_X$, we have

$$\begin{aligned} N(f, g) &= N(\tau(f)|_X, \tau(g)|_X) \\ &\leq N(\tau(f), \tau(g)) \\ &= N(p(f'), p(g')) \\ &\leq N(f', g') \\ &\leq N(f, g). \end{aligned}$$

Hence we have that τ is an isometric map. \square

Definition 4.4. (compare [12, Remark 7]) Let X be a subspace of an ultrametric space (Y, d_Y) . Then (Y, d_Y) is called a *um-tight* extension of X if for any ultrametric ρ on Y that satisfies $\rho \leq d_Y$ and agrees with d_Y on $X \times X$, we have that $\rho = d_Y$.

Remark 4.5. For any ultrametric *um-tight* extension Y_1 of X , any ultrametric extension (Y_2, d) of X and any nonexpansive map $\varphi : Y_1 \rightarrow Y_2$ satisfying $\varphi(x) = x$ whenever $x \in X$ must necessarily be an isometric map.

Indeed if that is not the case then the ultrametric $\rho : Y_1 \times Y_1 \rightarrow [0, \infty)$ defined by $(x, y) \mapsto \rho(x, y) = d(\varphi(x), \varphi(y))$ would contradict the *um-tightness* of the extension Y_1 of X .

As was shown above, the map $e_X : (X, d) \rightarrow (UT_X, N)$ from an ultrametric space (X, d) to its ultra-metrically injective hull (UT_X, N) defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric embedding.

We shall proceed now with the help of Lemma 4.1 to show that the embedding is *um-tight*, that is, UT_X is a *um-tight* extension of $e_X(X)$.

Proposition 4.6. *Let (X, d) be an ultrametric space and $e_X : X \rightarrow UT_X$ be as defined above. Then UT_X is a *um-tight* extension of $e_X(X)$.*

Proof. Let ρ be an ultrametric on UT_X such that $\rho \leq N$ and $\rho(f_x, f_y) = N(f_x, f_y)$ whenever $x, y \in X$. By Lemma 4.1 and the fact that $\rho \leq N$, for any $f, g \in UT_X$, we have

$$\begin{aligned} N(f, g) &= \sup_{x_1, x_2 \in X} \{N(f_{x_1}, f_{x_2}) : N(f_{x_1}, f_{x_2}) > N(f_{x_1}, f), N(g, f_{x_2})\} \\ &\leq \sup_{x_1, x_2 \in X} \{\rho(f_{x_1}, f_{x_2}) : \rho(f_{x_1}, f_{x_2}) > \rho(f_{x_1}, f), \rho(g, f_{x_2})\} \\ &\leq \rho(f, g) \text{ since } \rho(f_{x_1}, f_{x_2}) \leq \max\{\rho(f_{x_1}, f), \rho(f, g), \rho(g, f_{x_2})\}. \end{aligned}$$

Thus $\rho = N$. \square

Proposition 4.7. *(compare [14, Section 1.13]) Let (Y, d) be an ultrametric *um-tight* extension of X . Then the restriction map defined by $f \mapsto f|_X$ whenever $f \in UT_Y$ is a bijective isometric map $UT_Y \rightarrow UT_X$.*

Proof. Let us choose a retraction map $p : UP_X \rightarrow UT_X$ that satisfies the conditions of Proposition 4.2 and let $\varphi : UT_Y \rightarrow UT_X : f \mapsto p(f|_X)$ denote the composition of the restriction map with the retraction map p . It is easy to check that φ is nonexpansive. Thus by Remark 4.5, φ must be an isometry, because UT_Y is a *um-tight* extension of X (this is so since UT_Y is a *um-tight* extension of Y and Y is a *um-tight* extension of X).

We can find an isometric embedding $\tau : UT_X \rightarrow UT_Y$ such that $\tau(f)|_X = f$ for every $f \in UT_X$ (compare Proposition 4.3). We therefore have

$$\varphi(\tau(f)) = p(\tau(f)|_X) = p(f) = f \text{ for every } f \in UT_X.$$

This implies that φ is surjective. Injectivity of φ is clear. Thus φ is bijective. In this case, φ has to be the inverse of τ and hence for any $f \in UT_Y$, we have $f|_X = \tau(\varphi(f))|_X = \varphi(f) \in UT_X$, that is the map

$$UT_Y \rightarrow UP_X : f \mapsto f|_X$$

maps UT_Y onto UT_X , without it being composed with p . Hence for any ultrametric *um-tight* extension Y of X , the map

$$UT_Y \rightarrow UT_X : f \mapsto f|_X$$

is a bijective isometry between UT_X and UT_Y . \square

Theorem 4.8. *(compare [1, Proposition 5]) Let X be a subspace of the ultrametric space (Y, d) . Then the following are equivalent:*

(a) *Y is an ultrametric *um-tight* extension of X .*

(b) $d(y_1, y_2) = \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(x_2, y_2)\}$ whenever $y_1, y_2 \in Y$.

(c) $f_y|_X(x) = d(x, y)$, $x \in X$ is minimal on X whenever $y \in Y$ and the map $(Y, d) \rightarrow (UT_X, N)$ defined by $y \mapsto f_y|_X$ is an isometric embedding.

Proof. (a) \Rightarrow (b)

Let Y be an ultrametric *um*-tight extension of X . By Proposition 4.7, the restriction map $UT_Y \rightarrow UT_X$ is a bijective isometry between UT_Y and UT_X . Thus the extension $Y \subseteq UT_Y$ satisfies condition (b), since UT_X satisfies it by Lemma 4.1.

(b) \Rightarrow (c)

Let $x_1, x_2 \in X$ and $y_1 \in Y$. Then we have that $d(x_1, x_2) \leq \max\{d(x_1, y_1), d(y_1, x_2)\}$. Thus by condition (b) we have that $d(x_1, x_2) \leq d(y_1, x_2)$. Consequently for $y_1, y_2 \in Y$ we have by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(y_2, x_2)\} \\ &\leq \sup\{d(y_1, x_2) : x_2 \in X, d(y_1, x_2) > d(x_2, y_2)\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Similarly we have that $d(x_1, x_2) \leq \max\{d(x_1, y_2), d(y_2, x_2)\}$ whenever $x_1, x_2 \in X$ and $y_2 \in Y$ so that by condition (b) we get $d(x_1, x_2) \leq d(x_1, y_2)$. Thus for $y_1, y_2 \in Y$ we see by (b) that

$$\begin{aligned} d(y_1, y_2) &= \sup\{d(x_1, x_2) : x_1, x_2 \in X, d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(x_2, y_2)\} \\ &\leq \sup\{d(x_1, y_2) : x_1 \in X, d(x_1, y_2) > d(x_1, y_1)\} \\ &\leq d(y_1, y_2). \end{aligned}$$

Thus we conclude that $d(y_1, y_2) = N(f_{y_1}|_X, f_{y_2}|_X)$.

As we have above, for any $y_1, y_2 \in Y$

$$(4.1) \quad d(y_1, y_2) = \sup\{d(y_1, x_2) : x_2 \in X, d(y_1, x_2) > d(x_2, y_2)\}$$

and

$$(4.2) \quad d(y_1, y_2) = \sup\{d(x_1, y_2) : x_1 \in X, d(x_1, y_2) > d(x_1, y_1)\}.$$

Notice that if we substitute $x_1 \in X$ for y_1 and $x_2 \in X$ for y_2 , respectively, we obtain the following equations

$$(4.3) \quad f_{y_1}(x_2) = d(y_1, x_2) = \sup\{d(x_1, x_2) : x_1 \in X \text{ and } d(x_1, x_2) > d(x_2, y_2)\}$$

whenever $y_1 \in Y, x_2 \in X$ and

$$(4.4) \quad f_{y_2}(x_1) = d(x_1, y_2) = \sup\{d(x_1, x_2) : x_2 \in X \text{ and } d(x_1, x_2) > d(x_2, y_2)\}$$

whenever $y_2 \in Y, x_1 \in X$. We have therefore that the restriction $f_y|_X$ is minimal on X whenever $y \in Y$.

(c) \Rightarrow (a)

Let ρ be an ultrametric on Y such that $\rho(y_1, y_2) \leq d(y_1, y_2)$ whenever $y_1, y_2 \in Y$ and $\rho(x_1, x_2) = d(x_1, x_2)$ whenever $x_1, x_2 \in X$. Then according to part (c) and the fact that $f_y|_X$ is minimal whenever $y \in X$, we have

$$\begin{aligned} d(y_1, y_2) &= N(f_{y_1}|_X, f_{y_2}|_X) \\ &= \sup\{d(y_1, x) : x \in X, d(y_1, x) > d(x, y_2)\} \text{ by Equation (4.1)} \\ &= \sup\{d(x, y_2) : x \in X, d(x, y_2) > d(x, y_1)\} \text{ by Equation (4.2)}. \end{aligned}$$

By substituting

$$d(x_1, y_2) = \sup\{d(x_1, x_2) : x_2 \in X \text{ and } d(x_1, x_2) > d(x_2, y_2)\}$$

from Equation (4.4) into the formula

$$d(y_1, y_2) = \sup\{d(x_1, y_2) : x_1 \in X \text{ and } d(x_1, y_2) > d(x_1, y_1)\}$$

from Equation (4.2) we obtain

$$\begin{aligned} d(y_1, y_2) &= \\ &= \sup\{d(x_1, y_2) : x_1 \in X \text{ and } d(x_1, y_2) > d(x_1, y_1)\} \\ &= \sup\{d(x_1, x_2) : x_1, x_2 \in X \text{ and } d(x_1, x_2) > d(x_1, y_1), d(x_1, x_2) > d(x_2, y_2)\} \\ &= \sup\{\rho(x_1, x_2) : x_1, x_2 \in X \text{ and } \rho(x_1, x_2) > \rho(x_1, y_1), \rho(x_1, x_2) > \rho(x_2, y_2)\} \\ &\leq \rho(y_1, y_2) \end{aligned}$$

whenever $y_1, y_2 \in Y$. The last inequality holds by the light of the inequality

$$\rho(x_1, x_2) \leq \max\{\rho(x_1, y_1), \rho(y_1, y_2), \rho(x_2, y_2)\}$$

and the fact that $\rho(x_1, x_2) > \rho(x_1, y_1)$ and $\rho(x_1, x_2) > \rho(x_2, y_2)$. Thus, we have that $\rho(y_1, y_2) = d(y_1, y_2)$ whenever $y_1, y_2 \in Y$ and hence (a) follows. \square

Remark 4.9. We see from Theorem 4.8 that there is only one isometric embedding $\varphi : Y \rightarrow UT_X$ satisfying $\varphi(x) = f_x$ whenever $x \in X$, since for such an embedding we have

$$f_y|_X(x) = d(x, y) = N(\varphi(x), \varphi(y)) = N(f_x, \varphi(y)) = (\varphi(y))(x).$$

Therefore $f_y|_X = \varphi(y)$.

Thus one sees easily that the *um*-tight extension Y of X can be understood as a subspace of the extension UT_X of X . Hence UT_X is maximal among the ultrametric *um*-tight extensions of X .

REFERENCES

- [1] C.A. Agyingi, P. Haihambo and H.-P.A. Künzi, *Tight extensions of T_0 -quasi-metric spaces*, Ontos-Verlag: Logic, Computations, Hierarchies, (2014), 9–22.

- [2] N. Aronszajn and P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific Journal of Mathematics **6** (1956), 405–439.
- [3] R. Rammal, G. Toulouse, and M.A. Virasoro, *Ultrametricity for physicists*, Rev. Modern Phys. **58** (1986), 765–788.
- [4] J. H. Wells and L. R. Williams, *Extensions and embeddings in analysis*, Springer-Verlag, Berlin and New York, 1975.
- [5] G. P. Murphy, *A metric basis characterization of euclidean space*, Pacific J. Math. **60** (1975), 159–163.
- [6] J.R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39**:65–76, 1964.
- [7] J. R. Isbell, *s admits an injective space*, Proc. Amer. Math. Soc. **28** (1971), 259–261.
- [8] M.A. Khamsi and W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point theory*, John Wiley, New York, 2001.
- [9] M. M. Bonsangue, F. Van Breugel and J. J. M. M. Rutten, *Generalized ultrametric spaces: completion, topology, and powerdomains via the Yoneda embedding*, (1995), CWIreports/AP/CS-R9560.
- [10] J.M. Bayod, and J. Martínez-Maurica, *Ultrametrically injective spaces*, Proceedings of the American Mathematics Society, **101**, (1987) 571–576.
- [11] H.-P.A. Künzi and O.O. Otafudu, *The ultra-quasi-metrically injective hull of a T_0 -ultra-quasi-metric space*, Applied Categorical Structures (2012).
- [12] E. Kemajou, H.-P.A. Künzi and O.O. Otafudu, *The Isbell-hull of a di-space*, Topology and its Applications, **159** (2012), 2463–2475.
- [13] H.-P.A. Künzi and Manuel Sanchis, *The Katětov construction modified for a T_0 -quasi-metric space*, Topology and its Applications, **159** (2012), 711–720.
- [14] A.W.M. Dress, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Advances in Mathematics **53**, 321–402, 1984.

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