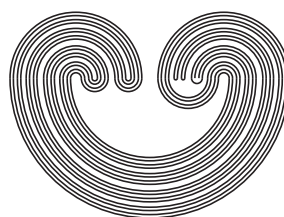


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ON THE SUBMETRIZABILITY NUMBER AND i -WEIGHT OF QUASI-UNIFORM SPACES AND PARATOPOLOGICAL GROUPS

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ABSTRACT. We derive many upper bounds on the submetrizability number and i -weight of paratopological groups and topological monoids with open shifts. In particular, we prove that each first countable Hausdorff paratopological group is submetrizable thus answering a problem of Arhangel'skii posed in 2002. Also we construct an example of a zero-dimensional (and hence regular) Hausdorff paratopological abelian group G with countable pseudocharacter which is not submetrizable. In fact, all results on the i -weight and submetrizability are derived from more general results concerning normally quasi-uniformizable and bi-quasi-uniformizable spaces.

INTRODUCTION

This paper was motivated by the following problem of Arhangel'skii [1, 3.11] (also repeated by Tkachenko in his survey [26, 2.1]): *Does every first countable Hausdorff paratopological group admit a weaker metrizable topology?* A surprisingly simple answer to this problem was given by the authors in [4]. We just observed that each Hausdorff paratopological group G carries a natural uniformity generated by the base consisting of entourages $\{(x, y) \in G \times G : y \in UxU^{-1} \cap U^{-1}xU\}$ where U runs over open neighborhoods of the unit e in G . In [4] this uniformity was called the *quasi-Roelcke uniformity* on G and denoted by \mathcal{Q} . If G is first-countable, then the quasi-Roelcke uniformity \mathcal{Q} is metrizable, which implies that the space G is submetrizable. Moreover, if the quasi-Roelcke uniformity \mathcal{Q} is

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ω -bounded, then the topology generated by the uniformity \mathcal{Q} is metrizable and separable, which implies that G has countable i -weight, i.e., admits a continuous injective map onto a metrizable separable space.

In fact, for the submetrizability of G it suffices to require the countability of the pseudocharacter $\psi(\mathcal{Q})$ of \mathcal{Q} , i.e., the existence of a countable subfamily $\mathcal{U} \subset \mathcal{Q}$ such that $\bigcap \mathcal{U} = \Delta_X$. So, the aim of the paper is to detect paratopological groups G whose quasi-Roelcke uniformity \mathcal{Q} has countable pseudocharacter. For this we shall find some upper bounds on the pseudocharacter $\psi(\mathcal{Q})$. These bounds will give us upper bounds on the submetrizability number $sm(G)$ and the i -weight $iw(G)$ of a paratopological group G . In fact, the obtained upper bounds on $sm(G)$ and $iw(G)$ have uniform nature and depends on the properties of the two canonical quasi-uniformities \mathcal{L} and \mathcal{R} on G called the left and right quasi-uniformities of G . These quasi-uniformities are studied in Sections 5 and 6. In Sections 3 and 4 we study properties of topological spaces whose topology is generated by two quasi-uniformities which are compatible in some sense (more precisely, are \pm -subcommuting or normally \pm -subcommuting). In Section 4 we prove that any two normally \pm -subcommuting quasi-uniformities are normal in the sense of [4]. This motivates the study of topological spaces whose topology is generated by a normal quasi-uniformity. For such spaces we obtain some upper bounds on the i -weight, which is done in Section 4. Section 1 has preliminary character. It contains the necessary information of topological spaces, quasi-uniform spaces, and their cardinal characteristics. In Section 7 we present two counterexamples to some natural conjectures concerning submetrizable paratopological groups.

1. PRELIMINARIES

In this section we collect known information on topological spaces, quasi-uniformities, and their cardinal characteristics. For a set X by $|X|$ we denote its cardinality. By ω we denote the set of all finite ordinals and by $\mathbb{N} = \omega \setminus \{0\}$ the set of natural numbers.

For a cardinal κ by $\log(\kappa)$ we denote the smallest cardinal λ such that $2^\lambda \geq \kappa$.

1.1. Topological spaces and their cardinal characteristics. For a subset A of a topological space X by \overline{A} , A° and \overline{A}° we denote the closure, interior and interior of the closure of the set A in X , respectively.

A family \mathcal{N} of subsets of a topological space X is called a *network* of the topology of X if each open set $U \subset X$ can be written as the union $\bigcup \mathcal{U}$ of some subfamily $\mathcal{U} \subset \mathcal{N}$. If each set $N \in \mathcal{N}$ is open in X , then \mathcal{N} is a *base* of the topology of X .

A subset D of a topological space X is called *strongly discrete* if each point $x \in D$ has a neighborhood $U_x \subset X$ such that the family $(U_x)_{x \in D}$ is *discrete* in the sense that each point $z \in X$ has a neighborhood that meets at most one set U_x , $x \in D$. It is easy to see that each strongly discrete subset of (a T_1 -space) X is discrete (and closed) in X . A topological space X is called (*strongly*) σ -*discrete* if X can be written as the countable union $X = \bigcup_{n \in \omega} X_n$ of (strongly) discrete subsets of X .

A topological space X is called

- *Hausdorff* if any two distinct points $x, y \in X$ have disjoint open neighborhoods $O_x \ni x$ and $O_y \ni y$;
- *collectively Hausdorff* if each closed discrete subset of X is strongly discrete in X ;
- *functionally Hausdorff* if for any two distinct points $x, y \in X$ there is a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x) \neq f(y)$;
- *regular* if for any point $x \in X$ and a neighborhood $O_x \subset X$ there is a neighborhood $V_x \subset X$ of x such that $\bar{V}_x \subset O_x$;
- *completely regular* if for any point $x \in X$ and a neighborhood $O_x \subset X$ there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f^{-1}([0, 1)) \subset O_x$;
- *quasi-regular* if each non-empty open set $U \subset X$ contains the closure \bar{V} of another non-empty open set $V \subset X$;
- *submetrizable* if X admits a continuous metric (or equivalently, admits a continuous injective map into a metrizable space).

It is clear that each submetrizable space is functionally Hausdorff.

In Section 7 we shall need the following property of strongly σ -discrete spaces.

Proposition 1.1. *Each strongly σ -discrete Tychonoff space X is zero-dimensional and submetrizable. Moreover, X admits an injective continuous map into the Cantor cube $\{0, 1\}^\kappa$ of weight $\kappa = \log(|X|)$.*

Proof. The proposition trivially holds if X is discrete. So, we assume that X is not discrete and hence infinite. Write X as the countable union $X = \bigcup_{n \in \omega} X_n$ of pairwise disjoint strongly discrete non-empty subsets X_n of X . Let βX be the Stone-Ćech compactification of X . Using the strong discreteness of each X_n , we can extend each continuous bounded function $f : X_n \rightarrow \mathbb{R}$ to a continuous bounded function on X . This implies that the closure \bar{X}_n of X_n in βX is homeomorphic to the Stone-Ćech compactification βX_n of the discrete space X_n and hence has covering dimension $\dim(\beta X_n) = 0$ (see [10, 3.6.7 and 7.1.17]). By the Countable Sum Theorem [11, 3.1.8] for covering dimension in normal spaces, the σ -compact (and hence normal) space $Z = \bigcup_{n \in \omega} \bar{X}_n$ has covering dimension $\dim(Z) = 0$, which implies that its subspace $X = \bigcup_{n \in \omega} X_n$ is zero-dimensional.

Now we prove that X is submetrizable. For every $n \in \omega$ and every $x \in X_n$ we can choose a closed-and-open neighborhood $U_x \subset X$ of x such that $U_x \cap \bigcup_{k < n} X_k = \emptyset$ and the indexed family $(U_x)_{x \in X_n}$ is discrete in X . Then the union $\bigcup_{x \in X_n} U_x$ is a closed-and-open subset in X and the function $d_n : X \times X \rightarrow \{0, 1\}$ defined by

$$d_n(x, y) = \begin{cases} 0, & \text{if } x, y \in U_x \text{ for some } x \in X_n \text{ or } x, y \notin \bigcup_{z \in X_n} U_z, \\ 1, & \text{otherwise,} \end{cases}$$

is a continuous pseudometric on X . Consequently, the function $d = \max_{n \in \omega} \frac{1}{2^n} d_n$ is a continuous metric on X , which implies that X is submetrizable.

It follows that the space X admits a continuous injective map into the countable product $\prod_{n \in \omega} D_n$ of discrete spaces D_n of cardinality $|D_n| = 1 + |X_n| \leq |X|$. By definition of the cardinal $\kappa = \log(|X|)$, every discrete space D_n , $n \in \omega$, admits an injective (and necessarily continuous) map into the Cantor cube $\{0, 1\}^\kappa$. Then $\prod_{n \in \omega} D_n$ and hence X also admits a continuous injective map into $\{0, 1\}^\kappa$. \square

For a cover \mathcal{U} of a set X and a subset $A \subset X$ we put $\mathcal{S}t^0(A; \mathcal{U}) = A$ and $\mathcal{S}t^{n+1}(A; \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap \mathcal{S}t^n(A; \mathcal{U}) \neq \emptyset\}$ for $n \geq 0$.

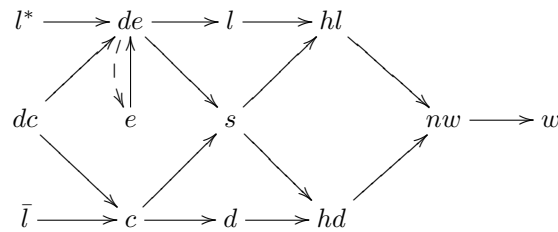
1.2. Cardinal characteristics of topological spaces, I. For a topological space X let

- $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network of the topology of } X\}$ be the *network weight* of X ;
- $d(X) = \min\{|A| : A \subset X, \overline{A} = X\}$ be the *density* of X ;
- $hd(X) = \sup\{d(Y) : Y \subset X\}$ the *hereditary density* of X ;
- $s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\}$ be the *spread* of X ;
- $e(X) = \sup\{|D| : D \text{ is a closed discrete subspace of } X\}$ be the *extent* of X ;
- $c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a disjoint family of non-empty open sets in } X\}$ be the *cellularity* of X ;
- $de(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a discrete family of non-empty sets in } X\}$ be the *discrete extent* of X ;
- $dc(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a discrete family of non-empty open sets in } X\}$ be the *discrete cellularity* of X ;
- $l(X)$, the *Lindelöf number* of X , be the smallest cardinal κ such that each open cover \mathcal{U} of X has a subcover $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \leq \kappa$;

- $\bar{l}(X)$, the *weak Lindelöf number* of X , be the smallest cardinal κ such that each open cover \mathcal{U} of X contains a subcollection $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \leq \kappa$ with dense union $\bigcup \mathcal{V}$ in X ;
- $l^*(X)$, the *weak extent* of X , be the smallest cardinal κ such that for each open cover \mathcal{U} of X there is a subset $A \subset X$ of cardinality $|A| \leq \kappa$ such that $X = St(A; \mathcal{U})$.

The cardinal characteristics nw, d, s, e, c, l are well-known in General Topology (see [10], [14]) whereas \bar{l}, l^* are relatively new and notations for these cardinal characteristics are not fixed yet. For example, the weak Lindelöf number \bar{l} often is denoted by wL , but in [14, §3] it is denoted by wc and called the *weak covering number*. According to [23], the weak extent l^* can be called the *star cardinality*. Spaces with countable weak extent are called *star-Lindelöf* in [22] and *strongly star-Lindelöf* in [9]. Observe that $e \leq de$ and $e(X) = de(X)$ for any T_1 -space X .

The relations between the above cardinal invariants are described in the following version of Hodel's diagram [14]. In this diagram an arrow $f \rightarrow g$ (resp $f \dashrightarrow g$) indicates that $f(X) \leq g(X)$ for any (T_1) -space X .



In fact, the cardinal characteristics d, l, \bar{l}, l^* are initial representatives of the hierarchy of cardinal characteristics l^{*n} and \bar{l}^{*n} , $n \in \frac{1}{2}\mathbb{N}$, describing star-covering properties of topological spaces (see the survey paper [22] of Matveev for more information on this subject).

For a topological space X and an integer number $n \geq 0$ let

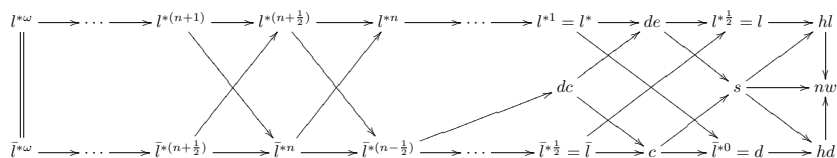
- $l^{*n}(X)$ be the smallest cardinal κ such that for every open cover \mathcal{U} of X there is a subset $A \subset X$ of cardinality $|A| \leq \kappa$ such that $St^n(A; \mathcal{U}) = X$;
- $\bar{l}^{*n}(X)$ be the smallest cardinal κ such that for every open cover \mathcal{U} of X there is a subset $A \subset X$ of cardinality $|A| \leq \kappa$ such that $St^n(A; \mathcal{U})$ is dense in X ;
- $l^{*n\frac{1}{2}}(X)$ be the smallest cardinal κ such that every open cover \mathcal{U} of X contains a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \leq \kappa$ such that $St^n(\bigcup \mathcal{V}; \mathcal{U}) = X$;

- $\bar{l}^{*n\frac{1}{2}}(X)$ be the smallest cardinal κ such that every open cover \mathcal{U} of X contains a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| \leq \kappa$ such that $St^n(\cup \mathcal{V}; \mathcal{U})$ is dense in X ;
- $l^{*\omega}(X) = \min_{n \in \omega} l^{*n}(X)$ and $\bar{l}^{*\omega} = \min_{n \in \omega} \bar{l}^{*n}(X)$.

Observe that $l^{*0} = |\cdot|$, $\bar{l}^{*0} = d$, $l^{*\frac{1}{2}} = l$, $\bar{l}^{*\frac{1}{2}} = \bar{l}$, and $l^{*1} = l^*$.

In [7] the cardinal characteristics l^{*n} and $l^{*n\frac{1}{2}}$ are denoted by st_n - l and $st_{n\frac{1}{2}}$ - l , respectively. In [9] spaces X with countable $l^{*n\frac{1}{2}}(X)$ and $l^{*n}(X)$ are called n -star-Lindelöf and *strongly n -star Lindelöf*, respectively.

The following diagram describes provable inequalities between cardinal characteristics l^{*n} , \bar{l}^{*n} , $l^{*n\frac{1}{2}}$, and $\bar{l}^{*n\frac{1}{2}}$ for $n \in \mathbb{N}$. For two cardinal characteristics f, g an arrow $f \rightarrow g$ indicates that $f(X) \leq g(X)$ for any topological space X .



The unique non-trivial inequalities $l^{*1} \leq de$ and $\bar{l}^{*1\frac{1}{2}} \leq dc$ in this diagram follow from the next proposition whose proof can be found in [5].

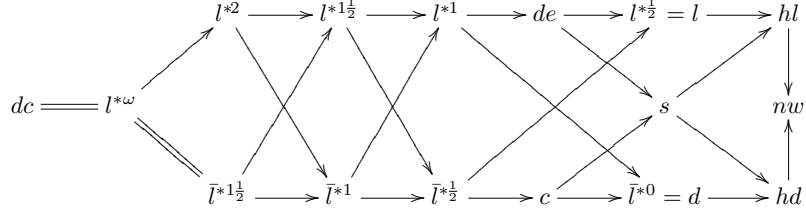
Proposition 1.2. *Any topological space X has $l^{*1}(X) \leq de(X)$ and $\bar{l}^{*1\frac{1}{2}}(X) \leq dc(X)$.*

For quasi-regular spaces many star-covering properties are equivalent. Let us recall that a topological space X is called *quasi-regular* if each non-empty open set $U \subset X$ contains the closure \bar{V} of another non-empty open set V in X . The following proposition was proved in [5] (and for regular spaces in [9]).

Proposition 1.3. *Let X be a quasi-regular space. Then*

- (1) $dc(X) = \bar{l}^{*1\frac{1}{2}}(X) = l^{*\omega}(X)$.
- (2) *If X is normal, then $dc(X) = \bar{l}^{*1}(X)$.*
- (3) *If X is perfectly normal, then $dc(X) = c(X) = \bar{l}^{*\frac{1}{2}}(X)$.*
- (4) *If X is collectively Hausdorff, then $dc(X) = de(X) = l^{*1}(X)$.*
- (5) *If X is paracompact, then $dc(X) = l(X)$.*
- (6) *If X is perfectly paracompact, then $dc(X) = hl(X)$.*

Proposition 1.3 implies that for quasi-regular spaces the diagram describing the relations between the cardinal characteristics simplifies to the following form.



Next, we consider some local cardinal characteristics of topological spaces. Let X be a topological space, x be a point of X , and \mathcal{N}_x be the family of all open neighborhoods of x in X .

- The *character* $\chi_x(X)$ of X at x is the smallest cardinality of a neighborhood base at x .
- The *pseudocharacter* $\psi_x(X)$ of X at x is the smallest cardinality of a subfamily $\mathcal{U} \subset \mathcal{N}_x$ such that $\bigcap \mathcal{U} = \bigcap \mathcal{N}_x$.
- The *closed pseudocharacter* $\bar{\psi}_x(X)$ of X at x is the smallest cardinality of a subfamily $\mathcal{U} \subset \mathcal{N}_x$ such that $\bigcap_{U \in \mathcal{U}} \bar{U} = \bigcap_{V \in \mathcal{N}_x} \bar{V}$.

It is easy to see that for any point x of a Hausdorff topological space X we get

$$\psi_x(X) \leq \bar{\psi}_x(X) \leq \chi_x(X).$$

The cardinals

$$\chi(X) = \sup_{x \in X} \chi_x(X), \quad \psi(X) = \sup_{x \in X} \psi_x(X), \quad \text{and} \quad \bar{\psi}(X) = \sup_{x \in X} \bar{\psi}_x(X)$$

are called the *character*, the *pseudocharacter*, and the *closed pseudocharacter* of X , respectively. It follows that

$$\psi(X) \leq \bar{\psi}(X) \leq \chi(X)$$

for any Hausdorff topological space X .

The (closed) pseudocharacter is upper bounded by the (closed) diagonal number defined as follows. Let X be a Hausdorff topological space. By $\Delta_X = \{(x, y) \in X \times X : x = y\}$ we denote the *diagonal* of the square $X \times X$.

- The *diagonal number* $\Delta(X)$ of X is the smallest cardinality of a family \mathcal{U} of open subsets of $X \times X$ such that $\bigcap \mathcal{U} = \Delta_X$.
- The *closed diagonal number* $\bar{\Delta}(X)$ of X is the smallest cardinality of a family \mathcal{U} of open subsets of $X \times X$ such that $\bigcap_{U \in \mathcal{U}} \bar{U} = \Delta_X$.

It is easy to see that $\psi(X) \leq \Delta(X) \leq \bar{\Delta}(X)$ and $\bar{\psi}(X) \leq \bar{\Delta}(X)$ for any Hausdorff space X .

Following [13, §2.1] we say that a space X has (*regular*) G_δ -*diagonal* if $\Delta(X) \leq \omega$ (resp. $\bar{\Delta}(X) \leq \omega$).

The (closed) diagonal number of a functionally Hausdorff space X is upper bounded by

- the *submetrizability number* $sm(X)$ of X , defined as the smallest number of continuous pseudometrics which separate points of X , and
- the *i -weight* $iw(X)$ of X , defined as the smallest number of continuous real-valued functions that separate points of X .

The following diagram describes relations between these cardinal characteristics. In this diagram for two cardinal characteristics f, g an arrow $f \rightarrow g$ indicates that $f(X) \leq g(X)$ for any functionally Hausdorff topological space X .

$$\begin{array}{ccccccc}
 & \psi & \longrightarrow & \Delta & & & \\
 & \downarrow & & \downarrow & & & \\
 \chi & \longleftarrow & \overline{\psi} & \longrightarrow & \overline{\Delta} & \longrightarrow & sm \longrightarrow iw \longrightarrow sm \cdot \log dc
 \end{array}$$

The unique non-trivial inequality $iw \leq sm \cdot \log dc$ in this diagram is proved in the following proposition.

Proposition 1.4. *Each infinite functionally Hausdorff space X has*

$$iw(X) \cdot \omega = sm(X) \cdot \log(dc(X)) \quad \text{and} \quad |X| \leq dc(X)^{\omega \cdot sm(X)} \leq 2^{\omega \cdot iw(X)}.$$

Proof. The inequality $sm(X) \cdot \log(dc(X)) \leq iw(X) \cdot \omega$ follows from the inequalities $sm(X) \leq iw(X)$ and $dc(X) \leq |X| \leq |[0, 1]^{iw(X)}| = 2^{iw(X) \cdot \omega}$, the latter of which implies $\log(dc(X)) \leq \log(2^{iw(X) \cdot \omega}) \leq iw(X) \cdot \omega$.

Now we prove the inequalities $iw(X) \cdot \omega \leq sm(X) \cdot \log(dc(X))$ and $|X| \leq dc(X)^{\omega \cdot sm(X)}$. The definition of the submetrizability number implies that X admits a continuous injective map $f : X \rightarrow \prod_{\alpha \in sm(X)} M_\alpha$ into the Tychonoff product of $sm(X)$ many metric spaces M_α . We lose no generality assuming that each metric space M_α is a continuous image of X and hence $d(M_\alpha) = dc(M_\alpha) \leq dc(X)$ and $|M_\alpha| \leq d(M_\alpha)^\omega$. Then

$$|X| \leq \prod_{\alpha \in sm(X)} |M_\alpha| \leq \prod_{\alpha \in sm(X)} d(M_\alpha)^\omega \leq \prod_{\alpha \in sm(X)} dc(X)^\omega = dc(X)^{\omega \cdot sm(X)}.$$

By [10, 4.4.9], for every $\alpha \in sm(X)$ the metric space M_α admits a topological embedding into the countable power H_κ^ω of the hedgehog $H_\kappa = \{(x_i)_{i \in \kappa} \in [0, 1]^\kappa : |\{i \in \kappa : x_i \neq 0\}| \leq 1\}$ with $\kappa = dc(X) \geq d(M_\alpha)$ many spines. The hedgehog H_κ can be thought as a cone over a discrete space D of cardinality κ . The discrete space D admits an injective continuous map into the Tychonoff cube $[0, 1]^{\log(\kappa)}$. Consequently, H_κ admits an injective continuous map into the cone over the Tychonoff cube $[0, 1]^{\log(\kappa)}$, which implies that $iw(H_\kappa) \leq \log(\kappa) = \log(dc(X))$ and $iw(H_\kappa^\omega) \leq \log(dc(X)) \cdot \omega = \log(dc(X))$. Then $iw(X) \leq sm(X) \cdot iw(H_\kappa^\omega) \leq sm(X) \cdot \log(dc(X))$. This completes the proof of the equality $iw(X) \cdot \omega = sm(X) \cdot \log(dc(X))$.

To complete the proof of the proposition, observe that

$$|X| \leq dc(X)^{\omega \cdot sm(X)} \leq (2^{\log dc(X)})^{\omega \cdot sm(X)} = 2^{\log(dc(X)) \cdot \omega \cdot sm(X)} = 2^{\omega \cdot iw(X)}.$$

□

1.3. Pre-uniform spaces and their cardinal characteristics. By an *entourage* on a set X we understand any subset $U \subset X \times X$ containing the diagonal $\Delta_X = \{(x, y) \in X \times X : x = y\}$ of $X \times X$. For an entourage U on X , point $x \in X$ and subset $A \subset X$ let $B(x; U) = \{y \in X : (x, y) \in U\}$ be the U -ball centered at x , and $B(A; U) = \bigcup_{a \in A} B(a; U)$ be the U -neighborhood of A in X .

Now we define some operations on entourages. For two entourages U, V on X let

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$$

be the *inverse* entourage and

$$UV = \{(x, z) \in X \times X : \exists y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}$$

be the *composition* of U and V . It is easy to see that $(UV)^{-1} = V^{-1}U^{-1}$. For every entourage U on X define its powers U^n , $n \in \mathbb{Z}$, by the formula: $U^0 = \Delta_X$ and $U^{n+1} = U^n U$, $U^{-n-1} = U^{-n} U^{-1}$ for $n \in \omega$. Define also the *alternating powers* $U^{\pm n}$ and $U^{\mp n}$ of U by the recursive formulas: $U^{\pm 0} = U^{\mp 0} = \Delta_X$, and $U^{\pm(n+1)} = U U^{\mp n}$, $U^{\mp(n+1)} = U^{-1} U^{\pm n}$ for $n \geq 0$. If U is an entourage on a topological space X , then put $\overline{U} = \bigcup_{x \in X} \overline{B(x; U)}$ be the closure of U in the product $X_d \times X$ where X_d is the set X endowed with the discrete topology.

The following lemma proved in [5] shows that the alternating power $U^{\mp 2}$ on an entourage U is equivalent to taking the star with respect to the cover $\mathcal{U} = \{B(x; U) : x \in X\}$.

Lemma 1.5. *For any entourage U on a set X and a point $x \in X$ we get $B(x; U^{-1}U) = St(x; \mathcal{U})$ where $\mathcal{U} = \{B(x; U) : x \in X\}$. Consequently, $B(x; U^{\mp 2n}) = B(x; (U^{-1}U)^n) = St^n(x; \mathcal{U})$ for every $n \in \mathbb{N}$.*

A family \mathcal{U} of entourages on a set X is called a *uniformity* on X if it satisfies the following four axioms:

- (U1) for any $U \in \mathcal{U}$, every entourage $V \subset X \times X$ containing U belongs to \mathcal{U} ;
- (U2) for any entourages $U, V \in \mathcal{U}$ there is an entourage $W \in \mathcal{U}$ such that $W \subset U \cap V$;
- (U3) for any entourage $U \in \mathcal{U}$ there is an entourage $V \in \mathcal{U}$ such that $VV \subset U$;
- (U4) for any entourage $U \in \mathcal{U}$ there is an entourage $V \in \mathcal{U}$ such that $V \subset U^{-1}$.

A family \mathcal{U} of entourages on X is called a *quasi-uniformity* (resp. *pre-uniformity*) on X if it satisfies the axioms (U1)–(U3) (resp. (U1)–(U2)). So, each uniformity is a quasi-uniformity and each quasi-uniformity is a pre-uniformity. Observe that a pre-uniformity is just a filter of entourages on X .

A subfamily $\mathcal{B} \subset \mathcal{U}$ is called a *base* of a pre-uniformity \mathcal{U} on X if each entourage $U \in \mathcal{U}$ contains some entourage $B \in \mathcal{B}$. Each base of a preuniformity satisfies the axiom (U2). Conversely, each family \mathcal{B} of entourages on X satisfying the axiom (U2) is a base of a unique pre-uniformity $\langle \mathcal{B} \rangle$ consisting of entourages $U \subset X \times X$ containing some entourage $B \in \mathcal{B}$. If the base \mathcal{B} satisfies the axiom (U3) (and (U4)), then the pre-uniformity $\langle \mathcal{B} \rangle$ is a quasi-uniformity (and a uniformity).

Next we define some operations over preuniformities. Given two pre-uniformities \mathcal{U}, \mathcal{V} on a set X put $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, $\mathcal{U} \wedge \mathcal{V} = \{U \cup V : U \in \mathcal{U}, V \in \mathcal{V}\}$, $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and let \mathcal{UV} be the pre-uniformity generated by the base $\{UV : U \in \mathcal{U}, V \in \mathcal{V}\}$. For every $n \in \omega$ let $\mathcal{U}^{\pm n}$, $\mathcal{U}^{\mp n}$, $\mathcal{U}^{\wedge n}$, $\mathcal{U}^{\vee n}$ be the pre-uniformities generated by the bases $\{U^{\pm n} : U \in \mathcal{U}\}$, $\{U^{\mp n} : U \in \mathcal{U}\}$, $\{U^{\pm n} \cup U^{\mp n} : U \in \mathcal{U}\}$, $\{U^{\pm n} \cap U^{\mp n} : U \in \mathcal{U}\}$, respectively. Observe that $\mathcal{U}^{\wedge n} = \mathcal{U}^{\pm n} \wedge \mathcal{U}^{\mp n}$ and $\mathcal{U}^{\vee n} = \mathcal{U}^{\pm n} \vee \mathcal{U}^{\mp n}$. For a pre-uniformity \mathcal{U} on a topological space X let $\overline{\mathcal{U}}$ be the pre-uniformity generated by the base $\{\overline{U} : U \in \mathcal{U}\}$.

The pre-uniformities $\mathcal{U}^{\pm n}$, $\mathcal{U}^{\mp n}$, $\mathcal{U}^{\wedge n}$, $\mathcal{U}^{\vee n}$ feet into the following diagram (in which an arrow $\mathcal{V} \rightarrow \mathcal{W}$ indicates that $\mathcal{V} \subset \mathcal{W}$):

$$\begin{array}{ccccc}
 & & \mathcal{U}^{\pm n} & & \\
 & \nearrow & & \searrow & \\
 \mathcal{U}^{\vee(n+1)} & \longrightarrow & \mathcal{U}^{\wedge n} & & \mathcal{U}^{\vee n} \longrightarrow \mathcal{U}^{\wedge(n-1)} \\
 & \searrow & & \nearrow & \\
 & & \mathcal{U}^{\mp n} & &
 \end{array}$$

We shall say that a preuniformity \mathcal{U} on X is

- $\pm n$ -separated if $\bigcap \mathcal{U}^{\pm n} = \Delta_X$;
- $\mp n$ -separated if $\bigcap \mathcal{U}^{\mp n} = \Delta_X$;
- n -separated if \mathcal{U} is both $\pm n$ -separated and $\mp n$ -separated.

Observe that for an odd number n a pre-uniformity \mathcal{U} is n -separated if and only if it is $\pm n$ -separated if and only if it is $\mp n$ -separated (this follows from the equality $(U^{\pm n})^{-1} = U^{\mp n}$ holding for every entourage U).

This equivalence does not hold for even n :

Example 1.6. For every $m \in \mathbb{N}$ consider the entourage $U_m = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : y \in \{x\} \cup [x+m, \infty)\}$ on the half-line $\mathbb{R}_+ = [0, \infty)$. The family $\{U_m\}_{m \in \mathbb{N}}$ is a base of a quasi-uniformity \mathcal{U} on \mathbb{R}_+ which is ∓ 2 -separated but not ± 2 -separated.

Each pre-uniformity \mathcal{U} on a set X generates a topology $\tau_{\mathcal{U}}$ consisting of all subsets $W \subset X$ such that for each point $x \in W$ there is an entourage $U \in \mathcal{U}$ with $B(x; U) \subset W$, see [8]. This topology $\tau_{\mathcal{U}}$ will be referred to as *the topology generated by the pre-uniformity \mathcal{U}* . If \mathcal{U} is a quasi-uniformity, then for each point $x \in X$ the family of balls $\{B(x; U) : U \in \mathcal{U}\}$ is a neighborhood base of the topology $\tau_{\mathcal{U}}$ at x . This implies that for a quasi-uniformity \mathcal{U} on a set X the topology $\tau_{\mathcal{U}}$ is Hausdorff if and only if for any distinct points $x, y \in X$ there is an entourage $U \in \mathcal{U}$ such that $B(x; U) \cap B(y; U) = \emptyset$ if and only if $\bigcap \mathcal{U}\mathcal{U}^{-1} = \Delta_X$ if and only if the quasi-uniformity \mathcal{U} is ± 2 -separated. It is known (see [18] or [19]) that the topology of each topological space X is generated by a suitable quasi-uniformity (namely, the *Pervin quasi-uniformity*, generated by the subbase consisting of the entourages $(U \times U) \cup ((X \setminus U) \times X)$ where U runs over open sets in X).

Now we consider some cardinal characteristics of pre-uniformities. Let \mathcal{U} be a pre-uniformity on a topological space X .

- The *boundedness number* $\ell(\mathcal{U})$ of \mathcal{U} is defined as the smallest cardinal κ such that for any entourage $U \in \mathcal{U}$ there is a subset $A \subset X$ of cardinality $|A| \leq \kappa$ such that $B(A; U) = X$;
- the *weak boundedness number* $\bar{\ell}(\mathcal{U})$ of \mathcal{U} is defined as the smallest cardinal κ such that for any entourage $U \in \mathcal{U}$ there is a subset $A \subset X$ of cardinality $|A| \leq \kappa$ such that $B(A; U)$ is dense in X ;
- the *character* $\chi(\mathcal{U})$ of \mathcal{U} is the smallest cardinality of a subfamily $\mathcal{V} \subset \mathcal{U}$ such that each entourage $U \in \mathcal{U}$ contains some entourage $V \in \mathcal{V}$;
- the *pseudocharacter* $\psi(\mathcal{U})$ of \mathcal{U} is the smallest cardinality of a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcap \mathcal{V} = \bigcap \mathcal{U}$;
- the *closed pseudocharacter* $\bar{\psi}(\mathcal{U})$ of \mathcal{U} is the smallest cardinality of a subfamily $\mathcal{V} \subset \mathcal{U}$ such that for every $x \in X$ we get $\bigcap_{V \in \mathcal{V}} B(x; V) = \bigcap_{U \in \mathcal{U}} B(x; U)$ (so, $\bar{\psi}(\mathcal{U}) = \psi(\bar{\mathcal{U}})$);
- the *local pseudocharacter* $\dot{\psi}(\mathcal{U})$ of \mathcal{U} is the smallest cardinal κ such that for every $x \in X$ there is a subfamily $\mathcal{V}_x \subset \mathcal{U}$ of cardinality $|\mathcal{V}_x| \leq \kappa$ such that $\bigcap_{V \in \mathcal{V}_x} B(x; V) = \bigcap_{U \in \mathcal{U}} B(x; U)$.

For any Hausdorff topological space X and a quasi-uniformity \mathcal{U} generating the topology of X we get the inequalities $\psi(X) = \dot{\psi}(\mathcal{U}) \leq \psi(\mathcal{U})$, $\bar{\psi}(X) \leq \bar{\psi}(\mathcal{U})$ and $\chi(X) \leq \chi(\mathcal{U})$, which fit into the following diagram (in which an arrow $a \rightarrow b$ indicates that $a \leq b$).

$$\begin{array}{ccccc}
 \psi(X) & \longrightarrow & \bar{\psi}(X) & \longrightarrow & \chi(X) \\
 \downarrow & & \downarrow & & \downarrow \\
 \psi(\mathcal{U}) & \longrightarrow & \bar{\psi}(\mathcal{U}) & \longrightarrow & \chi(\mathcal{U})
 \end{array}$$

The boundedness number $\ell(\mathcal{U})$ combined with the pseudocharacter $\psi^{\mp 2}(\mathcal{U})$ can be used to produce a simple upper bound on the cardinality of a ∓ 2 -separated pre-uniform space (cf. [6, 4.3]).

Proposition 1.7. *Any set X has cardinality $|X| \leq \ell(\mathcal{U})^{\psi^{\mp 2}(\mathcal{U})}$ for any ∓ 2 -separated pre-uniformity \mathcal{U} on a set X .*

Proof. The pre-uniformity $\mathcal{U}^{\mp 2}$, being separated, contains a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $|\mathcal{V}| = \psi(\mathcal{U}^{\mp 2})$ such that $\bigcap_{V \in \mathcal{V}} V^{-1}V = \Delta_X$. By the definition of the boundedness number $\ell(\mathcal{U})$, for every entourage $V \in \mathcal{V}$ there is a subset $L_V \subset X$ of cardinality $|L_V| \leq \ell(\mathcal{U})$ such that $X = B(L_V; V)$. For every $x \in X$ choose a function $f_x \in \prod_{V \in \mathcal{V}} L_V$ assigning to every entourage $V \in \mathcal{V}$ a point $f_x(V) \in L_V$ such that $x \in B(f_x(V); V)$. We claim that for any distinct points $x, y \in X$ the functions f_x, f_y are distinct. Indeed, the choice of the family \mathcal{V} yields an entourage $V \in \mathcal{V}$ such that $(x, y) \notin V^{-1}V$. Then $f_x(V) \neq f_y(V)$ and hence $f_x \neq f_y$. This implies that

$$|X| \leq \prod_{V \in \mathcal{V}} |L_V| \leq \ell(X)^{|\mathcal{V}|} = \ell(\mathcal{U})^{\psi(\mathcal{U}^{\mp 2})}. \quad \square$$

Following [4] we define a quasi-uniformity \mathcal{U} on a topological space X to be *normal* if for any subset $A \subset X$ and entourage $U \in \mathcal{U}$ we get $\overline{A} \subset \overline{B(A; U)}^\circ$. A topological space X is called *normally quasi-uniformizable* if the topology of X is generated by a normal quasi-uniformity. Normally quasi-uniformizable spaces possess the following important normality-type property proved in [4].

Theorem 1.8. *Let X be a topological space and \mathcal{U} be a normal quasi-uniformity generating the topology of X . Then for every subset $A \subset X$ and entourage $U \in \mathcal{U}$ there exists a continuous function $f : X \rightarrow [0, 1]$ such that $A \subset f^{-1}(0)$ and $f([0, 1)) \subset \overline{B(A; U)}^\circ$.*

1.4. Cardinal characteristics of topological spaces, II. Let X be a topological space. An entourage U on X is called a *neighborhood assignment* if for every $x \in X$ the U -ball $B(x; U)$ is a neighborhood of x . The family $\mathcal{p}\mathcal{U}_X$ of all neighborhood assignments on a topological space X is a pre-uniformity called the *universal pre-uniformity* on X . It contains any pre-uniformity generating the topology of X and is equal to the union of all pre-uniformities generating the topology of X .

The universal pre-uniformity $\mathcal{p}\mathcal{U}_X$ contains

- the *universal quasi-uniformity* $q\mathcal{U}_X = \bigcup \{\mathcal{U} \subset \mathcal{p}\mathcal{U}_X : \mathcal{U} \text{ is a quasi-uniformity on } X\}$, and
- the *universal uniformity* $\mathcal{U}_X = \bigcup \{\mathcal{U} \subset \mathcal{p}\mathcal{U}_X : \mathcal{U} \text{ is a uniformity on } X\}$

of X . It is clear that $\mathcal{U}_X \subset q\mathcal{U}_X \subset p\mathcal{U}_X$. The interplay between the universal pre-uniformities $p\mathcal{U}_X$, $q\mathcal{U}_X$ and \mathcal{U}_X are studied in [5].

Since the topology of any topological space is generated by a quasi-uniformity, the universal quasi-uniformity $q\mathcal{U}_X$ generates the topology of X . In contrast, the universal uniformity \mathcal{U}_X generates the topology of X if and only if the space X is completely regular.

Cardinal characteristics of the pre-uniformities $p\mathcal{U}_X$, $q\mathcal{U}_X$ and \mathcal{U}_X or their alternating powers can be considered as cardinal characteristics of the topological space X . In particular, for a Hausdorff space X we have the equalities:

$$\chi(X) = \chi(p\mathcal{U}_X), \quad \psi(X) = \psi(p\mathcal{U}_X), \quad \bar{\psi}(X) = \bar{\psi}(p\mathcal{U}_X), \quad \Delta(X) = \psi(p\mathcal{U}_X^{\mp 2}).$$

The last equality follows from Lemma 1.5. On the other hand, the boundedness number $\ell(p\mathcal{U}_X)$ of $p\mathcal{U}_X$ coincides with the Lindelöf number $l(X)$ of X .

Observe that for the universal pre-uniformity $p\mathcal{U}_X$ on a Hausdorff topological space X the upper bound $|X| \leq \ell(p\mathcal{U}_X)^{\psi(p\mathcal{U}_X^{\mp 2})}$ proved in Proposition 1.7 turns into the known upper bound $|X| \leq l(X)^{\Delta(X)}$.

Having in mind the equality $l(X) = \ell(p\mathcal{U}_X)$, for every $n \in \mathbb{N}$ let us define the following cardinal characteristics:

$$\begin{aligned} \ell^{\pm n}(X) &:= \ell(p\mathcal{U}_X^{\pm n}), & \bar{\ell}^{\pm n}(X) &:= \bar{\ell}(p\mathcal{U}_X^{\pm n}), & q\ell^{\pm n}(X) &:= \ell(q\mathcal{U}_X^{\pm n}), \\ \ell^{\mp n}(X) &:= \ell(p\mathcal{U}_X^{\mp n}), & \bar{\ell}^{\mp n}(X) &:= \bar{\ell}(p\mathcal{U}_X^{\mp n}), & q\ell^{\mp n}(X) &:= \ell(q\mathcal{U}_X^{\mp n}), \\ \ell^{\wedge n}(X) &:= \ell(p\mathcal{U}_X^{\wedge n}), & \bar{\ell}^{\wedge n}(X) &:= \bar{\ell}(p\mathcal{U}_X^{\wedge n}), & q\ell^{\wedge n}(X) &:= \ell(q\mathcal{U}_X^{\wedge n}), \\ \ell^{\vee n}(X) &:= \ell(p\mathcal{U}_X^{\vee n}), & \bar{\ell}^{\vee n}(X) &:= \bar{\ell}(p\mathcal{U}_X^{\vee n}), & q\ell^{\vee n}(X) &:= \ell(q\mathcal{U}_X^{\vee n}). \end{aligned}$$

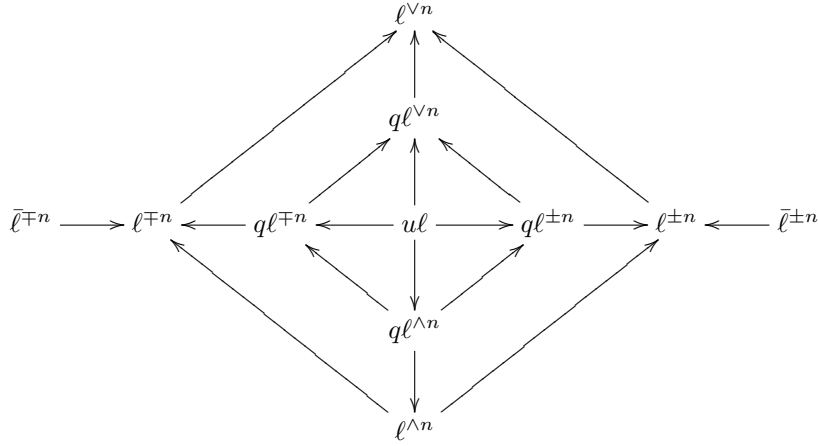
Let also

$$\ell^\omega(X) = \min_{n \in \mathbb{N}} \ell^{\vee n}(X), \quad q\ell^\omega(X) = \min_{n \in \mathbb{N}} q\ell^{\vee n}(X), \quad \text{and} \quad u\ell(X) = \ell(\mathcal{U}_X).$$

Observe that $u\ell(X) = \ell(\mathcal{U}_X^{\pm n}) = \ell(\mathcal{U}_X^{\mp n}) = \ell(\mathcal{U}_X^{\wedge n}) = \ell(\mathcal{U}_X^{\vee n})$ for every $n \in \mathbb{N}$ (this follows from the equality $\mathcal{U}_X = \mathcal{U}_X^{\pm n} = \mathcal{U}_X^{\mp n}$ holding for every $n \in \mathbb{N}$).

The above cardinal characteristics were introduced and studied in [5].

Some inequalities between the cardinal characteristics $\ell^{\pm n}$, $\ell^{\mp n}$, $\ell^{\wedge n}$, $\ell^{\vee n}$, $q\ell^{\pm n}$, $q\ell^{\mp n}$, $q\ell^{\wedge n}$, $q\ell^{\vee n}$, and $u\ell$ are described in the following diagram in which an arrow $a \rightarrow b$ indicates that $a(X) \leq b(X)$ for any topological space X .



It turns out that the cardinal invariants l^{*n} , $l^{*n\frac{1}{2}}$, \bar{l}^{*n} , and $\bar{l}^{*n\frac{1}{2}}$ can be expressed via the cardinal invariants $\ell^{\mp m}$, $\ell^{\pm m}$, $\ell^{\mp m}$, $\ell^{\pm m}$ for a suitable number m . The following proposition is proved in [5] (or can be easily derived from the definitions).

Proposition 1.9. *For every $n \in \omega$ we have the equalities:*

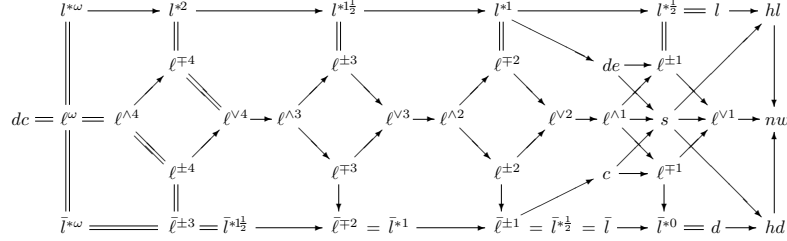
$$l^{*n} = \ell^{\mp 2n}, \quad \bar{l}^{*n} = \bar{\ell}^{\mp 2n}, \quad l^{*n\frac{1}{2}} = \ell^{\pm(2n+1)}, \quad \bar{l}^{*n\frac{1}{2}} = \bar{\ell}^{\pm(2n+1)}.$$

The following proposition (proved in [5]) describes the relation of the cardinal invariants $\ell^{\pm n}$, $\ell^{\mp n}$ to classical cardinal invariants.

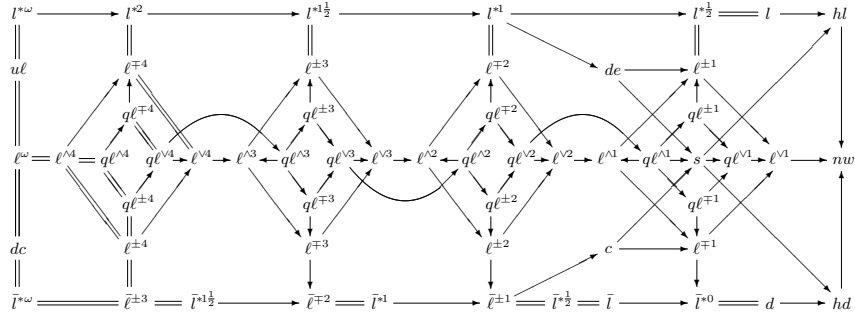
Proposition 1.10. *Let X be a topological space. Then*

- (1) $\ell^{\wedge 1}(X) \leq s(X) \leq q\ell^{\vee 1}(X) \leq \ell^{\vee 1}(X) \leq nw(X)$;
- (2) $e(X) \leq de(X) \leq q\ell^{\pm 1}(X) \leq \ell^{\pm 1}(X) = l(X)$;
- (3) $c(X) \leq q\ell^{\mp 1}(X) \leq \ell^{\mp 1}(X) \leq d(X)$;
- (4) *If X is quasi-regular, then $\bar{\ell}^{\pm 3}(X) = \bar{l}^{*1\frac{1}{2}}(X) = \ell^{\omega}(X) = dc(X)$;*
- (5) *If X is completely regular, then $q\bar{\ell}^{\pm 3}(X) = q\ell^{\omega}(X) = ul(X) = dc(X)$.*

Taking into account Propositions 1.3, 1.9 and 1.10, we see that for quasi-regular spaces the cardinal characteristics $\ell^{\pm n}$, $\ell^{\mp n}$, $\bar{\ell}^{\mp n}$, $\ell^{\wedge n}$, $\ell^{\vee n}$ relate to other cardinal characteristics of topological spaces as follows.



For Tychonoff spaces we can add to this diagram the cardinal characteristics $q\ell^{\pm n}$, $q\ell^{\mp n}$, $q\ell^{\vee n}$, and $u\ell$:



Question 1.11. Which cardinal characteristics in the above diagrams are pairwise distinct?

2. i -WEIGHT OF NORMALLY QUASI-UNIFORMIZABLE TOPOLOGICAL SPACES

In this section we apply Theorem 1.8 to derive some upper bounds on the i -weight of a normally quasi-uniformizable space.

Proposition 2.1. *Let X be a topological space whose topology is generated by a normal quasi-uniformity \mathcal{U} . The space X has i -weight $iw(X) \leq \kappa$ for some cardinal κ if there exists a family of subsets $\{A_\alpha\}_{\alpha \in \kappa}$ of X and a family of entourages $\{U_\alpha\}_{\alpha \in \kappa} \subset \mathcal{U}$ such that for any distinct points $x, y \in X$ there is $\alpha \in \kappa$ such that $x \in A_\alpha$ and $y \notin \overline{B(A_\alpha; U_\alpha)}$.*

Proof. For every $\alpha \in \kappa$ apply Theorem 1.8 to construct a continuous map $f_\alpha : X \rightarrow [0, 1]$ such that $f_\alpha(A_\alpha) \subset \{0\}$ and $f_\alpha^{-1}([0, 1]) \subset \overline{B(A_\alpha; U_\alpha)}$. It follows that the family of continuous maps $\{f_\alpha\}_{\alpha \in \kappa}$ separates points of X . So, $iw(X) \leq \kappa$. \square

This proposition will be used to prove:

Theorem 2.2. *A Hausdorff space X has i -weight $iw(X) \leq \overline{\psi}(\mathcal{A}^{-1}\mathcal{AU}) \cdot \ell(\mathcal{A})$ for any normal quasi-uniformity \mathcal{U} generating the topology of X and any pre-uniformity \mathcal{A} on X such that $\bigcap \overline{\mathcal{A}^{-1}\mathcal{AU}} = \Delta_X$.*

Proof. If the cardinal $\overline{\psi}(\mathcal{A}^{-1}\mathcal{AU})$ is finite, then $\overline{\psi}(\mathcal{A}^{-1}\mathcal{AU}) = 1$, which implies that $\mathcal{A}^{-1}\mathcal{AU} = \Delta_X = A = U$ for some $A \in \mathcal{A}$ and $U \in \mathcal{U}$. In this case $\ell(\mathcal{A}) = |X|$ and hence $iw(X) \leq |X| \leq \ell(\mathcal{A})$.

So, we assume that the cardinal $\kappa = \overline{\psi}(\mathcal{A}^{-1}\mathcal{AU})$ is infinite. Since $\bigcap \overline{\mathcal{A}^{-1}\mathcal{AU}} = \Delta_X$, we can choose subfamilies $(A_\alpha)_{\alpha \in \kappa} \subset \mathcal{A}$ and $(U_\alpha)_{\alpha \in \kappa} \subset \mathcal{U}$ such that $\bigcap_{\alpha < \kappa} \overline{B(x, A_\alpha^{-1}A_\alpha U_\alpha)} = \{x\}$ for every $x \in X$. For every $\alpha \leq \kappa$ choose a subset $Z_\alpha \subset X$ of cardinality $|Z_\alpha| \leq \ell(\mathcal{A})$ such that $X = B(Z_\alpha; A_\alpha)$. Consider the family of sets $\mathcal{Z} = \bigcup_{\alpha \in \kappa} \{B(z; A_\alpha) : z \in Z_\alpha\}$. We claim that for any distinct points $x, y \in X$ there is a set $Z \in \mathcal{Z}$ and ordinal $\alpha \in \kappa$ such that $x \in Z$ and $y \notin \overline{B(Z; U_\alpha)}$.

By the choice of the families (A_α) , (U_α) , for the points x, y there is an index $\alpha \in \kappa$ such that $y \notin \overline{B(x; A_\alpha^{-1}A_\alpha U_\alpha)}$. Since $X = B(Z_\alpha; A_\alpha)$, we can find a point $z \in Z_\alpha$ such that $x \in B(z; A_\alpha)$ and hence $z \in B(x; A_\alpha^{-1})$. We claim that the set $Z = B(z; A_\alpha) \in \mathcal{Z}$ has the required properties: $x \in Z$ and $y \notin \overline{B(Z; U_\alpha)}$. To derive a contradiction, assume that $y \in \overline{B(Z; U_\alpha)}$ which implies $y \in \overline{B(Z; U_\alpha)} = \overline{B(B(z; A_\alpha); U_\alpha)} = \overline{B(z; A_\alpha U_\alpha)} \subset \overline{B(B(x; A_\alpha^{-1}); A_\alpha U_\alpha)} = \overline{B(x; A_\alpha^{-1}A_\alpha U_\alpha)}$. But this contradicts the choice of the index α .

This contradiction allows us to apply Proposition 2.1 and conclude that

$$iw(X) \leq |\mathcal{Z}| \cdot \kappa \leq \sum_{\alpha \in \kappa} |Z_\alpha| \cdot \kappa \leq \kappa^2 \cdot \ell(\mathcal{A}) = \overline{\psi}(\mathcal{A}^{-1}\mathcal{AU}) \cdot \ell(\mathcal{A}). \quad \square$$

Applying Theorem 2.2 to some concrete pre-uniformities \mathcal{A} , we get the following corollary.

Corollary 2.3. *Let X be a functionally Hausdorff space and \mathcal{U} be a normal quasi-uniformity generating the topology of X . If for some $n \in \mathbb{N}$ the quasi-uniformity \mathcal{U} is*

- (1) $\pm(4n-2)$ -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm(4n-3)}) \cdot \ell(\mathcal{U}^{\vee(2n-1)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee(2n-1)}(X)$;
- (2) $\mp(4n-1)$ -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp(4n-2)}) \cdot \ell(\mathcal{U}^{\pm(2n-1)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\pm(2n-1)}(X)$;
- (3) $\pm(4n)$ -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm(4n-1)}) \cdot \ell(\mathcal{U}^{\vee(2n)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee(2n)}(X)$;
- (4) $\mp(4n+1)$ -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp(4n)}) \cdot \ell(\mathcal{U}^{\mp(2n)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\mp(2n)}(X)$.

Proof. 1. If \mathcal{U} is $\pm(4n-2)$ -separated, then for the pre-uniformity $\mathcal{A} = \mathcal{U}^{\pm(2n-1)} \vee \mathcal{U}^{\mp(2n-1)}$ we get

$$\mathcal{A}^{-1}\mathcal{A}\mathcal{U} \subset \mathcal{U}^{\pm(2n-1)}\mathcal{U}^{\pm(2n-1)}\mathcal{U} = \mathcal{U}^{\pm(4n-3)}\mathcal{U} = \mathcal{U}^{\pm(4n-3)}$$

and hence $\bigcap \overline{\mathcal{A}^{-1}\mathcal{A}\mathcal{U}} \subset \bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{U}\mathcal{U}^{-1} = \bigcap \mathcal{U}^{\pm(4n-2)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $\mathcal{A} = \mathcal{U}^{\vee(2n-1)}$, we get

$$iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm(4n-3)}) \cdot \ell(\mathcal{U}^{\vee(2n-1)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee(2n-1)}(X).$$

2. If \mathcal{U} is $\mp(4n-1)$ -separated, then for the pre-uniformity $\mathcal{A} = \mathcal{U}^{\pm(2n-1)}$ we get

$$\mathcal{A}^{-1}\mathcal{A}\mathcal{U} = \mathcal{U}^{\mp(2n-1)}\mathcal{U}^{\pm(2n-1)}\mathcal{U} = \mathcal{U}^{\mp(4n-2)}\mathcal{U} = \mathcal{U}^{\mp(4n-2)}$$

and hence $\bigcap \overline{\mathcal{A}^{-1}\mathcal{A}\mathcal{U}} \subset \bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{U}\mathcal{U}^{-1} = \bigcap \mathcal{U}^{\mp(4n-1)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $\mathcal{A} = \mathcal{U}^{\pm(2n-1)}$, we get

$$iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp(4n-2)}) \cdot \ell(\mathcal{U}^{\pm(2n-1)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\pm(2n-1)}(X).$$

3. If \mathcal{U} is $\pm(4n)$ -separated, then for the pre-uniformity $\mathcal{A} = \mathcal{U}^{\vee(2n)}$ we get

$$\mathcal{A}^{-1}\mathcal{A}\mathcal{U} \subset \mathcal{U}^{\pm(2n)}\mathcal{U}^{\mp(2n)}\mathcal{U} = \mathcal{U}^{\pm(4n-1)}\mathcal{U} = \mathcal{U}^{\pm(4n-1)}$$

and hence $\bigcap \overline{\mathcal{A}^{-1}\mathcal{A}\mathcal{U}} \subset \bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{U}\mathcal{U}^{-1} = \bigcap \mathcal{U}^{\pm(4n)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $\mathcal{A} = \mathcal{U}^{\vee(2n)}$, we get

$$iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm(4n-1)}) \cdot \ell(\mathcal{U}^{\pm(2n)} \vee \mathcal{U}^{\mp(2n)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee(2n)}(X).$$

4. If \mathcal{U} is $\mp(4n+1)$ -separated, then for the pre-uniformity $\mathcal{A} = \mathcal{U}^{\mp(2n)}$ we get

$$\mathcal{A}^{-1}\mathcal{A}\mathcal{U} = \mathcal{U}^{\mp(2n)}\mathcal{U}^{\mp(2n)}\mathcal{U} = \mathcal{U}^{\mp(4n)}$$

and hence $\bigcap \overline{\mathcal{A}^{-1}\mathcal{A}\mathcal{U}} \subset \bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{U}\mathcal{U}^{-1} = \bigcap \mathcal{U}^{\mp(4n+1)} = \Delta_X$. Applying Theorem 2.2 to the pre-uniformity $\mathcal{A} = \mathcal{U}^{\mp(2n)}$, we get

$$iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp(4n)}) \cdot \ell(\mathcal{U}^{\mp(2n)}) \leq \chi(\mathcal{U}) \cdot q\ell^{\mp(2n)}(X). \quad \square$$

Corollary 2.3 implies:

Corollary 2.4. *If X is a Hausdorff space and \mathcal{U} is a normal quasi-uniformity generating the topology of X , then the space X has i -weight $iw(X) \leq \overline{\psi}(\mathcal{U}) \cdot \ell(\mathcal{U} \vee \mathcal{U}^{-1}) \leq \chi(\mathcal{U}) \cdot \ell(\mathcal{U}^{\vee 1})$. Moreover, if the quasi-uniformity \mathcal{U} is*

- (1) ∓ 3 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp 2}) \cdot \ell(\mathcal{U}) \leq \chi(\mathcal{U}) \cdot q\ell^{\pm 1}(X)$;
- (2) ± 4 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm 3}) \cdot \ell(\mathcal{U}^{\vee 2}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee 2}(X)$;
- (3) ∓ 5 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp 4}) \cdot \ell(\mathcal{U}^{\mp 2}) \leq \chi(\mathcal{U}) \cdot q\ell^{\mp 2}(X)$;
- (4) ± 6 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm 5}) \cdot \ell(\mathcal{U}^{\vee 3}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee 3}(X)$;

- (5) ∓ 7 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp 6}) \cdot \ell(\mathcal{U}^{\pm 3}) \leq \chi(\mathcal{U}) \cdot q\ell^{\pm 3}(X)$;
- (6) ± 8 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm 7}) \cdot \ell(\mathcal{U}^{\vee 4}) \leq \chi(\mathcal{U}) \cdot q\ell^{\vee 4}(X)$;
- (7) ∓ 9 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\mp 8}) \cdot \ell(\mathcal{U}^{\mp 4}) \leq \chi(\mathcal{U}) \cdot q\ell^{\mp 4}(X)$;
- (8) ± 10 -separated, then $iw(X) \leq \overline{\psi}(\mathcal{U}^{\pm 9}) \cdot \ell(\mathcal{U}^{\vee 5}) \leq \chi(\mathcal{U}) \cdot dc(X)$.

3. BI-QUASI-UNIFORMIZABLE SPACES

In this section we introduce so-called bi-quasi-uniformizable spaces and obtain some upper bounds on the submetrizable number and i -weight of such spaces. As a motivation, consider the following characterization.

Proposition 3.1. *For two quasi-uniformities \mathcal{L} and \mathcal{R} on a set X the following conditions are equivalent:*

- (1) $\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$;
- (2) $\mathcal{R}\mathcal{L}^{-1} \subset \mathcal{L}^{-1}\mathcal{R}$;
- (3) $\mathcal{L}\mathcal{R}^{-1}$ is a quasi-uniformity;
- (4) $\mathcal{R}\mathcal{L}^{-1}$ is a quasi-uniformity.

Proof. (1) \Leftrightarrow (2) and (3) \Leftrightarrow (4): Since $(\mathcal{L}\mathcal{R}^{-1})^{-1} = \mathcal{R}\mathcal{L}^{-1}$, the inclusion $\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$ is equivalent to $\mathcal{R}\mathcal{L}^{-1} \subset \mathcal{L}^{-1}\mathcal{R}$. By the same reason, $\mathcal{L}\mathcal{R}^{-1}$ is a quasi-uniformity if and only if $\mathcal{R}\mathcal{L}^{-1}$ is a quasi-uniformity.

(1) \Rightarrow (3): If $\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$, then

$$\mathcal{L}\mathcal{R}^{-1} = (\mathcal{L}\mathcal{L})(\mathcal{R}^{-1}\mathcal{R}^{-1}) = \mathcal{L}(\mathcal{L}\mathcal{R}^{-1})\mathcal{R}^{-1} \subset \mathcal{L}(\mathcal{R}^{-1}\mathcal{L})\mathcal{R}^{-1} = (\mathcal{L}\mathcal{R}^{-1})(\mathcal{L}\mathcal{R}^{-1}),$$

which means that the pre-uniformity $\mathcal{L}\mathcal{R}^{-1}$ is a quasi-uniformity.

(3) \Rightarrow (1): If $\mathcal{L}\mathcal{R}^{-1}$ is a quasi-uniformity, then $\mathcal{L}\mathcal{R}^{-1} = \mathcal{L}\mathcal{R}^{-1}\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$. \square

Motivated by Proposition 3.1 let us introduce the following

Definition 3.2. Two quasi-uniformities \mathcal{L} and \mathcal{R} on a set X are called

- *commuting* if $\mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$;
- \pm -*subcommuting* if $\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$ and $\mathcal{R}\mathcal{L}^{-1} \subset \mathcal{L}^{-1}\mathcal{R}$.

A topological space X is defined to be *bi-quasi-uniformizable* if the topology of X is generated by two \pm -subcommuting quasi-uniformities.

Commuting pairs of (quasi-)uniformities were studied in [29], [30] (and [15]).

Theorem 3.3. *For any \pm -subcommuting quasi-uniformities \mathcal{L}, \mathcal{R} generating the topology τ of a topological space X the pre-uniformity $\mathcal{Q} = \mathcal{L}\mathcal{R}^{-1} \vee \mathcal{R}\mathcal{L}^{-1}$ is a uniformity generating a completely regular topology $\tau_{\mathcal{Q}}$, weaker than the topology τ of X . If the space X is Hausdorff, then*

the topology $\tau_{\mathcal{Q}}$ generated by the uniformity \mathcal{Q} is Tychonoff, the space X is functionally Hausdorff and has submetrizability number

$$sm(X) \leq \psi(\mathcal{Q}) \leq \chi(\mathcal{L}) \cdot \chi(\mathcal{R})$$

and i -weight

$$iw(X) \leq \psi(\mathcal{Q}) \cdot \log(\ell(\mathcal{Q})) \leq \chi(\mathcal{L}) \cdot \chi(\mathcal{R}) \cdot \log(dc(X)).$$

Proof. By Proposition 3.1, the pre-uniformity \mathcal{Q} is a quasi-uniformity. Since $\mathcal{Q}^{-1} = \mathcal{Q}$, it is a uniformity. Then the topology $\tau_{\mathcal{Q}}$ generated by the uniformity \mathcal{Q} is Tychonoff (see [10, 8.1.13]). Since $\mathcal{Q} \subset \mathcal{L}$, the topology $\tau_{\mathcal{Q}}$ is weaker than the topology $\tau_{\mathcal{L}} = \tau$ of the space X .

Now assume that the topology τ is Hausdorff. In this case for any distinct points $x, y \in X$ we can find entourages $L \in \mathcal{L}$ and $R \in \mathcal{R}$ such that $B(x; L) \cap B(y; R) = \emptyset$. Then $y \notin B(x; LR^{-1})$ and hence $(y, x) \notin \bigcap \mathcal{Q}$, which means that the uniformity \mathcal{Q} is separated and the topology $\tau_{\mathcal{Q}}$ generated by \mathcal{Q} is Tychonoff. Consequently, the space X is functionally Hausdorff.

To show that $sm(X) \leq \psi(\mathcal{Q})$, fix a subfamily $\mathcal{V} \subset \mathcal{Q}$ of cardinality $|\mathcal{V}| = \psi(\mathcal{Q})$ such that $\bigcap \mathcal{V} = \Delta_X$. By [10, 8.1.11], for every entourage $V \in \mathcal{V}$ there exists a continuous pseudometric d_V on X such that the entourage $[d_V]_{<1} = \{(x, y) \in X \times X : d_V(x, y) < 1\}$ is contained in V . Then the family of pseudometrics $\mathcal{D} = \{d_V\}_{V \in \mathcal{V}}$ separates points of X , which implies that $sm(X) \leq |\mathcal{D}| \leq |\mathcal{V}| = \psi(\mathcal{Q})$.

Taking into account that the topological weight of a metric space is equal to its boundedness number, which does not exceed the discrete cellularity, and applying Proposition 1.4, we conclude that

$$iw(X) \leq \psi(\mathcal{Q}) \cdot \log(\ell(\mathcal{Q})) \leq \chi(\mathcal{Q}) \cdot \log(dc(X)) \leq \chi(\mathcal{L}) \cdot \chi(\mathcal{R}) \cdot \log(dc(X)).$$

□

Theorem 3.3 implies:

Corollary 3.4. *Each Hausdorff bi-quasi-uniformizable topological space is functionally Hausdorff.*

We do not know if this corollary can be reversed.

Problem 3.5. *Is each functionally Hausdorff space bi-quasi-uniformizable?*

Proposition 3.6. *Let \mathcal{L}, \mathcal{R} be two \pm -subcommuting quasi-uniformities generating the same Hausdorff topology on X . If the quasi-uniformities $\mathcal{L}^{-1}, \mathcal{R}^{-1}$ generate the same topology on X , then the quasi-uniformities \mathcal{L} and \mathcal{R} are 3-separated.*

Proof. Given two distinct points $x, y \in X$ we shall find an entourage $R \in \mathcal{R}$ such that $(x, y) \notin R^{-1}RR^{-1}$. Since the topology generated by the quasi-uniformities \mathcal{L} and \mathcal{R} on X is Hausdorff, there are two entourages $L \in \mathcal{L}$ and $\hat{R} \in \mathcal{R}$ such that $B(x; \hat{R}) \cap B(y; LL) = \emptyset$ and hence $(x, y) \notin \hat{R}L^{-1}L^{-1}$. Replacing \hat{R} by a smaller entourage, we can additionally assume that $B(y; \hat{R}) \subset B(y; L)$. Then $B(x; \hat{R}) \cap B(y; \hat{R}L) = \emptyset$ and hence $y \notin B(x; \hat{R}L^{-1}\hat{R}^{-1})$. Since the quasi-uniformities \mathcal{L} and \mathcal{R} are \pm -subcommuting, for the entourages L and \hat{R} there are entourages $\tilde{L} \in \mathcal{L}$ and $\tilde{R} \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R} \subset \hat{R}L^{-1}$. Since quasi-uniformities \mathcal{L}^{-1} and \mathcal{R}^{-1} generate the same topology on X , for the entourage \tilde{L}^{-1} there is an entourage $\check{R} \in \mathcal{R}$ such that $B(x; \check{R}^{-1}) \subset B(x; \tilde{L}^{-1})$. Then for the entourage $R = \check{R} \cap \tilde{R} \cap \hat{R}$ we get $B(x; R^{-1}RR^{-1}) \subset B(x; \check{R}^{-1}\tilde{R}\hat{R}^{-1}) \subset B(x; \tilde{L}^{-1}\tilde{R}\hat{R}^{-1}) \subset B(x; \hat{R}L^{-1}\hat{R}^{-1})$ and hence $y \notin B(x; R^{-1}RR^{-1})$. So, $\bigcap \mathcal{R}^{-1}\mathcal{R}\mathcal{R}^{-1} = \Delta_X$ and after inversion, $\bigcap \mathcal{R}\mathcal{R}^{-1}\mathcal{R} = \Delta_X$, which means that the quasi-uniformity \mathcal{R} is 3-separated. By analogy we can prove that the quasi-uniformity \mathcal{L} is 3-separated. \square

4. NORMALLY BI-QUASI-UNIFORMIZABLE SPACES

Observe that for two quasi-uniformities \mathcal{L}, \mathcal{R} on a set X the inclusion $\mathcal{L}\mathcal{R}^{-1} \subset \mathcal{R}^{-1}\mathcal{L}$ is equivalent to the existence for every entourages $L \in \mathcal{L}$ and $R \in \mathcal{R}$ two entourages $\tilde{L} \in \mathcal{L}$ and $\tilde{R} \in \mathcal{R}$ such that $\tilde{R}^{-1}\tilde{L} \subset LR^{-1}$. Changing the order of quantifiers in this property we obtain the following notion.

Definition 4.1. A topological space X is called *normally bi-quasi-uniformizable* if its topology is generated by quasi-uniformities \mathcal{L} and \mathcal{R} satisfying the following properties:

- $\forall L \in \mathcal{L} \exists \tilde{L} \in \mathcal{L} \forall R \in \mathcal{R} \exists \tilde{R} \in \mathcal{R}$ such that $\tilde{R}^{-1}\tilde{L} \subset LR^{-1}$ and $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$;
- $\forall R \in \mathcal{R} \exists \tilde{R} \in \mathcal{R} \forall L \in \mathcal{L} \exists \tilde{L} \in \mathcal{L}$ such that $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$ and $\tilde{R}^{-1}\tilde{L} \subset RL^{-1}$.

In this case we shall say that the quasi-uniformities \mathcal{L} and \mathcal{R} are *normally \pm -subcommuting*.

By analogy we can introduce normally commuting quasi-uniformities.

Definition 4.2. Two quasi-uniformities \mathcal{L} and \mathcal{R} on a set X are defined to be *normally commuting* if they satisfy the following two conditions:

- $\forall L \in \mathcal{L} \exists \tilde{L} \in \mathcal{L} \forall R \in \mathcal{R} \exists \tilde{R} \in \mathcal{R}$ such that $\tilde{R}\tilde{L} \subset LR$ and $\tilde{L}\tilde{R} \subset RL$;
- $\forall R \in \mathcal{R} \exists \tilde{R} \in \mathcal{R} \forall L \in \mathcal{L} \exists \tilde{L} \in \mathcal{L}$ such that $\tilde{L}\tilde{R} \subset RL$ and $\tilde{R}\tilde{L} \subset RL$.

Proposition 4.3. *Any two normally \pm -subcommuting quasi-uniformities \mathcal{L}, \mathcal{R} generating the same topology on a set X are normal. Consequently, each normally bi-quasi-uniformizable topological space is normally quasi-uniformizable.*

Proof. To show that \mathcal{L} is normal, fix a subset $A \subset X$ and entourage $L \in \mathcal{L}$. Since \mathcal{L} and \mathcal{R} are normally \pm -subcommuting, for the entourage L there exists an entourage $\tilde{L} \in \mathcal{L}$ such that for every entourage $R \in \mathcal{R}$ there is an entourage $\tilde{R} \in \mathcal{R}$ with $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$. We claim that $B(\tilde{A}; \tilde{L}) \subset \overline{B(A; L)}$. Given any point $x \in B(\tilde{A}; \tilde{L})$, we need to show that $x \in \overline{B(A; L)}$. Given any neighborhood $O_x \subset X$ of x , find an entourage $R \in \mathcal{R}$ such that $B(x; R) \subset O_x$. By the choice of the entourage \tilde{L} , for the entourage R there is an entourage $\tilde{R} \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$. It follows from $x \in B(\tilde{A}; \tilde{L})$ that $B(x; \tilde{L}^{-1}) \cap \tilde{A} \neq \emptyset$ and hence $\emptyset \neq B(x; \tilde{L}^{-1}\tilde{R}) \cap A \subset B(x; RL^{-1}) \cap A$. Then $\emptyset \neq B(x; R) \cap B(A; L) \subset O_x \cap B(A; L)$, which means $x \in \overline{B(A; L)}$. So, $B(\tilde{A}; \tilde{L}) \subset \overline{B(A; L)}$ and hence $\tilde{A} \subset B(\tilde{A}; \tilde{L})^\circ \subset \overline{B(A; L)}^\circ$, which means that \mathcal{L} is normal. By analogy we can prove the normality of the quasi-uniformity \mathcal{R} . \square

Theorem 4.4. *If \mathcal{L} and \mathcal{R} are two normally \pm -subcommuting quasi-uniformities generating the topology of a Hausdorff topological space X , then the quasi-uniformities $\mathcal{L}\mathcal{R}^{-1}$ and $\mathcal{R}\mathcal{L}^{-1}$ are 1-separated and have pseudocharacter*

- (1) $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot q\ell^{\mp 1}(X)$;
- (2) $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot q\ell^{\pm 1}(X)$ if $\mathcal{L}^{-1}, \mathcal{R}^{-1}$ are normally \pm -subcommuting and generate the same topology on X ;
- (3) $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot q\ell^{\vee 2}(X)$ if the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting and $\bigcap \mathcal{L}\mathcal{L}^{-1}\mathcal{L} = \Delta_X$;
- (4) $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L}) \cdot \ell(\mathcal{A}) \cdot \ell^{\pm 2}(X)$ for any pre-uniformity \mathcal{A} on X such that $\bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{L} = \Delta_X$.

Proof. First we show that the quasi-uniformities $\mathcal{L}\mathcal{R}^{-1}$ and $\mathcal{R}\mathcal{L}^{-1}$ are 1-separated. Since the topology of X is Hausdorff, for any distinct points $x, y \in X$ we can find two disjoint open sets $O_x \ni x$ and $O_y \ni y$. Taking into account that the quasi-uniformities \mathcal{L} and \mathcal{R} generate the topology of X , we can find two entourages $L \in \mathcal{L}$ and $R \in \mathcal{R}$ such that $B(x; L) \subset O_x$ and $B(y; R) \subset O_y$. Then $B(x; L) \cap B(y; R) = \emptyset$ and hence $y \notin B(x; LR^{-1})$ and $x \notin B(y; RL^{-1})$, which implies that $\bigcap \mathcal{L}\mathcal{R}^{-1} = \Delta_X = \bigcap \mathcal{R}\mathcal{L}^{-1}$. So, the quasi-uniformities $\mathcal{L}\mathcal{R}^{-1}$ and $\mathcal{R}\mathcal{L}^{-1}$ are 1-separated. Taking into account that $(\mathcal{L}\mathcal{R}^{-1})^{-1} = \mathcal{R}\mathcal{L}^{-1}$ we conclude that $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1})$.

1. Now we shall prove the inequality $\psi(\mathcal{LR}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1})$. Fix a family of entourages $\Lambda \subset \mathcal{L}$ of cardinality $|\Lambda| \leq \psi(\mathcal{L}\mathcal{L}^{-1})$ such that $\bigcap_{L \in \Lambda} LL^{-1} = \Delta_X$. Replacing every $L \in \Lambda$ by a smaller entourage, we can assume that $\bigcap_{L \in \Lambda} (LL)(LL)^{-1} = \Delta_X$.

Since the quasi-uniformities \mathcal{L} and \mathcal{R} are normally \pm -subcommuting, for the entourage $L \in \mathcal{L}$ there exists an entourage $\tilde{L} \in \mathcal{L}$ such that for any entourage $R \in \mathcal{R}$ there exists an entourage $\tilde{R} \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$. Replacing \tilde{L} by $\tilde{L} \cap L$, we can assume that $\tilde{L} \subset L$. For the entourage \tilde{L} choose a subset $Z_L \subset X$ of cardinality $|Z_L| \leq \ell(\mathcal{L}^{-1})$ such that $X = B(Z_L; \tilde{L}^{-1})$. For every $z \in Z_L$ choose an entourage $R_z \in \mathcal{R}$ such that $B(z; R_z) \subset B(z; L)$. By the choice of \tilde{L} , for the entourage R_z there exists an entourage $\tilde{R}_z \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R}_z \subset R_z L^{-1}$. Consider the family

$$\mathcal{P} = \bigcup_{L \in \Lambda} \{(L, \tilde{R}_z) : z \in Z_L\} \subset \mathcal{L} \times \mathcal{R}.$$

We claim that for any distinct points $x, y \in X$ there is a pair $(L, \tilde{R}_z) \in \mathcal{P}$ such that $B(x; L) \cap B(y; \tilde{R}_z) = \emptyset$. By the choice of the family Λ , there is an entourage $L \in \Lambda$ such that $x \notin B(y; LLL^{-1}L^{-1})$. Since $y \in X = B(Z_L; \tilde{L}^{-1})$, there exists a point $z \in Z_L$ such that $y \in B(z; \tilde{L}^{-1})$ and hence $z \in B(y; \tilde{L})$. We claim that the pair $(\tilde{L}, \tilde{R}_z) \in \mathcal{P}$ has the desired property: $B(x; L) \cap B(y; \tilde{R}_z) = \emptyset$. Assuming that $B(x; L) \cap B(y; \tilde{R}_z) \neq \emptyset$, we would conclude that

$$\begin{aligned} x \in B(y; \tilde{R}_z L^{-1}) &\subset B(z; \tilde{L}^{-1}\tilde{R}_z L^{-1}) \subset B(z; R_z L^{-1}L^{-1}) \subset \\ &\subset B(z; LL^{-1}L^{-1}) \subset B(y; \tilde{L}LL^{-1}L^{-1}) \subset B(y, LLL^{-1}L^{-1}) \end{aligned}$$

which contradicts the choice of L . So $B(x; L) \cap B(y; \tilde{R}_z) = \emptyset$, which is equivalent to $y \notin B(x; L\tilde{R}_z^{-1})$. Then

$$\psi(\mathcal{LR}^{-1}) \leq |\mathcal{P}| \leq \sum_{L \in \Lambda} |Z_L| \leq |\Lambda| \cdot \ell(\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1}).$$

2. If the quasi-uniformities \mathcal{L}^{-1} and \mathcal{R}^{-1} are normally \pm -subcommuting and generate the same topology on X , then by Proposition 3.6, this topology is Hausdorff, which allows us to apply the first item to the quasi-uniformities $\mathcal{L}^{-1}, \mathcal{R}^{-1}$ and obtain the upper bound $\psi(\mathcal{L}^{-1}\mathcal{R}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L})$. The \pm -subcommutativity of \mathcal{L}^{-1} and \mathcal{R}^{-1} implies that $\psi(\mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}^{-1}\mathcal{R})$. So,

$$\psi(\mathcal{LR}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}^{-1}\mathcal{R}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot q\ell^{\pm 1}(X).$$

3. Next, assuming that the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting and $\bigcap \mathcal{L}\mathcal{L}^{-1}\mathcal{L} = \Delta_X$, we prove the inequality $\psi(\mathcal{R}\mathcal{L}^{-1}) = \psi(\mathcal{L}\mathcal{R}^{-1}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L})$. Fix a subfamily $\Lambda \subset \mathcal{L}$ of cardinality $|\Lambda| = \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L})$ such that $\bigcap_{L \in \Lambda} LL^{-1}L = \Delta_X$. Replacing every entourage $L \in \Lambda$ by a smaller entourage, we can assume that $\bigcap_{L \in \Lambda} L^2L^{-3}L = \Delta_X$.

Since the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting and normally \pm -subcommuting, for every entourage $L \in \Lambda$ there exists an entourage $\tilde{L} \in \mathcal{L}$, $\tilde{L} \subset L$, such that for every entourage $R \in \mathcal{R}$ there exists an entourage $\tilde{R} \in \mathcal{R}$ such that $\tilde{L}\tilde{R} \subset RL$ and $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$.

By the definition of the boundedness number $\ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L})$, for every $L \in \Lambda$ there exists a subset $A_L \subset X$ of cardinality $|A_L| \leq \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L})$ such that $X = B(A_L; \tilde{L}\tilde{L}^{-1} \cap \tilde{L}^{-1}\tilde{L})$.

For every point $a \in A_L$ choose an entourage $R_a \in \mathcal{R}$ such that $B(a; R_a) \subset B(a; L)$. By the choice of \tilde{L} for the entourage R_a there exists an entourage $\tilde{R}_a \in \mathcal{L}$ such that $\tilde{L}\tilde{R}_a \subset R_aL$, and for the entourage $\tilde{R}_a \in \mathcal{R}$ there is an entourage $\tilde{R}_a \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R}_a \subset \tilde{R}_aL^{-1}$. Consider the family of pairs

$$\mathcal{P} = \bigcup_{L \in \Lambda} \{(L, \tilde{R}_a) : a \in A_L\} \subset \mathcal{L} \times \mathcal{R}.$$

We claim that for any distinct points $x, y \in X$ there exists a pair $(L, R) \in \mathcal{P}$ such that $B(x; L) \cap B(y; R) = \emptyset$. Given two distinct points $x, y \in X$, find an entourage $L \in \Lambda$ such that $(x, y) \notin L^2L^{-3}L$.

Since $y \in X = B(A_L; \tilde{L}\tilde{L}^{-1} \cap \tilde{L}^{-1}\tilde{L})$, we can find a point $a \in A_L$ such that $y \in B(a; \tilde{L}\tilde{L}^{-1} \cap \tilde{L}^{-1}\tilde{L})$ and hence $y \in B(a; \tilde{L}\tilde{L}^{-1})$ and $a \in B(y; \tilde{L}^{-1}\tilde{L}) \subset B(y; L^{-1}L)$. We claim that $B(x; L) \cap B(y; \tilde{R}_a) = \emptyset$. To derive a contradiction, assume that $B(x; L) \cap B(y; \tilde{R}_a) \neq \emptyset$. Observe that

$$\begin{aligned} B(y; \tilde{R}_a) &\subset B(a; \tilde{L}\tilde{L}^{-1}\tilde{R}_a) \subset B(a; \tilde{L}\tilde{R}_aL^{-1}) \subset B(a; R_aLL^{-1}) \subset \\ &\quad B(a; LLL^{-1}) \subset B(y; L^{-1}LLLL^{-1}). \end{aligned}$$

Then $\emptyset \neq B(x; L) \cap B(y; \tilde{R}_a) \subset B(x; L) \cap B(y; L^{-1}LLLL^{-1})$ implies $y \notin B(x; L^2L^{-3}L)$, which contradicts the choice of the entourage L . This contradiction shows that $B(x; L) \cap B(y; \tilde{R}_a) = \emptyset$ and hence

$$\psi(\mathcal{R}\mathcal{L}^{-1}) = \psi(\mathcal{L}\mathcal{R}^{-1}) \leq |\mathcal{P}| \leq \sum_{L \in \Lambda} |A_L| \leq \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}).$$

4. Finally we prove that $\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq \overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L}) \cdot \ell(\mathcal{A}) \cdot \ell^{\pm 2}(X)$ for any pre-uniformity \mathcal{A} on X such that $\bigcap \mathcal{A}^{-1}\mathcal{A}\mathcal{U} = \Delta_X$. If $\overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L})$ is finite, then $\overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L}) = 1$, which implies that $\mathcal{A}^{-1}\mathcal{A}\mathcal{L} = \Delta_X = \mathcal{A} = L$ for some $\mathcal{A} \in \mathcal{A}$ and $L \in \mathcal{L}$. In this case $\ell(\mathcal{A}) = |X|$

and the topological space X is discrete. Then for every point $x \in X$ we can choose an entourage $R_x \in \mathcal{R}$ such that $B(x; R_x) = \{x\}$. Then $\bigcap_{x \in X} R_x L^{-1} = \bigcap_{x \in X} R_x = \Delta_X$ and hence $\psi(\mathcal{R}\mathcal{L}^{-1}) \leq |X| = \ell(\mathcal{A}) \leq \overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L}) \cdot \ell(\mathcal{A}) \cdot \ell^{\pm 2}(X)$.

So, we assume that the cardinal $\kappa = \overline{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{U})$ is infinite. Since $\bigcap \overline{\mathcal{A}^{-1}\mathcal{A}\mathcal{L}} = \Delta_X$, we can choose subfamilies $(A_\alpha)_{\alpha \in \kappa} \subset \mathcal{A}$ and $(L_\alpha)_{\alpha \in \kappa} \subset \mathcal{L}$ such that $\bigcap_{\alpha < \kappa} B(x, A_\alpha^{-1}A_\alpha L_\alpha^3) = \{x\}$ for every $x \in X$.

For every $\alpha < \kappa$ consider the entourage $A_\alpha \in \mathcal{A}$ and find a subset $Z_\alpha \subset X$ of cardinality $|Z_\alpha| \leq \ell(\mathcal{A})$ such that $X = B(Z_\alpha; A_\alpha)$. Since the quasi-uniformities \mathcal{L} and \mathcal{R} are normally \pm -subcommuting, for the entourage L_α there is an entourage \tilde{L}_α such that for every $R \in \mathcal{R}$ there is $\tilde{R}_\alpha \in \mathcal{R}$ such that $\tilde{L}_\alpha^{-1}\tilde{R}_\alpha \subset RL_\alpha^{-1}$.

Now fix any point $z \in Z_\alpha$. The normality of the quasi-uniformity \mathcal{L} (proved in Proposition 4.3) guarantees that $\overline{B(z; A_\alpha L_\alpha^2)} \subset \overline{B(z; A_\alpha L_\alpha^3)}^\circ$. Put $W_{\alpha,z} = \overline{B(z; A_\alpha L_\alpha^3)}^\circ$. For every point $y \in X \setminus W_{\alpha,z}$ choose an entourage $R_y \in \mathcal{R}$ such that $B(y; R_y R_y) \cap \overline{B(z; A_\alpha L_\alpha^2)} = \emptyset$ and hence $B(y; R_y^2 L_\alpha^{-1}) \cap B(z; A_\alpha L_\alpha) = \emptyset$. For every $y \in X \setminus \overline{B(z; A_\alpha L_\alpha^3)}$ we can replace R_y by a smaller entourage and assume additionally that $B(y; R_y)$ is disjoint with $\overline{B(z; A_\alpha L_\alpha^3)}$.

By the choice of the entourage \tilde{L}_α for every $y \in X \setminus W_{\alpha,z}$ there is an entourage $\tilde{R}_y \in \mathcal{R}$ such that $\tilde{R}_y \subset R_y$ and $\tilde{L}_\alpha^{-1}\tilde{R}_y \subset R_y L_\alpha^{-1}$. For every $y \in W_{\alpha,z}$ choose an entourage $\tilde{R}_y \in \mathcal{R}$ such that $B(y; \tilde{R}_y) \subset W_{\alpha,z}$. Now consider the neighborhood assignment $V = \bigcup_{y \in X} \{y\} \times B(y; \tilde{R}_y \cap \tilde{L}_\alpha)$. By the definition of $\ell^{\pm 2}(X)$, there exists a subset $A_{\alpha,z} \subset X$ of cardinality $|A_{\alpha,z}| \leq \ell^{\pm 2}(X)$ such that $X = B(A_{\alpha,z}; VV^{-1})$.

Consider the family $\mathcal{P} = \bigcup_{\alpha \in \kappa} \bigcup_{z \in Z_\alpha} \{(L_\alpha, \tilde{R}_a) : a \in A_{\alpha,z}\} \subset \mathcal{L} \times \mathcal{R}$. We claim that for any distinct points $x, y \in X$ there is a pair $(L, R) \in \mathcal{P}$ such that $B(x; L) \cap B(y; R) = \emptyset$.

Indeed, for the points $x, y \in X$ we can find an ordinal $\alpha \in \kappa$ such that $y \notin \overline{B(x; A_\alpha^{-1}A_\alpha L_\alpha^3)}$. Since $X = B(Z_\alpha; A_\alpha)$, there is a point $z \in Z_\alpha$ such that $x \in B(z; A_\alpha)$. Then $y \notin \overline{B(z; A_\alpha L_\alpha^3)}$ and hence $B(y, \tilde{R}_y) \subset B(y; R_y)$ is disjoint with $\overline{B(z; A_\alpha L_\alpha^3)}$ by the choice of the entourage R_y .

Since $y \in X = B(A_{\alpha,z}; VV^{-1})$, there is a point $a \in A_{\alpha,z}$ such that $y \in B(a; VV^{-1})$, which implies that $\emptyset \neq B(y; V) \cap B(a; V) = B(y; \tilde{R}_y \cap \tilde{L}_\alpha) \cap B(a; \tilde{R}_a \cap \tilde{L}_\alpha)$ and hence $y \in B(a; \tilde{R}_a \tilde{L}_\alpha^{-1})$. Since $B(y, \tilde{R}_y)$ is disjoint with $W_{\alpha,z}$, the choice of the entourage R_a guarantees that $a \notin W_{\alpha,z}$ and hence $B(a; R_a R_a) \cap \overline{B(z; A_\alpha L_\alpha^2)} = \emptyset$ and $B(a; R_a R_a L_\alpha^{-1}) \cap B(z; A_\alpha L_\alpha) = \emptyset$. Now observe that the \tilde{R}_a -ball $B(y; \tilde{R}_a) \subset B(a; VV^{-1}\tilde{R}_a) \subset B(a; R_a \tilde{L}_\alpha^{-1}\tilde{R}_a) \subset B(a; R_a R_a L_\alpha^{-1})$ is disjoint with the L_α -ball $B(x; L_\alpha) \subset B(z; A_\alpha L_\alpha)$.

The family \mathcal{P} witnesses that

$$\psi(\mathcal{L}\mathcal{R}^{-1}) = \psi(\mathcal{R}\mathcal{L}^{-1}) \leq |\mathcal{P}| \leq \bar{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{L}) \cdot \ell(\mathcal{A}) \cdot \ell^{\pm 2}(X). \quad \square$$

Taking into account that $\psi(\mathcal{L}\mathcal{R}^{-1} \vee \mathcal{R}\mathcal{L}^{-1}) \leq \psi(\mathcal{L}\mathcal{R}^{-1})$, and applying Theorem 4.4 we obtain:

Theorem 4.5. *Let X be a Hausdorff topological space and \mathcal{L}, \mathcal{R} be two normally \pm -subcommuting quasi-uniformities generating the topology of X . Then the uniformity $\mathcal{Q} = \mathcal{L}\mathcal{R}^{-1} \vee \mathcal{R}\mathcal{L}^{-1}$ has pseudocharacter:*

- (1) $\psi(\mathcal{Q}) \leq \bar{\psi}(\mathcal{L}) \cdot \ell(\mathcal{L} \vee \mathcal{L}^{-1}) \cdot \ell^{\pm 2}(X)$;
- (2) $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1}) \leq \psi(\mathcal{L}^{\pm 2}) \cdot q\ell^{\mp 1}(X)$.

Moreover, if the quasi-uniformity \mathcal{L} is

- (3) ∓ 3 -separated, then $\psi(\mathcal{Q}) \leq \bar{\psi}(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \cdot \ell^{\pm 2}(X) \leq \bar{\psi}(\mathcal{L}^{\mp 2}) \cdot \ell^{\pm 1}(X)$;
- (4) ± 4 -separated, then $\psi(\mathcal{Q}) \leq \bar{\psi}(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(X) \leq \bar{\psi}(\mathcal{L}^{\pm 3}) \cdot \ell^{\vee 2}(X)$;
- (5) ∓ 5 -separated, then $\psi(\mathcal{Q}) \leq \bar{\psi}(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(X) \leq \bar{\psi}(\mathcal{L}^{\mp 4}) \cdot q\ell^{\mp 2}(X) \cdot \ell^{\pm 2}(X)$;
- (6) ± 6 -separated, then $\psi(\mathcal{Q}) \leq \bar{\psi}(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(X) = \bar{\psi}(\mathcal{L}^{\pm 5}) \cdot \ell^{\pm 2}(X)$.

If the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting and 3-separated, then

$$(7) \quad \psi(\mathcal{Q}) \leq \psi(\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \leq \psi(\mathcal{L}^{\pm 3}) \cdot q\ell^{\vee 2}(X).$$

If the quasi-uniformities \mathcal{L}^{-1} , \mathcal{R}^{-1} are normally \pm -subcommuting and generate the same topology on X , then

- (8) $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \leq \psi(\mathcal{L}^{\mp 2}) \cdot q\ell^{\pm 1}(X)$ and
- (9) $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}) \leq \psi(\mathcal{L}^{\vee 2}) \cdot q\ell^{\pm 1}(X) \cdot q\ell^{\mp 1}(X)$.

Proof. 1. The first inequality follows from Theorem 4.4(4) applied to the pre-uniformity $\mathcal{A} = \mathcal{U} \vee \mathcal{U}^{-1}$.

2. The second item follows from Theorem 4.4(1).

3–6. The items (3)–(6) follow from Theorem 4.4(4) applied to the pre-uniformities \mathcal{L} , $\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}$, $\mathcal{L}^{-1}\mathcal{L}$, and $\mathcal{L}\mathcal{L}^{-1}$, respectively.

7. The seventh item follows from Theorem 4.4(3).

8, 9. Assume that the quasi-uniformities \mathcal{L}^{-1} , \mathcal{R}^{-1} are normally \pm -subcommuting and generate the same topology on X . The inequalities $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \leq \psi(\mathcal{L}^{\mp 2}) \cdot q\ell^{\pm 1}(X)$ follow from Theorem 4.4(2).

To prove that $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \cdot \ell(\mathcal{L}^{-1})$, fix a subset $\Lambda \subset \mathcal{L}$ of cardinality $|\Lambda| = \psi(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L})$ such that $\bigcap_{L \in \Lambda} LL^{-1} \cap L^{-1}L = \Delta_X$. Replacing every $L \in \Lambda$ by a smaller entourage, we can assume that $\bigcap_{L \in \Lambda} L^2L^{-2} \cap L^{-2}L^2 = \Delta_X$. Since the quasi-uniformities \mathcal{L}, \mathcal{R} are normally \pm -subcommuting and the quasi-uniformities $\mathcal{L}^{-1}, \mathcal{R}^{-1}$ are normally \pm -subcommuting, for every $L \in \Lambda$ there exists an entourage $\tilde{L} \in \mathcal{L}$ with $\tilde{L} \subset L$ such that for every $R \in \mathcal{R}$ there is $\tilde{R} \in \mathcal{R}$ such that $\tilde{L}^{-1}\tilde{R} \subset RL^{-1}$ and $\tilde{L}\tilde{R}^{-1} \subset R^{-1}L$.

For every $L \in \Lambda$ fix a subset $Z_L \subset X$ of cardinality $|Z_L| \leq \ell(\mathcal{L}) + \ell(\mathcal{L}^{-1})$ such that $X = B(Z_L; \tilde{L}) = B(Z_L; \tilde{L}^{-1})$. Since the quasi-uniformities \mathcal{L}, \mathcal{R} generate the same topology on X and $\mathcal{L}^{-1}, \mathcal{R}^{-1}$ generate the same topology on X , for every $z \in Z_L$ we can choose an entourage $R_z \in \mathcal{R}$ such that $B(z; R_z) \subset B(z; L)$ and $B(z; R_z^{-1}) \subset B(z; L^{-1})$. By the choice of \tilde{L} for the entourage R_z there is an entourage $\tilde{R}_z \in \mathcal{R}$ such that $\tilde{R}_z \subset R_z$, $\tilde{L}^{-1}\tilde{R}_z \subset R_zL^{-1}$ and $\tilde{L}\tilde{R}_z^{-1} \subset R_z^{-1}L$. For the entourage \tilde{R}_z there is an entourage $\check{R}_z \in \mathcal{R}$ with $\check{R}_z \subset \tilde{R}_z$ such that $\tilde{L}\check{R}_z^{-1} \subset \tilde{R}_z^{-1}L$, which is equivalent to $\check{R}_z\tilde{L}^{-1} \subset L^{-1}\check{R}_z$.

We claim that the family $\mathcal{P} = \{(\tilde{L}, \check{R}_z) : L \in \mathcal{L}, z \in Z_L\} \subset \mathcal{L} \times \mathcal{R}$ has $\bigcap_{(L, R) \in \mathcal{P}} LR^{-1} \cap RL^{-1} = \Delta_X$. Given any distinct points x, y find an entourage $L \in \Lambda$ such that $(x, y) \notin L^2L^{-2} \cap L^{-2}L^2$ and hence $(x, y) \notin L^2L^{-2}$ or $(x, y) \notin L^{-2}L^2$.

If $(x, y) \notin L^2L^{-2}$, then $B(y; L^2) \cap B(x; L^2) = \emptyset$. Since $y \in X = B(Z_L; \tilde{L})$, there is $z \in Z_L$ such that $y \in B(z; \tilde{L}^{-1}) \subset B(z; L^{-1})$. Then $z \in B(y; L)$ and the L -ball $B(z; L) \subset B(y; LL)$ does not intersect $B(x; L^2)$, which implies $B(z; LL^{-1}) \cap B(x; L) = \emptyset$. Observe that $B(y; \check{R}_z) \subset B(z; \tilde{L}^{-1}\check{R}_z) \subset B(z; R_zL^{-1}) \subset B(z; LL^{-1})$ and hence $B(y; \check{R}_z) \cap B(x; L) \subset B(z; LL^{-1}) \cap B(x; L) = \emptyset$. So, $(x, y) \notin L\check{R}_z^{-1}$ and hence $(x, y) \notin L\check{R}_z^{-1}$.

If $(x, y) \notin L^{-2}L^2$, then $B(y; L^{-2}) \cap B(x; L^{-2}) = \emptyset$. Since $y \in X = B(Z_L; \tilde{L})$, there is $z \in Z_L$ such that $y \in B(z; \tilde{L})$. Then $z \in B(y; \tilde{L}^{-1}) \subset B(y; L^{-1})$ and the L^{-1} -ball $B(z; L^{-1}) \subset B(y; L^{-2})$ does not intersect $B(x; L^{-2})$, which implies $B(z; L^{-1}L) \cap B(x; L^{-1}) = \emptyset$. Observe that $B(y; \check{R}_z^{-1}) \subset B(z; \tilde{L}\check{R}_z^{-1}) \subset B(z; R_z^{-1}L) \subset B(z; L^{-1}L)$ and hence $B(y; \check{R}_z^{-1}) \cap B(x; L^{-1}) \subset B(z; L^{-1}L) \cap B(x; L^{-1}) = \emptyset$. So, $(x, y) \notin L^{-1}\check{R}_z$. Since $\check{R}_z\tilde{L}^{-1} \subset L^{-1}\check{R}_z$, we get also $(x, y) \notin \check{R}_z\tilde{L}^{-1}$.

This completes the proof of the equality $\bigcap_{(L, R) \in \mathcal{P}} LR^{-1} \cap RL^{-1} = \Delta_X$, which implies the desired inequality

$$\psi(\mathcal{Q}) \leq |\mathcal{P}| \leq \sum_{L \in \Lambda} |Z_L| \leq \psi(\mathcal{L}\mathcal{L}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}). \quad \square$$

In Section 6 we shall need the following upper bound on the local pseudocharacters $\dot{\psi}(\mathcal{L}\mathcal{L}^{-1})$ and $\dot{\psi}(\mathcal{R}\mathcal{R}^{-1})$ of normally \pm -subcommuting quasi-uniformities \mathcal{L} and \mathcal{R} .

Proposition 4.6. *If the topology of a Hausdorff space X is generated by two normally \pm -subcommuting quasi-uniformities \mathcal{L} and \mathcal{R} , then $\dot{\psi}(\mathcal{L}\mathcal{L}^{-1}) \leq \bar{\psi}(X) \cdot \ell^{\pm 2}(X)$ and $\dot{\psi}(\mathcal{R}\mathcal{R}^{-1}) \leq \bar{\psi}(X) \cdot \ell^{\pm 2}(X)$.*

Proof. First we prove that $\dot{\psi}(\mathcal{L}\mathcal{L}^{-1}) \leq \bar{\psi}(X) \cdot \ell^{\pm 2}(X)$. Fix any point $x \in X$. Since the topology of X is generated by the quasi-uniformity \mathcal{R} , we can fix a subfamily $\mathcal{R}_x \subset \mathcal{R}$ of cardinality $|\mathcal{R}_x| \leq \bar{\psi}_x(X) \leq \bar{\psi}(X)$ such that $\bigcap_{R \in \mathcal{R}_x} \overline{B(x; RRR)} = \{x\}$.

By the normality of the quasi-uniformity \mathcal{R} , for every $R \in \mathcal{R}_x$ we get $\overline{B(x; RR)} \subset \overline{B(x; RRR)}^\circ$. Then for every point $z \in X \setminus \overline{B(x; RRR)}^\circ$ we can find an entourage $L_z \in \mathcal{L}$ such that $B(z; L_z L_z) \cap \overline{B(x; RR)} = \emptyset$. For every point $z \in \overline{B(x; RRR)}^\circ$ choose an entourage $L_z \in \mathcal{L}$ such that $B(z; L_z L_z) \subset \overline{B(x; RRR)}^\circ$. Since the quasi-uniformities \mathcal{L} and \mathcal{R} are normally \pm -subcommuting, for the entourage $R \in \mathcal{R}$ there is an entourage $\tilde{R} \in \mathcal{R}$ such that for every entourage $L \in \mathcal{L}$ there is an entourage $\tilde{L} \in \mathcal{L}$ such that $\tilde{R}^{-1} \tilde{L} \subset LR^{-1}$. In particular, for every $z \in Z$ there is an entourage $\tilde{L}_z \in \mathcal{L}$ such that $\tilde{R}^{-1} \tilde{L}_z \subset L_z R^{-1}$. Replacing \tilde{L}_z by a smaller entourage we can assume that $\tilde{L}_z \subset L_z$ and $B(x; \tilde{L}_z) \subset B(x; R)$.

By the definition of $\ell^{\pm 2}(X)$, for the neighborhood assignment $N_R = \bigcup_{z \in X} \{z\} \times B(z; \tilde{L}_z \cap \tilde{R})$ there is a subset $Z_R \subset X$ of cardinality $|Z_R| \leq \ell^{\pm 2}(X)$ such that $X = B(Z_R; N_R N_R^{-1})$.

We claim that the subfamily $\mathcal{L}' = \bigcup_{R \in \mathcal{R}_x} \{\tilde{L}_z : z \in Z_R\} \subset \mathcal{L}$ has the required property: $\bigcap_{L \in \mathcal{L}'} B(x; LL^{-1}) = \{x\}$. Given any point $y \in X \setminus \{x\}$, find an entourage $R \in \mathcal{R}_x$ such that $y \notin \overline{B(x; RRR)}$. Since $y \in X = B(Z_R; N_R N_R^{-1})$, there is a point $z \in Z_R$ such that $y \in B(z; N_R N_R^{-1})$ and hence $B(y; L_y \cap \tilde{R}) \cap B(z; L_z \cap \tilde{R}) = B(y; N_R) \cap B(z; N_R) \neq \emptyset$ and $y \in B(z; L_z \tilde{R}^{-1})$. Since $y \notin \overline{B(x; RRR)}^\circ$, the choice of the entourages L_y, L_z implies that $z \notin \overline{B(x; RRR)}^\circ$. We claim that $B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) = \emptyset$. To derive a contradiction, assume that $B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) \neq \emptyset$. Then

$$\emptyset \neq B(y; \tilde{L}_z) \cap B(x; \tilde{L}_z) \subset B(z; \tilde{L}_z \tilde{R}^{-1} \tilde{L}_z) \cap B(x; R) \subset B(z; \tilde{L}_z L_z R^{-1}) \cap B(x; R)$$

and hence $B(z; L_z L_z) \cap B(x; RR) \neq \emptyset$, which contradicts the choice of the entourage L_z . This contradiction completes the proof of the inequality $\dot{\psi}(\mathcal{L}\mathcal{L}^{-1}) \leq \bar{\psi}(X) \cdot \ell^{\pm 2}(X)$.

By analogy (or changing \mathcal{L} and \mathcal{R} by their places) we can prove that $\dot{\psi}(\mathcal{R}\mathcal{R}^{-1}) \leq \bar{\psi}(X) \cdot \ell^{\pm 2}(X)$. \square

5. QUASI-UNIFORMITIES ON TOPOLOGICAL MONOIDS

A *topological monoid* is a topological semigroup X possessing a (necessarily unique) two-sided unit $e \in X$. We shall say that a topological monoid S has *open shifts* if for any elements $a, b \in X$ the two-sided shift $s_{a,b} : X \rightarrow X$, $s_{a,b} : x \mapsto axb$, is an open map.

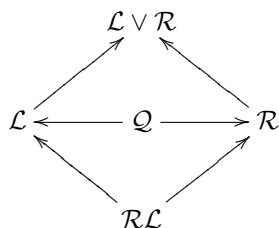
A typical example of a topological monoid with open shifts is a *paratopological group*, i.e., a group endowed with a topology making the group operation $G \times G \rightarrow G$, $(x, y) \mapsto xy$, continuous.

The closed half-line $[0, \infty)$ endowed the Sorgenfrey topology (generated by the base $\mathcal{B} = \{[a, b) : 0 \leq a < b < \infty\}$) and the operation of addition of real numbers is a topological monoid with open shifts, which is not a (paratopological) group.

Each topological monoid X carries five natural quasi-uniformities:

- the *left quasi-uniformity* \mathcal{L} , generated by the base $\{(x, y) \in X \times X : y \in xU\} : U \in \mathcal{N}_e\}$,
- the *right quasi-uniformity* \mathcal{R} , generated by the base $\{(x, y) \in X \times X : y \in Ux\} : U \in \mathcal{N}_e\}$,
- the *two-sided quasi-uniformity* $\mathcal{L} \vee \mathcal{R}$, generated by the base $\{(x, y) \in X \times X : y \in Ux \cap xU\} : U \in \mathcal{N}_e\}$,
- the *Roelcke quasi-uniformity* $\mathcal{RL} = \mathcal{LR}$, generated by the base $\{(x, y) \in X \times X : y \in UxU\} : U \in \mathcal{N}_e\}$, and
- the *quasi-Roelcke uniformity* $\mathcal{Q} = \mathcal{RL}^{-1} \vee \mathcal{LR}^{-1}$, generated by the base $\{(x, y) \in X \times X : Ux \cap yU \neq \emptyset \neq Uy \cap xU\} : U \in \mathcal{N}_e\}$.

Here by \mathcal{N}_e we denote the family of all open neighborhoods of the unit e in X . The quasi-uniformities \mathcal{L} , \mathcal{R} , $\mathcal{L} \vee \mathcal{R}$, and \mathcal{RL} are well-known in the theory of topological and paratopological groups (see [24, Ch.2], [2, §1.8]). The quasi-Roelcke uniformity was recently introduced in [4]. It should be mentioned that on topological groups the quasi-Roelcke uniformity coincides with the Roelcke (quasi-)uniformity. The following diagram describes the relation between these five quasi-uniformities (an arrow $\mathcal{U} \rightarrow \mathcal{V}$ in the diagram indicates that $\mathcal{U} \subset \mathcal{V}$).



If a topological monoid X has open shifts, then the quasi-uniformities \mathcal{L} , \mathcal{R} , $\mathcal{L} \vee \mathcal{R}$ and \mathcal{RL} generate the original topology of X (see [17], [20]) whereas the quasi-Roelcke uniformity \mathcal{Q} generates a topology $\tau_{\mathcal{Q}}$, which is (in general, strictly) weaker than the topology τ of X . If X is a paratopological group, then the topology $\tau_{\mathcal{Q}}$ on G coincides with the joint $\tau_2 \vee (\tau^{-1})_2$ of the second oscillator topologies considered by the authors in [3]. The topology $\tau_{\mathcal{Q}}$ turns the paratopological group into a quasi-topological group, i.e., a group endowed with a topology in which the inversion and all shifts are continuous (see Proposition 6.3).

Proposition 5.1. *On each topological monoid X with open shifts the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting, normally \pm -subcommuting, and normal. The topology of X is Hausdorff if and only if the quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{LR}^{-1} \vee \mathcal{RL}^{-1}$ on X is separated.*

Proof. To see that the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting and normally \pm -subcommuting, fix any entourage $L \in \mathcal{L}$ and find a neighborhood $U \subset G$ of the unit e such that $\tilde{L} = \{(x, y) \in X \times X : y \in xU\} \subset L$. Given any entourage $R \in \mathcal{R}$, find a neighborhood $V \subset G$ of the unit e such that $\tilde{R} = \{(x, y) \in X \times X : y \in Vx\} \subset R$. Then

$$\begin{aligned} \tilde{L}\tilde{R} &= \{(x, y) \in X \times X : \exists z \in X \text{ such that } (x, z) \in \tilde{L} \text{ and } (z, y) \in \tilde{R}\} = \\ &= \{(x, y) \in X \times X : \exists z \in X \text{ such that } z \in xU \text{ and } y \in Vz\} = \\ &= \{(x, y) \in X \times X : y \in V(xU)\} = \{(x, y) \in X \times X : y \in (Vx)U\} = \\ &= \tilde{R}\tilde{L} \subset RL \cap LR. \end{aligned}$$

This implies that the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting.

Next, we prove that $\tilde{L}^{-1}\tilde{R} \subset \tilde{R}\tilde{L}^{-1} \subset RL^{-1}$. Given any pair $(x, y) \in \tilde{L}^{-1}\tilde{R}$, find a point $z \in X$ such that $(x, z) \in \tilde{L}^{-1}$ and $(z, y) \in \tilde{R}$. Then $x \in zU$ and $y \in Vz$. So, we can find points $u \in U$ and $v \in V$ such that $x = zu$ and $y = vz$. Multiplying $x = zu$ by v , we get $vx = vzu = yu$ and hence $(x, vx) \in \tilde{R}$ and $(y, vx) = (y, yu) \in \tilde{L}$, which implies that $(x, y) \in \tilde{R}\tilde{L}^{-1} \subset RL^{-1}$. So, $\tilde{L}^{-1}\tilde{R} \subset \tilde{R}\tilde{L}^{-1} \subset RL^{-1}$. By analogy we can prove that $\tilde{R}^{-1}\tilde{L} \subset \tilde{L}\tilde{R}^{-1} \subset LR^{-1}$.

By Proposition 4.3, the quasi-uniformities \mathcal{L} and \mathcal{R} , being normally \pm -subcommuting, are normal.

If X is Hausdorff, then for any distinct points $x, y \in X$ we can find a neighborhood $U \subset X$ of the unit e such that $Ux \cap yU = \emptyset$. Then for the entourages $L = \{(x, y) \in X : y \in xU\} \in \mathcal{L}$ and $R = \{(x, y) \in X \times X : y \in Ux\}$ we get $y \notin B(x; RL^{-1}) \supset B(x; RL^{-1} \cap LR^{-1})$. This means that $\bigcap \mathcal{Q} = \Delta_X$ and the quasi-Roelcke uniformity \mathcal{Q} is separated.

Now assume that the quasi-Roelcke uniformity \mathcal{Q} is separated. Given two distinct points $x, y \in X$, find two entourages $L \in \mathcal{L}$ and $R \in \mathcal{R}$ such that $(x, y) \notin LR^{-1} \cap RL^{-1}$ and hence $(x, y) \notin LR^{-1}$ or $(x, y) \notin RL^{-1}$. For the entourages L, R , find a neighborhood $U \subset X$ of e such that $\{(x, y) \in X \times X : y \in xU\} \subset L$ and $\{(x, y) \in X \times X : y \in Ux\} \subset R$. If $(x, y) \notin LR^{-1}$, then $xU \cap Uy = \emptyset$. If $(x, y) \in RL^{-1}$, then $Ux \cap yU = \emptyset$. In both cases the points x, y has disjoint neighborhoods in X , which means that X is Hausdorff. \square

Proposition 5.1 and Theorem 3.3 imply:

Theorem 5.2. *Each Hausdorff topological monoid X with open shifts is functionally Hausdorff and has submetrizable number $sm(X) \leq \psi(\mathcal{Q}) \leq \chi(X)$ and i -weight $iw(X) \leq \psi(\mathcal{Q}) \cdot \log(\ell(\mathcal{Q})) \leq \chi(X) \cdot \log(dc(X))$.*

Observe that for a paratopological group G the quasi-Roelcke uniformity \mathcal{Q} generates the topology of G if and only if G is a topological group.

Problem 5.3. *Study properties of topological monoids S with open shifts whose topology is generated by the quasi-Roelcke uniformity \mathcal{Q} .*

6. THE SUBMETRIZABILITY NUMBER AND i -WEIGHT OF PARATOPOLOGICAL GROUPS

In this section we apply the results of the preceding sections to paratopological groups, i.e., groups G endowed with a topology making the group operation $G \times G \rightarrow G$, $(x, y) \mapsto xy$, continuous. It is easy to see that the inversion map $G \rightarrow G$, $x \mapsto x^{-1}$, is a uniform homeomorphism of the quasi-uniform spaces (G, \mathcal{L}^{-1}) and (G, \mathcal{R}) and also a uniform homeomorphism of the quasi-uniform spaces (G, \mathcal{R}^{-1}) and (G, \mathcal{L}) . This observation combined with Propositions 3.6 and 5.1 implies:

Proposition 6.1. *On each paratopological group G*

- (1) *the quasi-uniformities \mathcal{L} and \mathcal{R} are normally commuting, normally \pm -subcommuting, and normal;*
- (2) *the quasi-uniformities \mathcal{L}^{-1} and \mathcal{R}^{-1} are normally commuting, normally \pm -subcommuting, and generate the same topology on G .*

If the topology of G is Hausdorff, then the quasi-uniformities \mathcal{L} and \mathcal{R} are 3-separated and the quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{L}\mathcal{R}^{-1} \vee \mathcal{R}\mathcal{L}^{-1}$ is separated.

Next, we prove that a paratopological group endowed with the quasi-Roelcke uniformity is a uniform quasi-topological group.

Definition 6.2. A *uniform quasi-topological group* is a group G endowed with a uniformity \mathcal{U} such that the inversion $G \rightarrow G$, $x \mapsto x^{-1}$, is uniformly continuous and for every $a, b \in G$ the shifts $s_{a,b} : G \rightarrow G$, $s_{a,b} : x \mapsto axb$, is uniformly continuous.

Proposition 6.3. *Any paratopological group G endowed with the quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{LR}^{-1} \vee \mathcal{RL}^{-1}$ is a uniform quasi-topological group.*

Proof. Observe that for any neighborhood $V \in \mathcal{N}_e$ and points $x, y \in G$ the inclusion $y \in VxV^{-1} \cap V^{-1}xV$ is equivalent to $y^{-1} \in Vx^{-1}V^{-1} \cap V^{-1}x^{-1}V$, which implies that the inversion map $G \rightarrow G, x \mapsto x^{-1}$, is uniformly continuous.

Next, we show that for every $a, b \in G$ the shift $s_{a,b} : G \rightarrow G, s_{a,b} : x \mapsto axb$, is uniformly continuous. Fix any neighborhood $V \in \mathcal{N}_e$ of e . By the continuity of the shifts on G , there exists a neighborhood $U \subset V$ of e such that $aU \subset Va, Ub \subset bV, Ua^{-1} \subset a^{-1}V$, and $b^{-1}U \subset Vb^{-1}$. Inverting the two latter inclusions, we get $aU^{-1} \subset V^{-1}a$ and $U^{-1}b \subset bV^{-1}$. Then for any points $x, y \in G$ with $y \in U^{-1}xU \cap UxU^{-1}$, we get $ayb \in aU^{-1}xUb \cap aUxU^{-1}b \subset V^{-1}axbV \cap VaxbV^{-1}$, which means that the shift $s_{a,b}$ is uniformly continuous. \square

The following theorem is a partial case of Theorem 5.2.

Theorem 6.4. *Each Hausdorff paratopological group G is functionally Hausdorff and has submetrizability number $sm(G) \leq \psi(\mathcal{Q}) \leq \chi(G)$ and i -weight $iw(G) \leq \psi(\mathcal{Q}) \cdot \log(\ell(\mathcal{Q})) \leq \chi(G) \cdot \log(dc(G))$.*

In light of this theorem it is important to have upper bound on the pseudocharacter $\psi(\mathcal{Q})$ of the quasi-Roelcke uniformity. Such upper bounds are given in the following theorem, which unifies or generalizes the results of [25] and [21].

Theorem 6.5. *For any Hausdorff paratopological group G its quasi-Roelcke uniformity $\mathcal{Q} = \mathcal{LR}^{-1} \vee \mathcal{RL}^{-1}$ has pseudocharacter*

- (1) $\psi(\mathcal{Q}) \leq \min\{\psi(\mathcal{LL}^{-1}) \cdot \ell(\mathcal{L}^{-1}), \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L})\} \leq \overline{\psi}(G) \cdot \ell^{\pm 2}(G) \cdot \min\{\ell(\mathcal{L}), \ell(\mathcal{L}^{-1})\} \leq \overline{\psi}(G) \cdot \ell^{\pm 2}(G) \cdot \min\{q\ell^{\pm 1}(G), q\ell^{\mp 1}(G)\};$
- (2) $\psi(\mathcal{Q}) \leq \psi(\mathcal{LL}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}) \leq \psi(\mathcal{L}^{\vee 2}) \cdot q\ell^{\mp 1}(G) \cdot q\ell^{\pm 1}(G);$
- (3) $\psi(\mathcal{Q}) \leq \psi(\mathcal{LL}^{-1}\mathcal{L}) \cdot \ell(\mathcal{LL}^{-1} \vee \mathcal{L}^{-1}\mathcal{L}) \leq \psi(\mathcal{L}^{\pm 3}) \cdot q\ell^{\vee 2}(G).$

Moreover, if the quasi-uniformity \mathcal{L} is

- (4) ∓ 4 -separated, then $\psi(\mathcal{Q}) \leq \psi(\mathcal{L}^{-1}\mathcal{LL}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(X) \leq \psi(\mathcal{L}^{\mp 4}) \cdot q\ell^{\mp 2}(X) \cdot \ell^{\pm 2}(G);$
- (5) ± 6 -separated, then $\psi(\mathcal{Q}) \leq \overline{\psi}(\mathcal{LL}^{-1}\mathcal{LL}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(G) = \overline{\psi}(\mathcal{L}^{\pm 5}) \cdot \ell^{\pm 2}(G).$

Proof. 1. The inequality $\psi(\mathcal{Q}) \leq \psi(\mathcal{LL}^{-1}) \cdot \ell(\mathcal{L}^{-1})$ follows from Theorem 4.5(2), which also implies $\psi(\mathcal{Q}) \leq \psi(\mathcal{RR}^{-1}) \cdot \ell(\mathcal{R}^{-1}) = \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L})$. By Proposition 4.6, $\psi(\mathcal{LL}^{-1}) = \psi(\mathcal{L}\mathcal{L}^{-1}) \leq \overline{\psi}(G) \cdot \ell^{\pm 2}(G)$ and

$\psi(\mathcal{L}^{-1}\mathcal{L}) = \psi(\mathcal{R}\mathcal{R}^{-1}) = \dot{\psi}(\mathcal{R}\mathcal{R}^{-1}) \leq \bar{\psi}(G) \cdot \ell^{\pm 2}(G)$, which implies $\min\{\psi(\mathcal{L}\mathcal{L}^{-1}) \cdot \ell(\mathcal{L}^{-1}), \psi(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L})\} \leq \bar{\psi}(G) \cdot \ell^{\pm 2}(G) \cdot \min\{\ell(\mathcal{L}), \ell(\mathcal{L}^{-1})\}$.

2, 3. The upper bounds from the second and third items follow from Theorem 4.5(9,7) and Proposition 6.1.

4. Assume that the quasi-uniformity \mathcal{L} is ∓ 4 -separated. Then we can choose a subfamily $\mathcal{U} \subset \mathcal{N}_e$ of cardinality $|\mathcal{U}| = \psi(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L})$ such that $\bigcap_{U \in \mathcal{U}} U^{-1}UU^{-1}U = \{e\}$. Replacing every U by a smaller neighborhood of e , we can assume that $\bigcap_{U \in \mathcal{U}} U^{-2}UU^{-1}U = \{e\}$. Since $\overline{U^{-1}UU^{-1}U} \subset U^{-1}(U^{-1}UU^{-1}U)$, we conclude that $\bigcap_{U \in \mathcal{U}} \overline{U^{-1}UU^{-1}U} = \{e\}$ and $\bar{\psi}(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \leq |\mathcal{U}| = \psi(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L})$. Applying Theorem 4.4(4) to the pre-uniformity $\mathcal{A} = \mathcal{L}^{-1}\mathcal{L}$, we get the upper bound

$$\begin{aligned} \psi(\mathcal{Q}) &\leq \bar{\psi}(\mathcal{A}^{-1}\mathcal{A}\mathcal{U}) \cdot \ell(\mathcal{A}) \cdot \ell^{\pm 2}(G) = \\ \bar{\psi}(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(G) &= \psi(\mathcal{L}^{-1}\mathcal{L}\mathcal{L}^{-1}\mathcal{L}) \cdot \ell(\mathcal{L}^{-1}\mathcal{L}) \cdot \ell^{\pm 2}(G). \end{aligned}$$

5. The fifth item follows from Theorem 4.5(6). \square

7. TWO COUNTEREXAMPLES

In this section we construct two examples of paratopological groups that have some rather unexpected properties.

7.1. A paratopological group with countable pseudocharacter which is not submetrizable. In Theorem 6.5(1) we proved that for each Hausdorff paratopological group G its quasi-Roelcke uniformity has pseudocharacter $\psi(\mathcal{Q}) \leq \bar{\psi}(G) \cdot \ell^{\pm 2}(G) \cdot \min\{\ell(\mathcal{L}), \ell(\mathcal{L}^{-1})\}$. It is natural to ask if this upper bound can be improved to $\psi(\mathcal{Q}) \leq \bar{\psi}(G)$. In this section we show that this inequality is not true in general. Namely, we present an example of a zero-dimensional (and hence) Hausdorff abelian paratopological group which has countable pseudocharacter but is not submetrizable. Some properties of this group can be proved only under Martin Axiom [31], whose topological equivalent says that each countably cellular compact Hausdorff space is κ -Baire for every cardinal $\kappa < \mathfrak{c}$. We say that a topological space X is κ -Baire if for any family \mathcal{U} consisting of κ many open dense subsets of X the intersection $\bigcap \mathcal{U}$ is dense in X . Under Martin's Axiom for σ -centered posets, each separable compact Hausdorff space is κ -Baire for every cardinal $\kappa < \mathfrak{c}$. This implies that under Martin's Axiom (for σ -centered posets) the space \mathbb{Z}^κ endowed with the Tychonoff product topology is κ -Baire for every cardinal $\kappa < \mathfrak{c}$. Here \mathfrak{c} stands for the cardinality of continuum. In the statement (4) of the following theorem by \mathfrak{d} we denote the cofinality of the partially ordered set $(\mathbb{N}^\omega, \leq)$. It is known [28] that $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{c}$ under Martin's Axiom (for countable posets).

Let κ be an uncountable cardinal. On the group \mathbb{Z}^κ of all functions $g : \kappa \rightarrow \mathbb{Z}$ consider the shift-invariant topology τ_\uparrow whose neighborhood base at the zero function $e : \kappa \rightarrow \mathbb{Z}$ consists of the sets

$$W_{F,m} = \{g \in \mathbb{Z}^\kappa : g|F = 0, g(\kappa) \subset \{0\} \cup [m, \infty)\}$$

where $m \in \mathbb{N}$ and F runs over finite subsets of κ . The group \mathbb{Z}^κ endowed with the topology τ_\uparrow is a paratopological group, denoted by $\uparrow\mathbb{Z}^\kappa$. Since the group $\uparrow\mathbb{Z}^\kappa$ is abelian, the four standard uniformities of $\uparrow\mathbb{Z}^\kappa$ coincide (i.e., $\mathcal{L} = \mathcal{R} = \mathcal{L} \vee \mathcal{R} = \mathcal{RL}$) whereas the quasi-Roelcke uniformity \mathcal{Q} coincides with the pre-uniformities \mathcal{LL}^{-1} and \mathcal{RR}^{-1} .

Theorem 7.1. *For any uncountable cardinal κ the paratopological group $G = \uparrow\mathbb{Z}^\kappa$ has the following properties:*

- (1) G is a zero-dimensional (and hence regular) Hausdorff abelian paratopological group;
- (2) the topology on G induced by the quasi-Roelcke uniformity \mathcal{Q} coincides with the Tychonoff product topology τ on \mathbb{Z}^κ ;
- (3) $\psi(\mathcal{Q}) = \chi(G) = \kappa$ but $\psi(G) = \overline{\psi}(G) = \omega$;
- (4) $\ell(\mathcal{Q}) = \omega$ but $\ell(\mathcal{L}) \geq \mathfrak{d} > \omega$;
- (5) $c(G) \geq \kappa$ but $dc(G) = \omega$;
- (6) $iw(G) \cdot \omega = sm(G) \cdot \omega \geq \log(2^\kappa)$.
- (7) If $2^\kappa > \mathfrak{c}$, then G is not submetrizable.
- (8) If the space \mathbb{Z}^κ is κ -Baire, then G fails to have G_δ -diagonal and hence is not submetrizable.

Proof. 1. It is clear that the topology τ_\uparrow on $\uparrow\mathbb{Z}^\kappa$ is stronger than the Tychonoff product topology τ on \mathbb{Z}^κ . This implies that the paratopological group $G = \uparrow\mathbb{Z}^\kappa$ is Hausdorff. Observing that each basic neighborhood $W_{F,m}$ of the zero function $e \in \mathbb{Z}^\kappa$ is τ -closed, we conclude that it is τ_\uparrow -closed, which implies that the space $\uparrow\mathbb{Z}^\kappa$ is zero-dimensional and hence regular.

2. Observe that for every basic neighborhood $W_{F,m}$ of zero, the set $W_{F,m} - W_{F,m}$ coincides with the basic neighborhood $W_F = \{g \in \mathbb{Z}^\kappa : g|F = 0\}$ of zero in the Tychonoff product topology τ . This implies that τ coincides with the topology induced by the quasi-Roelcke uniformity \mathcal{Q} .

3. The equality $\chi(G) = \kappa = \psi(\mathcal{Q})$ easily follows from the definition of the topology τ_\uparrow and the fact that the quasi-Roelcke uniformity \mathcal{Q} generates the Tychonoff product topology on \mathbb{Z}^κ . To see that $\psi(G) = \overline{\psi}(G) = \omega$, observe that $\bigcap_{m \in \mathbb{N}} W_{\emptyset,m} = \{e\}$.

4. To see that $\ell(\mathcal{Q}) = \omega$, take any basic open neighborhood $W_{F,m}$ of zero in the group G and observe that $\mathbb{Z}^F = \{g \in \mathbb{Z}^\kappa : g|F = 0\}$ is a countable subgroup of G such that $G = \mathbb{Z}^F + (W_{F,m} - W_{F,m})$, which

implies that $\ell(\mathcal{Q}) \leq \omega$. On the other hand, the boundedness number $\ell(\mathcal{L})$ of the left quasi-uniformity on the paratopological group $\uparrow\mathbb{Z}^\kappa$ is equal to the cofinality of the partially ordered set $(\mathbb{N}^\kappa, \leq)$ which is not smaller than \mathfrak{d} , the cofinality of the partially ordered set $(\mathbb{N}^\omega, \leq)$.

5. For every $x \in \kappa$ denote by $\delta_x : \kappa \rightarrow \{0, 1\} \subset \mathbb{Z}$ the characteristic function of the singleton $\{x\}$ and let $U_x = \delta_x + W_{\{x\}, 2}$ be a basic neighborhood of δ_x . We claim that for any distinct points $x, y \in \kappa$ the sets U_x and U_y are disjoint. To derive a contradiction, assume that $U_x \cap U_y$ contains some function $f \in \mathbb{Z}^\kappa$. The inclusion $f \in U_x$ implies that $f(x) = \delta_x(x) = 1$. On the other hand, $f \in U_y$ implies $f(x) \in \{\delta_y(x)\} \cup [\delta_y(x) + 2, \infty) = \{0\} \cup [2, \infty) \not\ni 1$. So, the closed-and-open sets U_x , $x \in \kappa$, are pairwise disjoint and hence $c(G) \geq |\{U_x\}_{x \in \kappa}| = \kappa$.

By Proposition 1.10, $dc(G) = \ell^{\pm 4}(G)$. So, it suffices to prove that $\ell^{\pm 4}(G) = \omega$. Given a neighborhood assignment V on G , we need to find a countable subset $C \subset G$ such that $B(C; VV^{-1}VV^{-1}) = G$. Using Zorn's Lemma, find a maximal subset $C \subset G$ such that $B(x; VV^{-1}) \cap B(y; VV^{-1}) = \emptyset$ for any distinct points $x, y \in C$. By the maximality of C , for every $x \in G$ there is a point $c \in C$ such that $B(c; VV^{-1}) \cap B(x; VV^{-1}) \neq \emptyset$, which implies $x \in B(C; VV^{-1}VV^{-1})$ and hence $X = B(C; VV^{-1}VV^{-1})$. It remains to prove that the set C is countable. To derive a contradiction, assume that C is uncountable. For every $x \in G$ find a finite subset $F_x \subset \kappa$ and a positive number $m_x \in \mathbb{N}$ such that $x + W_{F_x, m_x} \subset B(x; V)$. By the Δ -system Lemma [16, 16.1], the uncountable set C contains an uncountable subset $D \subset C$ such that the family $(F_x)_{x \in D}$ is a Δ -system with kernel K , which means that $F_x \cap F_y = K$ for any distinct points $x, y \in D$. For every $n \in \mathbb{N}$ and $f \in \mathbb{Z}^K$ consider the subset $D_{n,f} = \{x \in D : x|K = f, m_x \leq n, \sup_{\alpha \in F_x} |x(\alpha)| \leq n\}$ of D and observe that $D = \bigcup_{n \in \mathbb{N}} \bigcup_{f \in \mathbb{Z}^K} D_{n,f}$. By the Pigeonhole Principle, for some $n \in \mathbb{N}$ and $f \in \mathbb{Z}^K$ the set $D_{n,f}$ is uncountable. Consider the clopen subset $\mathbb{Z}^\kappa(f) = \{x \in \mathbb{Z}^\kappa : x|K = f\}$ of \mathbb{Z}^κ . Since $\mathbb{Z}^\kappa(f)$ is a Baire space, for some $m \in \mathbb{N}$ the set $X_m = \{x \in \mathbb{Z}^\kappa(f) : m_x = m\}$ is not nowhere dense in $\mathbb{Z}^\kappa(f)$. Consequently, there is a finite subset $\bar{K} \subset \kappa$ containing K and a function $\bar{f} : \bar{K} \rightarrow \mathbb{Z}$ such that the set $X_m \cap \mathbb{Z}^\kappa(\bar{f})$ is dense in $\mathbb{Z}^\kappa(\bar{f}) = \{x \in \mathbb{Z}^\kappa : x|\bar{K} = \bar{f}\}$. Since the family $(F_x \setminus K)_{x \in D}$ is disjoint, the set $\{x \in D : (F_x \setminus K) \cap \bar{K} \neq \emptyset\}$ is finite, so we can find two functions $x, y \in D_{n,f}$ such that $(F_x \cup F_y) \cap \bar{K} = K$. Put $\tilde{K} = F_x \cup F_y \cup K$ and choose any function $\tilde{f} : \tilde{K} \rightarrow \mathbb{Z}$ such that $\tilde{f}|_{\bar{K}} = \bar{f}$ and $f(\alpha) < -n - m$ for any $\alpha \in \tilde{K} \setminus \bar{K}$. The function \tilde{f} determines a non-empty open set $\mathbb{Z}^\kappa(\tilde{f}) = \{z \in \mathbb{Z}^\kappa : z|\tilde{K} = \tilde{f}\}$, which contains some function $z \in X_m$ (by the density of $X_m \cap \mathbb{Z}^\kappa(\bar{f})$ in $\mathbb{Z}^\kappa(\bar{f})$). Choose a function $\tilde{z} \in \mathbb{Z}^\kappa$ such

that $\tilde{z}|_{F_x} = x|_{F_x}$ and $\tilde{z}(\alpha) \geq \max\{m + z(\alpha), m_x + x(\alpha)\}$ for every $\alpha \in \kappa \setminus F_x$. Then $\tilde{z} \in (z + W_{F_z, m}) \cap (x + W_{F_x, m_x}) \subset B(z; V) \cap B(x; V)$, which implies $z \in B(x; VV^{-1})$. By analogy we can prove that $z \in B(y; VV^{-1})$. So, $B(x; VV^{-1}) \cap B(y; VV^{-1}) \neq \emptyset$, which contradicts the choice of the set $C \ni x, y$. This contradiction shows that C is countable and hence $dc(G) = \ell^{\pm 4}(G) = \omega$.

6. By Proposition 1.4, $iw(G) \cdot \omega = sm(G) \cdot \log(dc(G)) = sm(G) \cdot \omega$. On the other hand, $2^\kappa = |G| \leq |[0, 1]^{iw(G)}| = |2^{iw(G) \cdot \omega}|$ implies that $\log(2^\kappa) \leq iw(G) \cdot \omega$.

7. If $2^\kappa > \mathfrak{c}$, then $sm(G) \cdot \omega \geq \log(2^\kappa) \geq \log(\mathfrak{c}^+) > \omega$, which implies that $sm(G) > \omega$ and hence G is not submetrizable.

8. Suppose that the space \mathbb{Z}^κ is κ -Baire. Assuming that the space $G = \uparrow \mathbb{Z}^\kappa$ has G_δ -diagonal, we can apply Theorem 2.2 in [13] and find a countable family $(\mathcal{U}_n)_{n \in \mathbb{N}}$ open covers of G , which separates the points of G in the sense that for every distinct points $f, g \in G$ there is $n \in \mathbb{N}$ such that no set $U \in \mathcal{U}_n$ contains both points f and g . Since the space G is zero-dimensional, we can assume that each set $U \in \bigcup_{n \in \omega} \mathcal{U}_n$ is closed-and-open in G . Put $\mathcal{U}_0 = \{G\}$.

We shall construct an increasing sequence $(F_n)_{n \in \omega}$ of finite subsets and a sequence $f_n \in \mathbb{Z}^{F_n}$, $n \in \omega$, of functions such that for every $n \in \omega$ the clopen set $\mathbb{Z}^\kappa(f_n) = \{f \in \mathbb{Z}^\kappa : f|_{F_n} = f_n\}$ is contained in $U_n \cap \mathbb{Z}^\kappa(f_{n-1})$ for some set $U_n \in \mathcal{U}_n$.

We start the inductive construction letting $F_0 = \emptyset$ and $f_0 : \emptyset \rightarrow \mathbb{Z}$ be the unique function. Then $\mathbb{Z}^\kappa(f_0) = \mathbb{Z}^\kappa \in \mathcal{U}_0$. Assume that for some $n \in \mathbb{Z}$ we have defined a finite set $F_{n-1} \subset \kappa$ and a function $f_{n-1} \in \mathbb{Z}^{F_{n-1}}$ such that $\mathbb{Z}^\kappa(f_{n-1}) \subset U_{n-1}$ for some $U_{n-1} \in \mathcal{U}_{n-1}$.

The \mathcal{F} being the family of all triples (F, f, m) where F is a finite subset of κ containing F_{n-1} , $f : F \rightarrow \mathbb{Z}$ is a function extending the function f_{n-1} and $m \in \mathbb{N}$ is a positive integer. Observe that $|\mathcal{F}| = \kappa$. For every function $g \in \uparrow \mathbb{Z}^\kappa$ choose a closed-and-open subset $U_g \in \mathcal{U}_n$ containing g and choose a finite subset $F_g \subset \kappa$ containing F_{n-1} and a number m_g such that $g + W_{F_g, m_g} \subset U_g$. For every triple $(F, f, m) \in \mathcal{F}$ consider the subset $Z_{(F, f, m)} = \{g \in \uparrow \mathbb{Z}^\kappa : (F_g, g|_{F_g}, m_g) = (F, f, m)\}$ and observe that $\mathbb{Z}^\kappa(f_{n-1}) = \bigcup_{(F, f, m) \in \mathcal{F}} Z_{(F, f, m)}$. Since the space $\mathbb{Z}^\kappa(f_{n-1})$ is κ -Baire, there is a triple $(F, f, m) \in \mathcal{F}$ such that the set $Z_{(F, f, m)}$ is not nowhere dense in $\mathbb{Z}^\kappa(f_{n-1})$. Consequently we can find a finite set $F_n \subset \kappa$ and a function $f_n \in \mathbb{Z}^{F_n}$ such that for the basic open set $\mathbb{Z}^\kappa(f_n) = \{g \in \mathbb{Z}^\kappa : g|_{F_n} = f_n\}$ the intersection $\mathbb{Z}^\kappa(f_n) \cap Z_{(F, f, m)}$ is dense in $\mathbb{Z}^\kappa(f_n)$. It follows that $F_n \supset F \supset F_{n-1}$ and $f_n|_F = f$. Choose any point $g \in Z_{(F, f, m)} \cap \mathbb{Z}^\kappa(f_n)$.

We claim that $\mathbb{Z}^\kappa(f_n) \subset U_g \in \mathcal{U}$. Assuming that $\mathbb{Z}^\kappa(f_n) \not\subset U_g$, choose a function $h \in \mathbb{Z}^\kappa(f_n) \setminus U_g$ and find a basic neighborhood $h + W_{E,l} \subset \mathbb{Z}^\kappa(f_n) \setminus U_g$ of h . It follows from the inclusion $h + W_{E,l} \subset \mathbb{Z}^\kappa(f_n)$ that $E \supset F_n \supset F$ and $h|_{F_n} = f_n$. Then $h|_F = f_n|_F = f$. Choose a function $\tilde{h} : \kappa \rightarrow \mathbb{Z}$ such that $\tilde{h}|_E = h|_E$ and $\tilde{h}(x) \geq \max\{g(x) + m, h(x) + l\}$ for every $x \in \kappa \setminus E$. Then $\tilde{h} \in (h + W_{E,l}) \cap (g + W_{F,m}) \subset (\mathbb{Z}^\kappa(f_n) \setminus U_g) \cap U_g = \emptyset$, which is a desired contradiction completing the inductive step.

After completing the inductive construction, consider the countable set $F_\omega = \bigcup_{n \in \omega} F_n$ and the function $f_\omega : F_\omega \rightarrow \mathbb{Z}$ such that $f_\omega|_{F_n} = f_n$ for all $n \in \omega$. Since the complement $\kappa \setminus F_\omega$ is not empty, the “cube” $\mathbb{Z}^\kappa(f_\omega) = \{g \in \mathbb{Z}^\kappa : g|_{F_\omega} = f_\omega\}$ contains two distinct functions f, g . By the choice of the family $(\mathcal{U}_n)_{n \in \omega}$ there is a number $n \in \omega$ such that no set $U \in \mathcal{U}_n$ contains both points f and g . On the other hand, by the inductive construction, $f, g \in \mathbb{Z}^\kappa(f_\omega) \subset \mathbb{Z}^\kappa(f_n) \subset U_n$ for some set $U_n \in \mathcal{U}$, which is a desired contradiction completing the proof of the theorem. \square

Corollary 7.2. *For every cardinal $\kappa \geq \mathfrak{c}$ the paratopological group $\uparrow\mathbb{Z}^\kappa$ has countable pseudocharacter but fails to be submetrizable.*

It is known [31] that under Martin’s Axiom the space \mathbb{Z}^κ is κ -Baire for every cardinal $\kappa < \mathfrak{c}$. This fact combined with Theorem 7.6 (7, 8) implies the following MA-improvement of Corollary 7.2.

Corollary 7.3. *Under Martin’s Axiom, for any uncountable cardinal κ the paratopological group $\uparrow\mathbb{Z}^\kappa$ has countable pseudocharacter but fails to be submetrizable.*

Problem 7.4. *Can the space $\uparrow\mathbb{Z}^{\omega_1}$ be submetrizable in some model of ZFC?*

In Theorem 7.1 we proved that the paratopological group $G = \uparrow\mathbb{Z}^\kappa$ has $d(G) \geq c(G) \geq \kappa$ and $dc(G) = \omega$. By Propositions 1.3 and 1.10, $\ell^{\pm 4}(G) = \tilde{\ell}^{*1\frac{1}{2}}(G) = dc(G) = \omega$. It would be interesting to know the values of some other cardinal characteristics of G , intermediate between $dc(G)$ and $c(G)$.

Problem 7.5. *For the paratopological group $G = \uparrow\mathbb{Z}^\kappa$ calculate the values of cardinal characteristics $\ell^{\pm n}(G)$, $\ell^{\mp n}(G)$, $\ell^{\wedge n}(G)$, $\ell^{\vee n}(G)$ for all $n \in \mathbb{N}$.*

7.2. A submetrizable paratopological group whose quasi-Roelcke uniformity has uncountable pseudocharacter. By Theorem 6.4, each Hausdorff paratopological group G has submetrizability number $sm(G) \leq \psi(\mathcal{Q})$. This inequality can be strict as shown by an example constructed in this subsection.

Given an uncountable cardinal κ in the paratopological group $\uparrow\mathbb{Z}^\kappa$ consider the subgroup $H = \{f \in \uparrow\mathbb{Z}^\kappa : |\text{supp}(f)| < \omega\}$ consisting of functions $f : \kappa \rightarrow \mathbb{Z}$ that have finite support $\text{supp}(f) = \{\alpha \in \kappa : f(\alpha) \neq 0\}$. A neighborhood base of H at zero consists of the sets

$$W_{F,m} = \{h \in H : h|_F = 0, h(\kappa) \in \{0\} \cup [m, \infty)\}$$

where F runs over finite subsets of κ and $m \in \mathbb{N}$.

Theorem 7.6. *For any uncountable cardinal κ the paratopological group H has the following properties:*

- (1) H is a zero-dimensional (and hence regular) Hausdorff abelian paratopological group;
- (2) H is strongly σ -discrete and submetrizable;
- (3) $iw(H) \cdot \omega = \log(\kappa)$;
- (4) $\psi(\mathcal{Q}) = \chi(H) = \kappa$ but $\psi(H) = \overline{\psi}(H) = \omega$;
- (5) $\ell(\mathcal{Q}) = \omega$ but $\ell(\mathcal{L}) = dc(H) = \kappa$.

Proof. The items (1), (4), (5) follow (or can be proved by analogy with) the corresponding items of Theorem 7.1.

(2)–(3): To see that the space H is strongly σ -discrete, write $H = \bigcup_{n,m \in \omega} H_{n,m}$ where $H_{n,m} = \{h \in \uparrow\mathbb{Z}^\kappa : |\text{supp}(h)| = n, \|h\| \leq m\}$ and $\|h\| = \sup_{\alpha \in \kappa} |h(\alpha)|$. We claim that each set $H_{n,m}$ is strongly discrete in H . To each function $h \in H_{n,m}$ assign the neighborhood $U_h = h + W_{\text{supp}(h), m+1}$. Given any two distinct functions $g, h \in H_{n,m}$, we shall prove that $U_g \cap U_h = \emptyset$. Assuming that $U_g \cap U_h$ contains some function $f \in H$, we would conclude that $f|_{\text{supp}(g)} = g|_{\text{supp}(g)}$ and $f|_{\text{supp}(h)} = h|_{\text{supp}(h)}$. So, $g|_{\text{supp}(g)} \cap \text{supp}(h) = h|_{\text{supp}(g)} \cap \text{supp}(h)$ and $g \neq h$ implies that $\text{supp}(g) \neq \text{supp}(h)$. Since $|\text{supp}(g)| = |\text{supp}(h)| = n$, there is $\alpha \in \text{supp}(g) \setminus \text{supp}(h)$ such that $g(\alpha) \neq 0 = h(\alpha)$. Then $f(\alpha) \in \{g(\alpha)\} \cap [m+1, \infty) \subset [-m, m] \cap [m+1, \infty) = \emptyset$, which is a contradiction showing that the indexed family $(U_h)_{h \in H_{n,m}}$ is disjoint.

To show that this family $(U_h)_{h \in H_{n,m}}$ is discrete, for every function $g \in H \setminus \bigcup_{h \in H_{n,m}} U_h$ consider its neighborhood $U_g = g + W_{\text{supp}(g), m+1}$. We claim that $U_g \cap U_h = \emptyset$ for every $h \in H_{n,m}$. Assume conversely that for some $h \in H_{n,m}$ the intersection $U_g \cap U_h$ contains a function $f \in H$. Then $f|_{\text{supp}(g)} = g|_{\text{supp}(g)}$ and $f|_{\text{supp}(h)} = h|_{\text{supp}(h)}$, which implies $\text{supp}(g) \neq \text{supp}(h)$. If $\text{supp}(h) \setminus \text{supp}(g) \neq \emptyset$, then we can find $\alpha \in \text{supp}(h) \setminus \text{supp}(g)$ and conclude that $f(\alpha) = h(\alpha) \neq 0 = g(\alpha)$ and hence $f(\alpha) \in \{h(\alpha)\} \in [-m, m] \cap [m+1, \infty) = \emptyset$, which is a contradiction. So, $\text{supp}(h) \subset \text{supp}(g)$ and $g|_{\text{supp}(h)} = h|_{\text{supp}(h)}$. It follows from $g \notin U_h$ that for some $\alpha \in \kappa \setminus \text{supp}(h)$ we get $g(\alpha) \notin \{0\} \cup [m+1, \infty)$. Then $\alpha \in \text{supp}(g)$ and $f(\alpha) = g(\alpha) \notin [m+1, \infty)$. On the other hand, the inclusion $f \in U_h$ and $f(\alpha) \neq 0 = h(\alpha)$ implies $f(\alpha) \in [m+1, \infty)$.

This contradiction completes the proof of the equality $U_g \cap U_h = \emptyset$, which shows that the family $(U_h)_{h \in H_n}$ is discrete in H and the set $H_{n,m}$ is strongly discrete in H . Then the space $H = \bigcup_{n,m \in \omega} H_{n,m}$ is strongly σ -discrete. By Proposition 1.1 it is submetrizable and has i -weight $iw(H) \cdot \omega = \log(|H|) = \log(\kappa)$. \square

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