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TOPOLOGY PROCEEDINGS

Volume 47, 2016

Pages 261–271

http://topology.nipissingu.ca/tp/

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Electronically published on October 19, 2015

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 47 (2016)
Pages 261-271

E-Published on October 19, 2015

CONCERNING GENERALIZED QUASIMETRIC AND QUASI-UNIFORMITY FOR TOPOLOGICAL SPACES

M. N. MUKHERJEE, A. DEBRAY, AND S. SINHA

Abstract. In this paper, we give a construction of generalized quasimetric for an arbitrary quasi-uniform space. Here we also note that the generalized quasimetric in [5] for an arbitrary topological space gives the Pervin's quasi-uniformity for the space. Finally, we study some categorical relations between quasi-uniform spaces and generalized quasimetric spaces.

1. Introduction

It is well known that not every topological space is generated from some metric space, i.e., there are topological spaces which are not metrizable. Hence, untill now many mathematicians have constructed different kinds of generalizations of metrics and have represented arbitrary topological spaces in terms of those generalized versions of metrics (see [4], [5]). In [5], Kopperman gave such a generalization and that was further modified in [7]. Here, in this paper, we have considered this modified version and have defined the quasi-uniformity, which arises quite naturally from this generalized version of metric. Pervin [10] proved that the topology τ of any topological space (X, τ) is induced by some transitive quasi-uniformity, a subbase for which is given by the collection, $\{T(G, X \setminus G) : G \in \tau\}$, where $T(G, X \setminus G)$ stands for $(X \times X \setminus (G \times (X \setminus G)))$. In this context we have observed that the particular generalized quasimetric construction for any arbitrary topological space in [5], actually gives rise to the Pervin's quasi-uniformity for that topological space. For detailed discussion regarding quasi-uniformization of topological spaces [2] and [9] may be consulted.

2010 *Mathematics Subject Classification.* 54E15, 54E35, 54A05, 54B30.
Key words and phrases. Quasi-uniformity, generalized quasimetric, value lattice.
The third author is supported by University Grant Commission, Government of India, under research grant UGC/540/Jr. Fellow(Sc.), dated 13.06.2012.

At this point we have a generalized quasimetric structure for an arbitrary topological space and this generalized quasimetric gives a quasi-uniformity for the topological space. Again it is known that an arbitrary topological space possesses a compatible quasi-uniform structure, e.g. the Pervin's quasi-uniformity. Then a natural question arises whether an arbitrary quasi-uniform space possesses a generalized quasimetric structure, compatible with the given quasi-uniformity. We answer it in the affirmative and give a construction.

In the last section we consider three categories, namely QU of all quasi-uniform spaces with quasi-uniformly continuous functions, QM of all generalized quasimetric spaces with continuous functions and QM^* of all generalized quasimetric spaces with uniformly continuous functions and study some interrelations among these three. To arrive at these descriptions, we briefly undertake towards the end of Section 3, certain discussion on continuity and uniform continuity of functions between two generalized quasimetric spaces, and quasi-uniform continuity of functions between two quasi-uniform spaces.

2. Preliminaries

The concept of continuous lattice has been discussed in detail in [1] and [3].

Definition 2.1. A lattice is a non-empty set L with a relation '$\leq$' such that

(1) $l \leq l$, $\forall l \in L$. (reflexive)
(2) $l \leq m, m \leq l$ implies, $l = m$, $\forall l, m \in L$. (antisymmetric)
(3) $l \leq m, m \leq n$ implies, $l \leq n$, $\forall l, m, n \in L$. (transitive)
(4) any two elements of L have least upper bound.

Definition 2.2. A lattice $(L, \leq)$ is said to be a complete lattice if for any non-empty subset A of L, supremum of A exists in A.

Definition 2.3. A non-empty set D with a relation '$\leq$' is said to be directed if

(1) '$\leq$' is reflexive.
(2) '$\leq$' is transitive.
(3) for $a, b \in D$, $\exists c \in D$ such that $a \leq c$, $b \leq c$.

Notation 2.4. *In a complete lattice* $(L, \leq)$,

(1) *'$l \ll m$' means if $m \leq supD$, for some directed subset D of L, then $\exists d \in D$ such that $l \leq d$.*
(2) $\{l \in L : l \ll m\} = \Downarrow m$.

Definition 2.5. A complete lattice $(L, \leq)$ is said to be a continuous lattice if for each $m \in L$, $\Downarrow m$ is directed and m is its supremum.

Definition 2.6 ([6])**.** A set V with a binary operation '$+$' and a relation '$\leq$' is said to be a value lattice if

(1) $(V, \geq)$ is a continuous lattice, where $a \geq b$ iff $b \leq a$, for $a, b \in V$.
(2) $(V, +)$ is a commutative semigroup.
(3) $v + 0 = v$, $\forall v \in V$, where '0' is the least element of $(V, \leq)$.
(4) $v + infA = inf\{v + a : a \in A\}$, $\forall v \in V$ and $A \subseteq V$ with $A \neq \phi$.

Lemma 2.7 ([7])**.** *The greatest element '∞' of $(V, \leq)$ is the absorbing element of $(V, +)$ i.e., $v + \infty = \infty$, $\forall v \in V$.*

Definition 2.8 ([7])**.** Let $(V, \leq)$ be a poset and $P \subseteq V$. Then P is said to

(1) be an upper subset if $p \in P$ and $p \leq q$ implies, $q \in P$, $\forall p, q \in V$.
(2) be separating if $(V, \leq, +)$ is a value lattice and P is an upper subset, and if $a \leq b + p$, $\forall p \in P$ implies, $a \leq b$, $\forall a, b \in V$.
(3) filtered if $p, q \in P$ implies, $\exists r \in P$ such that $r \leq p, r \leq q$.
(4) have halves if $(V, \leq, +)$ is a value lattice and if for each $p \in P$, $\exists q \in P$ such that $q + q \leq p$.
(5) be a set of positives if it is a filtered upper set with halves.

Definition 2.9 ([7])**.** A V-quasimetric space is a quadruple (X, V, P, d) such that X is a non-empty set, V is a value lattice, P is a separating upper subset of V and $d : X \times X \longrightarrow V$ such that

(1) $d(x, x) = 0$, $\forall x \in X$.
(2) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

Here d is called a V-quasimetric.

Definition 2.10 ([7])**.** A generalized quasimetric space is a V-quasimetric space for some value lattice V.

Definition 2.11 ([7])**.** Let (X, V, P, d) be a V-quasimetric space with P a separating set of positives. Then, $\tau(d) = \{G \subseteq X : x \in G \Rightarrow \exists p \in P$ such that $N_p(x) = \{y : d(x, y) \leq p\} \subseteq G\}$ is a topology on X, and it is called the topology generated by the generalized quasimetric d.

Lastly we recall a result (see [5], [7] for details) that is crucial for our purpose. We also reproduce here a sketch of its proof, as we will need to refer it, in the sequel, to the construction of the concerned generalized quasimetric.

Theorem 2.12 ([7],[5])**.** *Every topology is generated by some generalized quasimetric.*

Proof. Let, (X, τ) be a topological space. Then consider the sets $V = \mathbb{R}^\tau$, $P = \{p = (p_G)_{G \in \tau} : p_G \in (0, \infty], \forall G \in \tau$ and $p_G = \infty$ for all but a finite number of G's$\}$ and $\forall x, y \in X$, the function $d : X \times X \longrightarrow \mathbb{R}^\tau$ as,

$$(d(x,y))_G = d_G(x,y) = \begin{cases} q, & \text{if } x \in G, y \notin G \\ 0, & \text{otherwise} \end{cases}$$

where $q \in (0, \infty]$ is arbitrarily fixed.

Now, it can be shown that $(V, +, \leq)$ is a value lattice and P a separating set of positives of V, where '$+$' and '$\leq$' are defined coordinatewise. Then (X, V, P, d) becomes a V-quasimetric space and the generalized quasimetric d generates the topology τ. □

In the rest of this paper we require only those generalized quasi-metric spaces (X, V, P, d), where P is a separating set of positives. So, henceforth by a generalized quasi-metric space we shall mean a generalized quasimetric space with separating set of positives.

3. Generalized Quasimetric and Quasi-Uniformity

Theorem 3.1. *Every generalized quasimetric space gives rise to a quasi-uniform space and the corresponding topologies on the underlying set for both the quasimetric and the quasi-uniformity are same.*

Proof. Let, (X, V, P, d) be a generalized quasimetric space. Then it is easy to see that the collection $\mathcal{U} = \{\{(x,y) \in X \times X : d(x,y) \leq p\} : p \in P\}$ forms a base for some quasi-uniformity on X. It also is a routine check that both d and $\mathcal{U}$ generate the same topology on X. □

The following result examplifies the above theorem.

Theorem 3.2. *The generalized quasimetric of Theorem 2.12 induces Pervin's quasi-uniformity on any topological space (X, τ).*

Proof. Consider the generalized quasimetric space described in Theorem 2.12. Now take $p = (p_G)_{G \in \tau} \in P$ and consider a typical basic member $U = \{(x,y) \in X \times X : d(x,y) \leq p\}$ of the quasi-uniformity $\mathcal{U}$ (as in the proof of the above theorem). Then, $U = \{(x,y) \in X \times X : d_G(x,y) \leq p_G, \forall G \in \tau\} = \bigcap\limits_{G \in \tau} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\} = (\bigcap\limits_{\substack{G \in \tau \\ p_G \neq \infty}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\}) \bigcap (\bigcap\limits_{\substack{G \in \tau \\ p_G = \infty}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\}) = (\bigcap\limits_{\substack{G \in \tau \\ p_G \neq \infty}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\}) \bigcap (X \times X) = (\bigcap\limits_{\substack{G \in \tau \\ p_G \neq \infty}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\} = (\bigcap\limits_{\substack{G \in \tau \\ p_G < q}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\}) \bigcap (\bigcap\limits_{\substack{G \in \tau \\ p_G \geq q}} \{(x,y) \in X \times X : d_G(x,y) \leq p_G\}) =$

$(\bigcap\limits_{\substack{G\in\tau\\ p_G<q}} \{(x,y) \in X\times X : d_G(x,y) \leq p_G\}) \bigcap (X\times X) = \bigcap\limits_{\substack{G\in\tau\\ p_G<q}} \{(x,y) \in X\times X : d_G(x,y) \leq p_G\} = \bigcap\limits_{\substack{G\in\tau\\ p_G<q}} (X \times X \setminus (G \times (X \setminus G))) = \bigcap\limits_{\substack{G\in\tau\\ p_G<q}} T(G, X \setminus G)$, writing $T(G, X \setminus G)$ for $(X \times X \setminus (G \times (X \setminus G)))$. Now, as $\{p_G : p_G < q\}$ is finite, $S = \{T(G, X \setminus G) : G \in \tau\}$ turns out to be a subbase for the quasi-uniformity $\mathcal{U}$, induced by d and hence $\mathcal{U}$ is the Pervin's quasi-uniformity on (X, τ). □

Henceforth the quasi-uniformity generated by a generalized quasimetric space (X, V, P, d) will be denoted by $\mathcal{U}(d)$.

Theorem 3.3. *Each quasi-uniformity is generated by some generalized quasimetric.*

Proof. Let $(X, \mathcal{U})$ be a quasi-uniform space, with a base $\mathcal{B}$. Then consider the sets $V = \mathbb{R}^{\mathcal{B}}$, $P = \{p = (p_B)_{B\in\mathcal{B}} : p_B \in (0, \infty], \forall B \in \mathcal{B}$ and $p_B = \infty$ for all but a finite number of B's$\}$ and $\forall x, y \in X$, the function $d : X \times X \longrightarrow \mathbb{R}^{\mathcal{B}}$ as,

$$(d(x,y))_B = d_B(x,y) = \begin{cases} q(1 - 1/2^{n-1}), & \text{if } (x,y) \in B^n \setminus B^{n-1} \\ & \text{for some } n \in \mathbb{N} \\ q, & \text{if } (x,y) \notin B^n \text{ for all } n \in \mathbb{N} \end{cases}$$

where $0 < q < \infty$ is arbitrarily fixed.

Then (X, V, P, d) is a V-quasimetric space. Let the quasi-uniformity generated by d be $\mathcal{U}(d)$. Now take $p = (p_B)_{B\in\mathcal{B}} \in P$ and consider a typical basic member $W = \{(x,y) \in X \times X : d(x,y) \leq p\}$ of $\mathcal{U}(d)$. Then, $W = \{(x,y) \in X \times X : d_B(x,y) \leq p_B, \forall B \in \mathcal{B}\} = \bigcap\limits_{B\in\mathcal{B}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\} = (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B\neq\infty}} \{(x,y) \in X\times X : d_B(x,y) \leq p_B\}) \bigcap (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B=\infty}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\}) = (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B\neq\infty}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\}) \bigcap (X\times X) = \bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B\neq\infty}} \{(x,y) \in X\times X : d_B(x,y) \leq p_B\} = (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B<q}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\}) \bigcap (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B\geq q}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\}) = (\bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B<q}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\}) \bigcap (X \times X) = \bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B<q}} \{(x,y) \in X \times X : d_B(x,y) \leq p_B\} = \bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B<q}} \{(x,y) \in X \times X : (x,y) \in B^{n_B}$, for some $n_B \in \mathbb{N}\} = \bigcap\limits_{\substack{B\in\mathcal{B}\\ p_B<q}} B^{n_B}$, where $n_B \in \mathbb{N}, \forall B \in \mathcal{B}$. Now, $\{p_B : p_B < q\}$ is finite and hence $W \in \mathcal{U}$. Thus $\mathcal{U}(d) \subseteq \mathcal{U}$. Again take an arbitrary $B \in \mathcal{B}$ and $p = (p_{B'})_{B' \in \mathcal{B}} \in P$ such that

$$p_{B'} = \begin{cases} q/4, & \text{if } B' = B \\ \infty, & \text{if } B' \neq B \end{cases}$$

Then clearly $B = \{(x,y) \in X \times X : d(x,y) \leq p\} \in \mathcal{U}(d)$. Thus, $\mathcal{U} \subseteq \mathcal{U}(d)$. Hence d induces the same quasi-uniformity $\mathcal{U}$ on X. □

Remark 3.4. If in the above Theorem, $\mathcal{B}$ is transitive then d takes the following simple form:

$$(d(x,y))_B = d_B(x,y) = \begin{cases} 0, & \text{if } (x,y) \in B \\ q, & \text{if } (x,y) \notin B^n \text{ for any } n \in \mathbb{N} \end{cases}$$

Proof. The result follows trivially from the fact that as $\mathcal{B}$ is transitive, if $B \in \mathcal{B}$ then $B^n = B, \forall n \in \mathbb{N}$. □

Definition 3.5 ([2])**.** Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be two quasi-uniform spaces. A function $f : (X,\mathcal{U}_X) \longrightarrow (Y,\mathcal{U}_Y)$ is said to be quasi-uniformly continuous if for each $U_Y \in \mathcal{U}_Y, \exists U_X \in \mathcal{U}_X$ such that $(x_1, x_2) \in \mathcal{U}_X \Longrightarrow (f(x_1), f(x_2)) \in \mathcal{U}_Y$.

Definition 3.6 ([5], [7])**.** Let (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) be two generalized quasimetric spaces. A function $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is said to be continuous if for each $x \in X$ and $r_Y \in P_Y, \exists r_X \in P_X$ such that $d_X(x, x') \leq r_X \Longrightarrow d_Y(f(x), f(x')) \leq r_Y, \forall x' \in X$.

Let us now define as follows.

Definition 3.7. Let (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) be two generalized quasimetric spaces. A function $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is said to be uniformly continuous if for each $r_Y \in P_Y, \exists r_X \in P_X$ such that $d_X(x_1, x_2) \leq r_X \Longrightarrow d_Y(f(x_1), f(x_2)) \leq r_Y$.

Theorem 3.8. *Let (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) be two generalized quasimetric spaces. If $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is uniformly continuous then it is continuous.*

Proof. Immediate. □

Theorem 3.9. *Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be two quasi-uniform spaces, and (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) be two generalized quasimetric spaces, such that $\mathcal{U}_X = \mathcal{U}(d_X)$ and $\mathcal{U}_Y = \mathcal{U}(d_Y)$. If $f : (X,\mathcal{U}_X) \longrightarrow (Y,\mathcal{U}_Y)$ is quasi-uniformly continuous, then $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is uniformly continuous and thus continuous.*

Proof. Let $r_Y \in P_Y$. Put, $\{(y_1, y_2) \in Y \times Y : d_Y(y_1, y_2) \leq r_Y\} = U_Y$. So $U_Y \in \mathcal{U}_Y$. Now as $f : (X,\mathcal{U}_X) \longrightarrow (Y,\mathcal{U}_Y)$ is quasi-uniformly continuous, $\exists U_X \in \mathcal{U}_X$ such that $(a, b) \in U_X \implies (f(a), f(b)) \in U_Y$. Again, as $\mathcal{U}_X = \mathcal{U}(d_X)$, $\exists r_X \in P_X$ such that $\{(a, b) \in X \times X : d_X(a, b) \leq r_X\} \subseteq U_X$, i.e. $d_X(a, b) \leq r_X \implies d_Y(f(a), f(b)) \leq r_Y$. Thus, $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is uniformly continuous and hence continuous by Theorem 3.8. □

Theorem 3.10. *Let (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) be two generalized quasimetric spaces and $(X,\mathcal{U}(d_X))$ and $(Y,\mathcal{U}(d_Y))$ be the corresponding quasi-uniform spaces generated by d_X and d_Y respectively. If $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is uniformly continuous then $f : (X,\mathcal{U}(d_X)) \longrightarrow (Y,\mathcal{U}(d_Y))$ is quasi-uniformly continuous.*

Proof. Let $U_Y \in \mathcal{U}(d_Y)$. Then as $\mathcal{U}(d_Y)$ is generated by d_Y, $\exists r_Y \in P_Y$ such that $W = \{(a, b) \in Y \times Y : d_Y(a, b) \leq r_Y\} \subseteq U_Y$. Now as $f : (X, V_X, P_X, d_X) \longrightarrow (Y, V_Y, P_Y, d_Y)$ is uniformly continuous, $\exists r_X \in P_X$ such that $d_X(x_1, x_2) \leq r_X \implies d_Y(f(x_1), f(x_2) \leq r_Y$. Put, $\{(x_1, x_2) \in X \times X : d_X(x_1, x_2) \leq r_X\} = U_X$. Then clearly $U_X \in \mathcal{U}(d_X)$. Thus $(x_1, x_2) \in U_X \implies (f(x_1), f(x_2)) \in W \subseteq U_Y$. So, $f : (X,\mathcal{U}(d_X)) \longrightarrow (Y,\mathcal{U}(d_Y))$ is quasi-uniformly continuous. □

Theorem 3.11. *Let (X, V, P, d) be a generalized quasimetric space and $(X,\mathcal{U}(d))$ be the quasi-uniform space, generated by the generalized quasimetric d. If (X, V_X, P_X, d_X) is the generalized quasimetric space generated by the quasi-uniformity $\mathcal{U}(d)$ (as in Theorem 3.3) then the identity function $1_X : (X, V_X, P_X, d_X) \longrightarrow (X, V, P, d)$ is uniformly continuous.*

Proof. Let $r \in P$. Then, $U = \{(x, y) \in X \times X : d(x, y) \leq r\} \in \mathcal{U}(d)$. Again, $\mathcal{U}(d) = \mathcal{U}(d_X)$. So, $\exists s \in P_X$ such that $U_s = \{(x, y) \in X \times X : d_X(x, y) \leq s\} \subseteq U$ i.e., $d_X(x, y) \leq s \implies d(x, y) \leq r$. Thus, 1_X is uniformly continuous. □

Note 3.12. From some of the above results it may seem that "quasi-uniformly continuous $\implies$ continuous" can be handled in a better way by generalized quasimetric than by quasi-uniformities and their induced topologies. However, we do not subscribe to that, and our intention here is to describe a few results with their brief demonstrations, as prerequisites for the development of the next section, where we will be concerned with the notions of generalized quasimetric and its compatible quasi-uniformity without giving much importance to the ambient topologies of the spaces, under consideration. As a matter of fact, one does not need to go for generalized quasimetric for proof of such a result.

4. Interrelations among QU, QM and QM^*

In this section we define certain categories and discuss their interrelations. Before we proceed, we list a few facts that can be verified easily.

Fact 4.1. (1) *Composition of any two quasi-uniformly continuous functions between two quasi-uniform spaces is quasi-uniformly continuous.*
(2) *The identity function on a quasi-uniform space is quasi-uniformly continuous.*

Fact 4.2. (1) *Composition of any two continuous* (*uniformly continuous*) *functions between generalized quasi-metric spaces is continuous* (*respectively uniformly continuous*).
(2) *The identity function on a generalized quasimetric space is uniformly continuous* (*and hence continuous*).

In view of Facts 4.1 and 4.2 we get the following categories:

(1) QU, the category of all quasi-uniform spaces and quasi-uniformly continuous functions among them.
(2) QM, the category of all generalized quasimetric spaces and continuous functions among them.
(3) QM^*, the category of all generalized quasimetric spaces and uniformly continuous functions among them.

Theorem 4.3. *$F : QU \longrightarrow QM$ described by*

$$\begin{array}{ccc} (X,\mathcal{U}_X) \longmapsto & & (X, V_X, P_X, d_X) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle F(f)=f} \\ (Y,\mathcal{U}_Y) \longmapsto & & (Y, V_Y, P_Y, d_Y) \end{array}$$

is a faithful functor, where $\mathcal{U}_X = \mathcal{U}(d_X)$, $\mathcal{U}_Y = \mathcal{U}(d_Y)$ and the generalized quasi metric space (X, V_X, P_X, d_X) and (Y, V_Y, P_Y, d_Y) are constructed as in Theorem 3.3.

Proof. Follows from Theorem 3.9. □

Remark 4.4. Consider the real line $\mathbb{R}$. Clearly the usual metric d on $\mathbb{R}$ is a generalized quasimetric and the uniformity generated by d is a quasi-uniformity, say $\mathcal{U}$, on $\mathbb{R}$. There are plenty of real-valued continuous functions on $(\mathbb{R}, d)$ which are not uniformly continuous. So the functor F as described in Theorem 4.3 is not full in general.

Theorem 4.5. *QM^* is a subcategory, but not in general a full subcategory of QM.*

Proof. From Theorem 3.8 it follows that QM^* is a subcategory of QM and by similar argument as in Remark 4.4, it follows that QM^* is not a full subcategory of QM. □

Theorem 4.6. *The functor $F : QU \longrightarrow QM^*$ as described in Theorem 4.3, is a fully faithful functor.*

Proof. It follows immediately from Theorem 3.10. □

Theorem 4.7. *$G : QM^* \longrightarrow QU$ described by*

$$\begin{array}{ccc} (X, V_X, P_X, d_X) \longmapsto & & (X, \mathcal{U}_X) \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle G(g)=g} \\ (Y, V_Y, P_Y, d_Y) \longmapsto & & (Y, \mathcal{U}_Y) \end{array}$$

is a fully faithful functor, where $\mathcal{U}_X = \mathcal{U}(d_X)$ and $\mathcal{U}_Y = \mathcal{U}(d_Y)$.

Proof. Follows from Theorem 3.9 and Theorem 3.10. □

Definition 4.8 ([8])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a triple $< S, T, \phi >: \mathcal{C} \to \mathcal{D}$, where $S : \mathcal{C} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{C}$ are two functors and ϕ is a function which assigns to each pair of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and each arrow $f : S(C) \to D$, an arrow $\phi(f) : C \to T(D)$ in such a way that $\phi(k \circ f) = T(k) \circ \phi(f)$ and $\phi(f \circ S(h)) = \phi(f) \circ h$ hold for all f and all arrows $h : C' \to C$ and $k : D \to D'$.

In such a case, S and T are called adjoint to each other.

Theorem 4.9. *The two functors F and G described in Theorem 4.6 and 4.7 are adjoint to each other.*

Proof. Let $(X, \mathcal{U}_X)$ and (M, V_M, P_M, d_M) be two objects of QU and QM^* respectively. Now consider $g : F(X, \mathcal{U}_X) \longrightarrow (M, V_M, P_M, d_M)$. Then $g : (X, V_X.P_X, d_X) \longrightarrow (M, V_M, P_M, d_M)$ is a uniformly continuous function. Now take $\phi(g) : (X, \mathcal{U}_X) \longrightarrow G(M, V_M, P_M, d_M)$, i.e., $\phi(g) : (X, \mathcal{U}_X) \longrightarrow (M, \mathcal{U}(d_M))$ as $\phi(g) = g$. Then by Theorem 3.10, $\phi(g)$ is quasi-uniformly continuous. Now let $h : (X, \mathcal{U}_X) \longrightarrow (Y, \mathcal{U}_Y)$ be a quasi-uniformly continuous function and $k : (M, V_M, P_M, d_M) \longrightarrow (N, V_N, P_N, d_N)$ be a uniformly continuous function. Then using Theorem 3.10 it can be easily verified that, $\phi(k \circ f) = G(k) \circ \phi(f)$ and $\phi(f \circ F(h)) = \phi(f) \circ h$. So, by Definition 4.8, F and G are adjoint to each other. □

Definition 4.10 ([8])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $S, T : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\tau : S \dot{\to} T$ is a function which assigns to each object C of $\mathcal{C}$ an arrow $\tau_C : S(C) \to T(C)$ of $\mathcal{D}$ in such a way that for every arrow $f : C \to C'$ in $\mathcal{C}$, the following diagram is commutative.

$$\begin{tikzcd} C \arrow[d, "f"'] & S(C) \arrow[r, "\tau_C"] \arrow[d, "S(f)"'] & T(C) \arrow[d, "T(f)"] \\ C' & S(C') \arrow[r, "\tau_{C'}"'] & T(C') \end{tikzcd}$$

Definition 4.11 ([8])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories; $S : \mathcal{C} \to \mathcal{D}$ be a functor and D be an object of $\mathcal{D}$. A universal arrow from D to S is a pair $< C, g >$ consisting of an object C of $\mathcal{C}$ and an arrow $g : D \to S(C)$ of $\mathcal{D}$ such that every pair $< C', f >$ with an object C' of $\mathcal{C}$ and an arrow $f : D \to S(C')$ of $\mathcal{D}$, there is a unique arrow $f' : C \to C'$ of $\mathcal{C}$ such that $S(f') \circ g = f$.

Theorem 4.12. *In Theorem 4.9, GF and FG are functors on QU and QM^* respectively.*

Proof. It follows from Theorem 3.11 and the fact that $GF(X,\mathcal{U}_X) = (X,\mathcal{U}_X) = I_{QU}(X,\mathcal{U}_X)$. □

Theorem 4.13. *Consider the functors F and G as described in Theorem 4.6 and 4.7. Then the following hold:*

(1) *For each object $(X,\mathcal{U}_X)$ of QU, if $\eta_X : (X,\mathcal{U}_X) \to GF(X,\mathcal{U}_X)$ is the identity 1_X then η_X is a universal arrow from $(X,\mathcal{U}_X)$ to G and $\eta : I_{QU} \dot{\to} GF$ is a natural transformation.*

(2) *For each object (M, V_M, P_M, d_M) of QM^*, if $\varepsilon_M : FG(M, V_M, P_M, d_M) \to (M, V_M, P_M, d_M)$ is the identity 1_M then ε_M is a universal arrow from F to (M, V_M, P_M, d_M) and $\varepsilon : FG \dot{\to} I_{QM^*}$ is a natural transformation.*

(3) *$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$ and $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$ are identities.*

Proof. (1) From definition it follows that η_X is a universal arrow to G from $(X,\mathcal{U}_X)$.

Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be two objects of QU and $f : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ be quasi-uniformly continuous. Then $GF(X,\mathcal{U}_X) = (X,\mathcal{U}_X) = I_{QU}(X,\mathcal{U}_X)$, for all object $(X,\mathcal{U}_X)$ of QU. Also, $I_{QU}(f) = f = GF(f)$. Clearly, the following diagram commutes.:

$$\begin{tikzcd} (X,\mathcal{U}_X) \arrow[d, "f"'] & I_{QU}(X,\mathcal{U}_X) = (X,\mathcal{U}_X) \arrow[r, "\eta_X = 1_X"] \arrow[d, "f = I_{QU}(f)"'] & (X,\mathcal{U}_X) = GF(X,\mathcal{U}_X) \arrow[d, "f = GF(f)"] \\ (Y,\mathcal{U}_Y) & I_{QU}(Y,\mathcal{U}_Y) = (Y,\mathcal{U}_Y) \arrow[r, "\eta_Y = 1_Y"'] & (Y,\mathcal{U}_Y) = GF(Y,\mathcal{U}_Y) \end{tikzcd}$$

Thus η is a natural transformation.
(2) and (3) can be done similarly and hence omitted.

□

Remark 4.14. In view of Theorem 4.13 we may conclude that η is the unit and ε is the counit of the adjunction $< F, G, \phi >$ of Theorem 4.9.

Acknowledgement. The authors are grateful to the learned referee for meticulous reading of the manuscript and for some constructive suggestions towards improvement of the paper.

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