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TOPOLOGY PROCEEDINGS

Volume 47, 2016

Pages 273–278

http://topology.nipissingu.ca/tp/

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Electronically published on October 21, 2015

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

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TOPOLOGY PROCEEDINGS
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Pages 273-278

E-Published on October 21, 2015

REMARKS ON METRIZABILITY OF DUAL GROUPS

FRÉDÉRIC MYNARD AND MIKHAIL TKACHENKO

Abstract. We examine sufficient conditions for the dual of a topological group to be metrizable and locally compact, improving on results of [4].

1. Introduction and preliminaries

If G is an abelian topological group, its dual $\widehat{G}$ is the set of continuous group homomorphisms into the one-dimensional torus $\mathbb{T}$, endowed with the compact-open topology.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set X are said to *mesh,* in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Thus, if the set $\mathbb{F}X$ of filters on a set X is ordered by inclusion, two filters $\mathcal{F}$ and $\mathcal{G}$ admit a supremum $\mathcal{F} \vee \mathcal{G}$ if and only if $\mathcal{F}\#\mathcal{G}$. We do not distinguish between a sequence $(x_n)_{n\in\omega}$ and the filter generated by its tails. In particular, if $\mathcal{H}$ is a family of subsets of X and $(x_n)_{n\in\omega}$ is a sequence on X, the notation $\mathcal{H}\#(x_n)_{n\in\omega}$ means that $H \cap \{x_n : n \geq k\} \neq \varnothing$ for every $H \in \mathcal{H}$ and $k \in \omega$.

Recall that a topological space X is *sequential* if every sequentially closed subset is closed. The space is *Fréchet-Urysohn,* or simply *Fréchet,* if whenever $x \in X$ and $A \subset X$ with $x \in \operatorname{cl} A$, there is a sequence on A that converges to x. It is *strongly Fréchet* if whenever $(A_n)_{n\in\omega}$ is a decreasing sequence of subsets of X with $x \in \bigcap_{n\in\omega} \operatorname{cl} A_n$ there is $x_n \in A_n$ for each n with $x_n \to_n x$, equivalently if

$$\forall \mathcal{H} \in \mathbb{F}_1 X,\ \operatorname{adh} \mathcal{H} \subseteq \operatorname{adh}_{\mathrm{Seq}} \mathcal{H},$$

2010 *Mathematics Subject Classification.* 54H11; 54A20; 54C35; 22A05.

Key words and phrases. Topological group; metrizability; Fréchet-Urysohn; strongly Fréchet; productively Fréchet; hemicompact; locally compact; dual group.

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where $\mathbb{F}_1 X$ denotes the set of countably based filters on X, $\operatorname{adh}\mathcal{H} := \bigcap_{H\in\mathcal{H}} \operatorname{cl} H$ is the adherence of the filter $\mathcal{H}$, and

$$\operatorname{adh}_{\mathrm{Seq}}\mathcal{H} := \bigcup_{(x_n)_n \# \mathcal{H}} \lim (x_n)_n$$

is its sequential adherence. We will also denote by $\operatorname{cl}_{\mathrm{Seq}}$ the sequential closure, that is, the closure in the coarsest sequential topology that is finer than the original.

We are interested in metrization results for dual groups akin to:

Theorem 1.1. [4] *Let G be a metrizable topological group. If $\widehat{G}$ is Fréchet, then it is metrizable and locally compact.*

To this end, let us review its simple proof: A product of a strongly Fréchet space and a metrizable space is (strongly) Fréchet (e.g., [10]), and $\widehat{G}$ is strongly Fréchet as a Fréchet topological group [13], so that $G \times \widehat{G}$ is Fréchet, hence sequential. It is in particular a k-space[1], so that, in view of [9, Proposition 1.2], $\widehat{G}$ is locally compact. But a locally compact topological group with countable tightness (in particular a sequential locally compact one) is metrizable [1]. Hence $\widehat{G}$ is locally compact and metrizable.

Thus, the proof hinges on $G \times \widehat{G}$ being a k-space, and $\widehat{G}$ being of countable tightness. In this note, which can be considered as an appendix to [4], we will consider sufficient conditions on G and $\widehat{G}$ to achieve this result that are different and often weaker than those in Theorem 1.1.

2. Strongly sequential groups

The proof outlined above relies on the following:

Theorem 2.1. [10] *Let X be a topological space. The following are equivalent:*

1. *X is strongly Fréchet;*
2. *$\lim \mathcal{F} = \bigcap_{\mathcal{H}\in\mathbb{F}_1 X} \operatorname{adh}_{\mathrm{Seq}}\mathcal{H}$ for every filter $\mathcal{F}$ on X;*
3. *$X \times Y$ is strongly Fréchet for every bisequential space Y;*
4. *$X \times Y$ is Fréchet for every metrizable atomic[2] topology.*

Strongly sequential spaces are in some sense to sequential spaces like strongly Fréchet spaces are to Fréchet spaces:

[1] A topological space X is a k-space if a subset C is closed whenever $C \cap K$ is closed for every compact subset K of X.

[2] We use the terminology *atomic topology* for a topology with at most one non-isolated point.

Theorem 2.2. [11] *Let X be a T_1 topological space. The following are equivalent:*

(1) *X is strongly sequential;*
(2) *$\lim \mathcal{F} = \bigcap_{\mathcal{H} \in \mathbb{F}_1 X} \mathrm{cl}_{\mathrm{Seq}}(\mathrm{adh}_{\mathrm{Seq}}\, \mathcal{H})$ for every filter $\mathcal{F}$ on X;*
(3) *$X \times Y$ is strongly sequential for every bisequential space Y;*
(4) *$X \times Y$ is sequential for every metrizable atomic topology.*

The proof outlined in the Introduction thus applies virtually unchanged to the effect that:

Proposition 2.3. *Let G be a metrizable topological group. If $\widehat{G}$ is strongly sequential, then it is metrizable and locally compact.*

Since Fréchet topological groups are strongly Fréchet and strongly Fréchet spaces are strongly sequential, Proposition 2.3 formally generalizes Theorem 1.1.

Note that while Theorems 2.1 and 2.2 highlight similarities between strongly Fréchet and strongly sequential spaces, Proposition 2.3 points to an important difference: While Fréchet-Urysohn topological groups are automatically strongly Fréchet, a sequential topological group does not need to be strongly sequential. Indeed, there are metrizable groups G such that $\widehat{G}$ is sequential but not metrizable [4]. In view of Proposition 2.3, such dual groups are sequential but not strongly sequential.

We will now see that Proposition 2.3 generalizes Theorem 1.1 only formally, as both are in fact instances of a general metrization theorem for topological groups that is not specific to dual groups. Indeed, if G is metrizable, then $\widehat{G}$ is a hemicompact k-space [3], [2]. Thus, Theorem 1.1 follows from:

Theorem 2.4. [8] *A hemicompact Fréchet-Urysohn topological group is locally compact and Polish.*

The local compactness (hence also metrization) part of this result follows in turn from a more general fact.

Recall from [12] that a topological space X is called a *Tanaka space* if

(Tanaka condition) $\qquad \mathcal{H} \in \mathbb{F}_1 X,\ \mathrm{adh}\, \mathcal{H} \neq \varnothing \Longrightarrow \mathrm{adh}_{\mathrm{Seq}}\, \mathcal{H} \neq \varnothing.$

Of course, every strongly sequential space, in particular every strongly Fréchet space, is Tanaka. Thus every Fréchet topological group is Tanaka. Therefore Theorem 2.4 is an instance of Corollary 2.7 below, and Proposition 2.3 follows from Corollary 2.7 as well, showing that Theorem 1.1 and Proposition 2.3 are essentially the same.

Theorem 2.5. *A Hausdorff hemicompact Tanaka space has a point with a compact neighborhood.*

Proof. Let $(K_n)_{n\in\omega}$ be an *increasing* sequence of compact subsets such that each compact subset of X is contained in some K_n. If there is n with $\operatorname{int} K_n \neq \varnothing$, we are done. Otherwise, each K_n has empty interior, so that its complement is dense. Thus $(X\setminus K_n)_{n\in\omega}$ is a decreasing sequence of sets with $\bigcap_{n\in\omega} \operatorname{cl}(X \setminus K_n) = X$, but there is no convergent sequence meshing with $(X \setminus K_n)_{n\in\omega}$. Indeed, if there was such a sequence, there would be a subsequence $(x_n)_{n\in\omega}$ with $x_n \in X \setminus K_n$ for each n that converges to a point l. But $K := \{x_n : n \in \omega\} \cup \{l\}$ would then be a compact subset of X that is not contained in any of the K_n's. Thus X is not Tanaka. □

Corollary 2.6. *A homogeneous Hausdorff hemicompact Tanaka space is locally compact.*

Corollary 2.7. *A Hausdorff hemicompact topological group of countable tightness is locally compact and metrizable whenever it is Tanaka.*

Proof. This follows from Corollary 2.6 and the fact that a locally compact topological group of countable tightness is metrizable [1]. □

3. Productively Fréchet groups

Productively Fréchet spaces are those whose product with every strongly Fréchet space is (strongly) Fréchet [6]. To describe them, we first need to define strongly Fréchet filters. A filter $\mathcal{F}$ on X is *strongly Fréchet* if

$$\mathcal{H} \in \mathbb{F}_1 X,\ \mathcal{H}\#\mathcal{F} \Longrightarrow \exists \mathcal{G} \in \mathbb{F}_1 : \mathcal{G} \geq \mathcal{H} \vee \mathcal{F}.$$

Let $\mathbb{F}_1^{\triangle} X$ denote the set of strongly Fréchet filters on X. A topological space X is *productively Fréchet* if

$$\forall \mathcal{H} \in \mathbb{F}_1^{\triangle} X,\ \operatorname{adh} \mathcal{H} \subseteq \operatorname{adh}_{\mathrm{Seq}} \mathcal{H}.$$

Theorem 3.1. [6] *The following are equivalent:*

(1) *X is productively Fréchet;*
(2) *$X \times Y$ is strongly Fréchet for every strongly Fréchet space Y;*
(3) *$X \times Y$ is Fréchet for every strongly Fréchet space Y.*

Thus it is clear that the proof of Theorem 1.1 applies with virtually no change to the effect that:

Theorem 3.2. (1) *If G is a productively Fréchet topological group and $\widehat{G}$ is Fréchet-Urysohn, then $\widehat{G}$ is metrizable and locally compact.*
(2) *If G is a Fréchet topological group and $\widehat{G}$ is productively Fréchet, then $\widehat{G}$ is metrizable and locally compact.*

It is observed in [6] that the Σ-product of $\mathfrak{c}$ many copies of $\mathbb{R}$ is a non-metrizable productively Fréchet topological group.

Note that since there are productively Fréchet groups whose dual is not hemicompact, Theorem 3.2 no longer follows from Theorem 2.4. To see that such groups exist, first note that the Σ-product of $\mathfrak{c}$ many copies of $\mathbb{R}$ is a locally convex Fréchet-Urysohn topological vector space, in particular a locally quasi-convex (by, e.g., [2, Proposition 6.5]) k-group. It is therefore *subreflexive*, that is,

$$\begin{array}{rcl} i : G & \to & \widehat{\widehat{G}} \\ g & \mapsto & \langle g, \cdot \rangle \end{array}$$

is an embedding.

Moreover, we have:

Proposition 3.3. [5, Proposition 5.10] *A subreflexive topological group G is metrizable if and only if $\widehat{G}$ is hemicompact.*

As already noted, the Σ-product of $\mathfrak{c}$ many copies of $\mathbb{R}$ is a productively Fréchet non-metrizable group. In view of Proposition 3.3, its dual is not hemicompact.

In conclusion, while Theorem 1.1 and Proposition 2.3 are not specific to dual groups (but are true for any hemicompact group), the variants given in Theorem 3.2 seem to be specific to dual groups, even though the proof is essentially identical.

References

[1] A. V. Arhangel'skii and V. I. Malykhin. *Metrizability of topological groups*. Moscow Univ. Math. Bull., **51**, 3:9–11, 1996.

[2] L. Aussenhofer, *Contributions to the duality theory of abelian topoligcal groups and to the theory of nuclear groups*, Diss. Math. CCCLXXXIV (1999).

[3] M. J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math., **70** (1998): 22–28.

[4] M. J. Chasco, E. Martín-Peinador and V. Tarieladze, *A class of angelic sequential non-Fréchet-Urysohn topological groups*, Top. Appl., **154** (2007): 741–748.

[5] C. Chis, M. Vincenta Ferrer, S. Hernandez, and B. Tsaban, *The character of topological groups, via Pontryagin-Van Kampen duality,* Journal of Algebra **420** (2014): 86–119.

[6] F. Jordan and F. Mynard, *Productively Fréchet spaces*, Czech. Math. J. **54**(4) (2004): 981–990.

[7] ———, *Compatible relations on filters and stability of local topological properties under supremum and product*, Top. Appl., **153** (2006): 2386–2412.

[8] J. Kąkol and M. López Pellicier, *A note on Fréchet-Urysohn locally convex spaces*, Rev. R. Acad. Cien. Serie A. Mat. **101**(2) 2007: 127–131.

[9] E. Martín-Peinador and V. Tarieladze, *A property of Dunford-Pettis type in topological groups*, Proc. Amer. Math. Soc. **132** (2004): 1827–1834.
[10] E. Michael, *A quintuple quotient quest*, Gen. Topology Appl. **2** (1972): 91–138.
[11] F. Mynard, *Strongly sequential spaces*, Comment. Math. Univ. Carolinae, **41** (1)(2000): 143–153.
[12] ———, *More on strongly sequential spaces*, Comment. Math. Univ. Carolinae, **43** (3) (2002): 525–530.
[13] P.J. Nyikos, *Metrizability and the Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83**(4) (1981): 793–801.
[14] E. Ordman and B. Smith-Thomas, *sequential conditions and free topological groups,* Proc. Amer. Math. Soc. **79**(2) (1980): 319–326.
[15] Y. Tanaka, *0n products of k-spaces*, Top. Appl. 146-147 (2005): 593–602.
[16] E.G. Zelenyuk and I.V. Protasov, *Topologies on abelian groups*, Math. USSR Izvestiya, **37**(2) (1991): 445–460.

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