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NEIGHT: THE NESTED WEIGHT OF A TOPOLOGICAL SPACE

WILLIAM R. BRIAN

Abstract. The neight (nested weight) of a topological space X is the smallest number of nests in X whose union provides a subbasis for X. We explore some basic properties of this function, emphasizing the connections of neight with the small inductive dimension, weight, character, and density of a space.

1. Definitions and Preliminaries

Let X be a topological space. A set $\mathcal{N}$ of subsets of X is called a **nest** if $\mathcal{N}$ is totally ordered by $\subseteq$. In this paper we consider the question: *How many nests does it take to generate a given topology?*

For example, if X is a LOTS (Linearly Ordered Topological Space with the order topology) then

$$\mathcal{N}_L = \{(-\infty, a) \colon a \in X\} \quad \text{and} \quad \mathcal{N}_R = \{(a, \infty) \colon a \in X\}$$

are two nests in X and $\mathcal{N}_L \cup \mathcal{N}_R$ provides a subbasis for X. A stronger result, given as Theorem 2.2 in [2], is that (for T_1 spaces) the topology of X is generated by two nests if and only if X is a GO space (recall that a GO space, or Generalized Order space, is any space that is homeomorphic to a subspace of a LOTS).

It is trivially true that every topological space X has a subbasis that can be written as a union of nests: for each open subset U of X, $\{U\}$ is

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a nest in X. Therefore there is a least cardinal κ such that the topology on X is generated by a subbasis that can be written as the union of κ nests in X. This cardinal is called the **nested weight** or **neight** of X, which we denote $\mathfrak{N}(X)$. As any two homeomorphic spaces have the same neight, $\mathfrak{N}$ is a cardinal function in the usual sense of the term.

We will show in Section 2 that $\mathfrak{N}(\mathbb{R}^n) = n+1$ for every n. This fact, together with the above observation about GO spaces, might lead one to consider the function $X \mapsto \mathfrak{N}(X) - 1$ to be a measure of dimension. Even more suggestive is the so-called halfdirectional dimension of J. Deák, which we will consider in Section 2: as we will see, it is a topological measure of dimension whose definition is very similar to that of neight.

In addition to exploring a few of the basic properties of nested weight, we will also consider the question: *In what ways is the function* $\mathfrak{N}(X) - 1$ *a measure of dimension?* In exploring this question we will focus mostly on spaces with finite neight. Such spaces will be called **FUN spaces**, since their topologies are generated by a **F**inite **U**nion of **N**ests. We will use the shorthand

$$\mathfrak{N}^-(X) = \mathfrak{N}(X) - 1$$

whenever X is a FUN space.

Very roughly, the neight of a space tells us the complexity of convergence in that space. This is borne out by the connections between neight and dimension discussed in Section 2, and by the connections between neight and other cardinal functions like weight, character, and density, which will be discussed in Section 4.

The word "neight" was coined by Yurovetskiĭ in [12], who explored some of the basic properties of this function (his results are outlined below). Similar notions have been defined and explored in detail by E. Deák and J. Deák (see [3], [4], and [5]), and some of these will be discussed in Section 2.

We begin with a lemma summarizing the work of Yurovetskiĭ on the basic properties of $\mathfrak{N}$:

Lemma 1.1 (Yurovetskiĭ)**.**
(i) $\mathfrak{N}(X) = 0$ *if and only if* X *is indiscrete, and, if* X *is* T_1*, then* $|X| \geq 2$ *implies* $\mathfrak{N}(X) \geq 2$.
(ii) If $Y \subseteq X$ *then* $\mathfrak{N}(Y) \leq \mathfrak{N}(X)$.
(iii) $\mathfrak{N}^-(X \times Y) \leq \mathfrak{N}^-(X) + \mathfrak{N}^-(Y) + 1$.
(iv) $\mathfrak{N}(\prod_{\alpha \in I} X_\alpha) \leq \sum_{\alpha \in I} \mathfrak{N}(X_\alpha)$.
(v) If $Y = \bigsqcup_{\alpha \in I} X_\alpha$*, and some* X_α *has* $\mathfrak{N}(X_\alpha) \geq 2$*, then* $\mathfrak{N}(Y) = \sup_{\alpha \in I} \mathfrak{N}(X_\alpha)$.
(vi) If X *is metrizable then* $\mathfrak{N}(X) \leq \aleph_0$.

Proof. We will prove (i) in order to give the reader a taste for an elementary proof concerning the nested weight. Proofs for $(ii)-(vi)$ can be found in [12].

$\mathfrak{N}(X) = 0$ if and only if $\emptyset$ is a subbasis for the topology on X if and only if X is indiscrete. Now suppose that X is T_1 and that there is some nest $\mathcal{N}$ that provides a subbasis for the topology on X. If a, b are two distinct points of X then there are three possibilities:

Case 1: There is some $U \in \mathcal{N}$ such that $a \in U$ and $b \notin U$. Since $\mathcal{N}$ is a nest, there is no member of $\mathcal{N}$ which contains b but does not contain a. It follows that any finite intersection of members of $\mathcal{N}$ containing b must also contain a. Since $\mathcal{N}$ is a subbasis for X, this contradicts the fact that X is T_1.

Case 2: There is some $U \in \mathcal{N}$ such that $b \in U$ and $a \notin U$. This case is handled like Case 1.

Case 3: For each $U \in \mathcal{N}$, either $a \in U$, $b \in U$ or $a \notin U$, $b \notin U$. In this case, any finite intersection of members of $\mathcal{N}$ either contains both a and b or it contains neither a nor b. Since $\mathcal{N}$ is a subbasis for X, this contradicts the fact that X is T_0.

Thus any T_1 space X with $\mathfrak{N}(X) \leq 1$ consists of at most one point. $\square$

Although (iii) is a special case of (iv), we have listed it separately to emphasize that $\mathfrak{N}^-$ is not necessarily additive under products. In Theorem 3.1 we will exhibit two fairly well-behaved spaces such that the equality in (iii) holds. This tells us one way in which $\mathfrak{N}^-$ is badly behaved as a measure of dimension. Part (i) gives us another: if X is a zero-dimensional T_1 space with more than one point, then $\mathfrak{N}^-(X) \geq 1$.

If $\mathcal{C}$ is a collection of nests in X, we define

$$\#\mathcal{C} = \left\{ \bigcap_{i=0}^{n} U_i \colon n \in \mathbb{N}, U_0, ..., U_n \in \bigcup \mathcal{C} \right\}.$$

We say that $\mathcal{C}$ **generates** the topology on X if and only if $\#\mathcal{C}$ is a basis for X, which is true if and only if $\bigcup \mathcal{C}$ is a subbasis for X.

2. Dim, ind, and $\mathfrak{N}^-$

J. Deák defines a **halfdirection** on X to be a set $\mathcal{H}$ of open subsets of X such that (i) $\mathcal{H}$ is totally ordered by the relation $U \leq V \Leftrightarrow \overline{U} \subseteq V$ and (ii) If $\mathcal{H}' \subseteq \mathcal{H}$ then $\bigcup \mathcal{H}' \in \mathcal{H}$. (see [4]). Evidently, a halfdirection is a special kind of nest that is closed under suprema and in which, intuitively speaking, the "below" order has been replaced by a "way below" order. When X has a topology that is generated by a finite collection of halfdirections, the **halfdirectional dimension** of X, denoted $\mathrm{Dim}(X)$, is defined so that $\mathrm{Dim}(X) + 1$ is the least n such that some collection of n halfdirections on X generates its topology; otherwise $\mathrm{Dim}(X) = \infty$.

This definition is essentially the same as our definition of $\mathfrak{N}^-$, only with "nest" replaced by "halfdirection"; thus we have

Proposition 2.1. $\mathfrak{N}^-(X) \leq \mathrm{DIM}(X)$ *for every topological space* X.

One of the most fundamental theorems concerning the halfdirectional dimension, indeed the one justifying its name, is the following:

Theorem 2.2 (J. Deák)**.** *If* X *is a separable metrizable space then* X *embeds in* $\mathbb{R}^n$ *if and only if* $\mathrm{DIM}(X) \leq n$.

Proof. See [4], pp. 255-256. □

In addition to the halfdirectional dimension, J. Deák studies several other related measures of dimension, most notably Dim, the directional dimension, and O-Dim, the orderly directional dimension (see [5] for an overview of these and more). The directional dimension was defined originally by E. Deák and satisfies a theorem identical to Theorem 2.2. All of these dimensions are similar in that they are all defined in terms of subbases. Moreover, for any topological space X, we have $\mathfrak{N}^-(X) \leq \mathrm{DIM}(X) \leq \mathrm{Dim}(X) \leq \text{O-Dim}(X)$.

$\mathfrak{N}^-$, DIM, Dim, and O-Dim all have in common that it is typically very easy to find an upper bound for these functions on a particular space (one only needs to produce a subbasis witnessing the upper bound), but lower bounds are often harder to find. In a sense, our study of neight is justified by this fact and by the inequality given in the previous paragraph. In this section and the two following, we will prove and apply several theorems, each of which gives a lower bound for $\mathfrak{N}$ under certain circumstances. Since $\mathfrak{N}^-$ is in turn a lower bound for DIM, Dim, and O-Dim, each of these theorems also gives lower bounds for DIM, Dim, and O-Dim.

Before moving on to finding lower bounds for $\mathfrak{N}$, we point out that DIM and $\mathfrak{N}^-$ agree on which T_1 spaces have dimension 1:

Proposition 2.3. *If* X *is a* T_1 *space then* $\mathfrak{N}^-(X) = 1$ *if and only if* $\mathrm{DIM}(X) = 1$.

Proof. Assume X is T_1. As always, $\mathfrak{N}^-(X) \leq \mathrm{DIM}(X)$. Also, $\mathfrak{N}^-(X) < 1$ implies X is either empty or a singleton; in either case, $\mathrm{DIM}(X) < 1$. It suffices, then, to prove that if $\mathfrak{N}^-(X) = 1$ then $\mathrm{DIM}(X) \leq 1$. For this it suffices to prove that if two nests generate the topology on X then these two nests are halfdirections.

Let $\mathcal{L}$, $\mathcal{R}$ be two nests on X such that $\mathcal{L} \cup \mathcal{R}$ is a subbasis for X. Begin by expanding $\mathcal{L}$ and $\mathcal{R}$, if necessary, so that they are closed under suprema, i.e., so that $\mathcal{L}$ and $\mathcal{R}$ each contain all unions of subsets of themselves. This does not change the topology generated by $\mathcal{L} \cup \mathcal{R}$. By results of van Dalen and Wattel (see [2], proof of Lemma 3.1), $\mathcal{L}$ and $\mathcal{R}$ induce an order $\leq$ on X given by $x \leq y \Leftrightarrow \exists U \in \mathcal{L}(x \in U \wedge y \notin U)$; moreover, the topology of X is at least as fine as the topology induced by this order.

Using this fact, it is easy to see that if $U \in \mathcal{L}$ then $\overline{U}^X$ is either U itself or U together with one extra point. This implies that if $U, V \in \mathcal{L}$ with $U \subseteq V$ and $U \neq V$ then $\overline{U} \subseteq V$. Thus $\mathcal{L}$ is a halfdirection for X and, by a similar argument, so is $\mathcal{R}$. □

The following example shows that the condition in proposition 2.3 that X be T_1 is necessary:

Proposition 2.4. *There is a T_0 space X such that $\mathfrak{N}^-(X) = 1$ and* $\textsc{Dim}(X) = \infty$.

Proof. Let $X = \omega + 1$ with the following basis: $\{\omega\}$ is open in X and, for each n, $\{n, \omega\}$ is open in X.

X has an infinite discrete subspace, so $\mathfrak{N}^-(X) \geq 1$ by Lemma 1.1 (i) and (ii). The two nests $\{n \cup \{\omega\} \colon n \in \omega\}$ and $\{X \setminus \alpha \colon \alpha \in \omega + 1\}$ together provide a subbasis for X, so $\mathfrak{N}^-(X) = 1$. (Here, as elsewhere, we identify an ordinal number with the set of its predecessors.)

Any nonempty open subset of X is dense in X. Thus any halfdirection in X can contain at most one set other than $\emptyset$ and X. Using this fact, it is easy to see that $\textsc{Dim}(X) = \infty$. □

The argument of Proposition 2.3 does not extend to higher-dimensional spaces, even if the condition that X is T_1 is replaced with the much stronger condition of metrizability:

Theorem 2.5. *There is a metric space X such that $\mathfrak{N}^-(X) = 2$ and* $\textsc{Dim}(X) = \infty$.

The proof of Theorem 2.5 is rather long and requires several auxiliary definitions and lemmas that we do not use elsewhere, so we have postponed the proof until Section 5. The metric space used in the proof of Theorem 2.5 is non-separable. It remains an open problem either to find a separable metric space on which $\mathfrak{N}^-$ and $\textsc{Dim}$ disagree, or to prove that there is none. That is, it remains an open problem to show whether J. Deák's Theorem 2.2 holds under the weaker hypothesis that replaces $\textsc{Dim}$ with $\mathfrak{N}^-$.

Recall the definition of the **small inductive dimension** (see, e.g., [10]): a space is n-dimensional if it has a basis of sets with $(n-1)$-dimensional boundaries. Formally, we begin by defining $\operatorname{ind}(\emptyset) = -1$. A space X satisfies $\operatorname{ind}(X) \leq n$ if and only if there is a basis $\mathcal{B}$ for X such that each $U \in \mathcal{B}$ satisfies $\operatorname{ind}(\partial U) \leq n - 1$. We say that the small inductive dimension of X is equal to n, and write $\operatorname{ind}(X) = n$, whenever it is true that $\operatorname{ind}(X) \leq n$ but it is false that $\operatorname{ind}(X) \leq m$ for $m < n$. If it is not true for any n that $\operatorname{ind}(X) \leq n$, then we write $\operatorname{ind}(X) = \infty$.

J. Deák proves that ind is a lower bound for DIM (see [4]); for separable metric spaces, this is just a corollary of Theorem 2.2. In this section we prove this result for $\mathfrak{N}^-$ as well, but with a few additional (fairly mild) conditions.

Lemma 2.6. *Let X be a separable metric space.*
(i) If $Y \subseteq X$ then $\mathrm{ind}(Y) \leq \mathrm{ind}(X)$.
(ii) If $C_1, ..., C_n$ are closed subsets of X, then

$$\mathrm{ind}\left(\bigcup_{i=1}^{n} C_i\right) = \max\{\mathrm{ind}(C_i) : i = 1, ..., n\}.$$

Proof. See any reference on dimension theory, e.g. [10]. □

Lemma 2.7. *Let X be a regular space and let a collection $\mathcal{C}$ of nests generate the topology on X. If $\mathcal{N} \in \mathcal{C}$ and $U \in \mathcal{N}$, then $\mathcal{C} \setminus \{\mathcal{N}\}$ generates the topology on ∂U, that is,*

$$\mathcal{S} = \{V \cap \partial U : V \in \mathcal{M} \in \mathcal{C} \setminus \{\mathcal{N}\}\}$$

is a subbasis for ∂U. In particular, if $\mathcal{C}$ is a collection of nests in X of minimal cardinality such that $\bigcup \mathcal{C}$ is a subbasis for X, and if $U \in \bigcup \mathcal{C}$, then $\mathfrak{N}(\partial U) \leq \mathfrak{N}(X) - 1$.

Proof. Let X be a regular space and let $\mathcal{C}$ be a collection of nests in X that generates the topology on X. Let $U \in \mathcal{N} \in \mathcal{C}$ and $x \in \partial U$. We show that x has a neighborhood basis (in ∂U) consisting of finite intersections of elements of $\mathcal{S}$. Since $\#\mathcal{C}$ is a basis for X, it suffices to show that, for any $V \in \#\mathcal{C}$ such that $x \in V$, there is some $W \in \#\mathcal{S}$ such that $x \in W \cap \partial U \subseteq V \cap \partial U$.

Since X is regular and $\#\mathcal{C}$ is a basis for X, there is some $V' \in \#\mathcal{C}$ such that V' is a neighborhood of x and $\overline{V'} \subseteq V$. Now

$$V' = U_0 \cap U_1 \cap ... \cap U_n$$

where, without loss of generality, $U_0 \in \mathcal{N} \cup \{X\}$ and $U_1, ..., U_n \notin \mathcal{N}$ (recall that $\mathcal{N}$, hence $\mathcal{N} \cup \{X\}$, is a nest and thus is closed under finite intersections). We claim that

$$(U_1 \cap ... \cap U_n) \cap \partial U \subseteq V \cap \partial U.$$

Suppose this is not the case and let $z \in (U_1 \cap ... \cap U_n \cap \partial U) \setminus V$. Since $\mathcal{N}$ is a nest, $U, U_0 \in \mathcal{N} \cup \{X\}$, $x \in U_0$, and $x \notin U$, we have $U \subseteq U_0$. Thus, because $z \in \partial U$, either $z \in U_0$ or $z \in \partial U_0$. On the one hand, if $z \in U_0$ then $z \in U_0 \cap U_1 \cap ... \cap U_n = V'$ and $z \notin V$, contradicting the fact that $V' \subseteq V$. On the other hand, if $z \in \partial U_0$ and $z \in U_1 \cap ... \cap U_n$, then $z \in \partial(U_0 \cap U_1 \cap ... \cap U_n) = \partial V'$. Since $z \notin V$, this contradicts the fact that $\overline{V'} \subseteq V$. □

Theorem 2.8. *If X is a nonempty FUN separable metric space then*

$$\operatorname{ind}(X) \leq \mathfrak{N}^{-}(X).$$

Proof. The proof is by induction on $\mathfrak{N}^{-}(X)$. If $\mathfrak{N}^{-}(X) = 0$ then, by Lemma 1.1(i), X consists of at most one point, in which case $\operatorname{ind}(X) \leq 0$. Now assume that the result is true for separable metric spaces Y such that $\mathfrak{N}^{-}(Y) < \mathfrak{N}^{-}(X)$. Let $\mathcal{C}$ be a collection of nests that generates the topology on X with $|\mathcal{C}| = \mathfrak{N}(X)$. We show that, for each $U \in \#\mathcal{C}$, $\mathfrak{N}^{-}(\partial U) \leq \mathfrak{N}^{-}(X) - 1$. Let $U = U_0 \cap ... \cap U_n$, where $U_0, ..., U_n \in \mathcal{C}$. Clearly

$$\partial U \subseteq \partial U_0 \cup ... \cup \partial U_n.$$

By Lemma 2.7, $\mathfrak{N}^{-}(\partial U_i) \leq \mathfrak{N}^{-}(X) - 1$ for each $i = 0, ..., n$. By the inductive hypothesis, $\operatorname{ind}(\partial U_i) \leq \mathfrak{N}^{-}(X) - 1$ for each $i = 0, ..., n$. It now follows from Lemma 2.6 that $\operatorname{ind}(\partial U) \leq \mathfrak{N}^{-}(X) - 1$. Since this is true for an arbitrary $U \in \#\mathcal{C}$, it follows from the definition of the small inductive dimension that $\operatorname{ind}(X) \leq \mathfrak{N}^{-}(X)$. □

Corollary 2.9. *For each $n \in \mathbb{N}$, $\mathfrak{N}^{-}(\mathbb{R}^n) = n$.*

Proof. It follows from Theorem 2.8 that $\mathfrak{N}^{-}(\mathbb{R}^n) \geq \operatorname{ind}(\mathbb{R}^n) = n$. Therefore it suffices to find, for each n, a collection of $n+1$ nests whose union is a subbasis for $\mathbb{R}^n$. This was done in [12]; even before that, J. Deák showed in [4] that we may use $n + 1$ halfdirections to generate the topology of $\mathbb{R}^n$: simply take the n nests

$$\{\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i < r\} : r \in \mathbb{R}\}$$

for $1 \leq i \leq n$, plus the one additional nest

$$\{\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \ldots x_n > r\} : r \in \mathbb{R}\}.$$ □

In general, we say that the **finite sum theorem** holds for X if, whenever A and B are closed subsets of X,

$$\operatorname{ind}(A \cup B) = \max\{\operatorname{ind}(A), \operatorname{ind}(B)\}.$$

Lemma 2.6(i) says that the finite sum theorem holds for separable metric spaces. However, the finite sum theorem holds in many spaces that are not separable metric spaces. This, along with the observation that Theorem 2.8 really only uses the fact that X satisfies the finite sum theorem (not that X is a separable metric space), leads to the following strengthening of Theorem 2.8:

Corollary 2.10. *Let X be a regular FUN space and suppose that the finite sum theorem holds in X. Then $\operatorname{ind}(X) \leq \mathfrak{N}^{-}(X)$.*

3. Bad behavior under products

We now prove that, even for fairly well-behaved spaces, the inequality in Lemma 1.1(*iii*) cannot be improved. This example is similar to an example used by Gerlits in the study of Dim (see [7], Example 3.1), and the main idea of our proof is already present in his paper.

Theorem 3.1. *There are compact LOTS X and Y such that*

$$\mathfrak{N}^-(X \times Y) = \mathfrak{N}^-(X) + \mathfrak{N}^-(Y) + 1.$$

Proof. Let X be the totally ordered set obtained by adding a reversed copy of ω_1 to the end of $\omega + 1$, sometimes denoted $(\omega+1)^\frown\omega_1^*$. More explicitly, $X = \{0\} \times (\omega+1) \cup \{1\} \times \omega_1$ and

$$(i,\alpha) \leq (j,\beta) \quad \text{if and only if} \quad \begin{array}{l} i < j \ \text{ or} \\ i = j = 0 \text{ and } \alpha < \beta \ \text{ or} \\ i = j = 1 \text{ and } \beta < \alpha. \end{array}$$

Let X have the topology induced by $\leq$. Similarly, let $Y = (\omega_2+1)^\frown\omega_3^*$, with the usual order topology. Both X and Y are compact LOTS. We have already seen that every nontrivial LOTS has neight 2, so $\mathfrak{N}^-(X) = \mathfrak{N}^-(Y) = 1$.

We now show that $\mathfrak{N}^-(X \times Y) = 3$. We already know from Lemma 1.1 (*iii*) that $\mathfrak{N}^-(X \times Y) \leq 3$, so it suffices to show that no collection of only 3 nests can produce a subbasis for $X \times Y$.

To this end, suppose that $\mathcal{N}_0, \ldots, \mathcal{N}_n$ are nests in $X \times Y$ and that $\bigcup_{i=0}^n \mathcal{N}_i$ is a subbasis for $X \times Y$. Let $p = ((0,\omega),(0,\omega_2))$ and, for each $i \leq n$, let

$$\mathcal{A}_i = \langle \{U \in \mathcal{N}_i \colon p \in U\}, \supseteq \rangle.$$

Each $\mathcal{A}_i$ is a linear order, and we will show $n \geq 3$ by a detailed consideration of the cofinalities of these orders.

For each $n \in \omega$, let $O_m = ((0,m),\infty) \times Y$ (as is customary with a LOTS, we take $((0,m),\infty) = \{x \in X : (0,m) \leq x\}$). $\langle O_m \rangle_{m<\omega}$ is a nested sequence of open subsets of $X \times Y$. We now define simultaneously by induction a sequence $\langle U_m^i \colon i \leq \omega \rangle$ in $\mathcal{A}_i$ for each $i \leq n$. Take $U_0^i \in \mathcal{A}_i$ arbitrarily. Suppose we have $U_{m-1}^i \in \mathcal{A}_i$ for each i; since $\bigcup_{i\leq n} \mathcal{N}_i$ is a subbasis for $X \times Y$, there are $U_n^i \in \mathcal{A}_i$ such that $U_m^i \subseteq U_{m-1}^i$ for each i and $p \in \bigcap_{i\leq n} U_m^i \subseteq O_m$.

For each $i \leq n$, $m \mapsto U_m^i$ is a nondecreasing map $\omega \to \mathcal{A}_i$. We claim that, for at least one value of i, this map must be cofinal. If not, there is for each i some $U_\infty^i \in \mathcal{A}_i$ such that $U_m^i \supseteq U_\infty^i$ for each $m \in \omega$. But then $\bigcap_{i\leq n} U_\infty^m$ is an open neighborhood of p contained in each O_m.

This is impossible because p is not in the interior of $\bigcap_{m\in\omega} O_m$. Thus one of these maps is cofinal, so that one of the $\mathcal{A}_i$ has cofinality ω.

Next, for each $\alpha \in \omega_1$, let $O_\alpha = (-\infty, (1,\alpha)) \times Y$. As before, $\langle O_\alpha \rangle_{\alpha<\omega_1}$ is a nested sequence of open subsets of $X \times Y$. We will once again pick sequences of sets from our subbasis to get inside the O_α, but now we must do so in two stages.

Let $I = \{i \leq n \colon \mathrm{cf}(\mathcal{A}_i) \leq \omega\}$ and $J = \{i \leq n \colon \mathrm{cf}(\mathcal{A}_i) > \omega\}$. For each $i \in I$, fix some cofinal sequence $\langle W^i_m \colon m < \omega \rangle$ in $\mathcal{A}_i$ (it is clear that we lose no generality by assuming that $\mathrm{cf}(\mathcal{A}_i)$ is infinite for every i). For each $\alpha < \omega_1$, choose a tuple $t_\alpha \in \omega^I$ such that there exist sets U^j_α, $j \in J$, with

$$\bigcap_{i\in I} W^i_{t_\alpha(i)} \cap \bigcap_{j\in J} U^j_\alpha \subseteq O_\alpha.$$

Because ω^I is countable there is some $t \in \omega^I$ such that $t = t_\alpha$ for uncountably many α. But then, for any $\alpha < \omega_1$, there exist sets U^j_α, $j \in J$, such that $\bigcap_{i\in I} W^i_{t(i)} \cap \bigcap_{j\in J} U^j_\alpha \subseteq O_\alpha$. For all $\alpha < \omega_1$ and $i \in I$, set $V^i_\alpha = W^i_{t(i)}$.

We now proceed to define $V^j_\alpha \in \mathcal{A}_j$ for each $j \in J$ using transfinite recursion. Pick V^j_0 arbitrarily for each $j \in J$. Assuming V^j_β has been chosen for each $\beta < \alpha$ and $j \in J$, choose V^j_α such that

$$V^j_\alpha \subseteq \bigcap_{\beta<\alpha} V^j_\beta \text{ for each } j \text{ and } \bigcap_{j\leq n} V^j_\alpha \subseteq O_\alpha.$$

This is possible by our choice of the V^i_α for $i \in I$ and because $\langle V_\beta \colon \beta < \alpha \rangle$ cannot be cofinal in $\mathcal{A}_j$ if $\alpha < \omega_1$.

For each $j \in J$, $\alpha \mapsto V^j_\alpha$ is a nondecreasing map $\omega_1 \to \mathcal{A}_i$. We claim that, for at least one value of j, this map must be cofinal. If not, there is for each $j \in J$ some $V^j_\infty \in \mathcal{A}_j$ such that $V^j_\alpha \supseteq V^j_\infty$ for each $\alpha < \omega_1$. Then

$$\bigcap_{i\in I} V^i_0 \cap \bigcap_{j\in J} V^j_\infty \subseteq O_\alpha$$

for each α, which implies $p \in \mathrm{Int}\big(\bigcap_{\alpha<\omega_1} O_\alpha\big)$, a contradiction. Thus some $\mathcal{A}_i$ has cofinality ω_1.

Using the same technique, we can prove that some $\mathcal{A}_i$ has cofinality ω_2 and that some $\mathcal{A}_i$ has cofinality ω_3. Thus $n \geq 3$. $\square$

The proof of Theorem 3.1 generalizes in an obvious way to prove the following more general result:

Theorem 3.2. *Let $p \in X$ and suppose that, for $i = 0, ..., n$, $\mathcal{A}_i$ is a nest of open sets in X such that, for each i, p is in $\bigcap \mathcal{A}_i$ but is not in the interior of $\bigcap \mathcal{A}_i$. If $\mathrm{cf}(\mathcal{A}_i) \neq \mathrm{cf}(\mathcal{A}_j)$ whenever $i \neq j$ then $\mathfrak{N}^-(X) \geq n$.*

One would hope that a topological measure of dimension is additive under products, or least products of sufficiently well-behaved spaces. Theorem 3.1 says that $\mathfrak{N}^-$ fails this test, at least for compact LOTS. The following few corollaries demonstrate a few other similar pathologies.

Corollary 3.3. *For arbitrarily large n, there is a compact Hausdorff space X with $\mathfrak{N}^-(X) = n$ and a point $x \in X$ such that $\mathfrak{N}^-(X \setminus \{x\}) = 1$.*

Proof. Fix n and let Y be the disjoint union

$$Y = \bigsqcup_{i=0}^{n} (\omega_i + 1) \times \{i\}.$$

Let X be the quotient space obtained from Y by identifying the $n+1$ points $(\omega_0, 0), (\omega_1, 1), ..., (\omega_n, n)$. X is clearly a compact Hausdorff space, and it follows from Theorem 3.2 that $\mathfrak{N}^-(X) \geq n$ (in fact, $\mathfrak{N}^-(X) = n$: Gerlits shows this in Example 3.1(a) of [7]). Let x be the point in X whose pre-image in Y is the set $\{(\omega_0, 0), (\omega_1, 1), ..., (\omega_n, n)\}$. $X \setminus \{x\}$ is (homeomorphic to) a disjoint union of LOTS, and hence is a GO space. Thus $\mathfrak{N}^-(X \setminus \{x\}) = 1$. □

Corollary 3.4. *$\mathfrak{N}^-$ does not satisfy the finite sum theorem for compact Hausdorff spaces. That is, for every $n \geq 1$ there is a compact Hausdorff space X, and closed subsets $C_0, ..., C_n$ of X, such that*

$$\mathfrak{N}(X) > \max\{\mathfrak{N}(C_0), ..., \mathfrak{N}(C_n)\}.$$

Proof. The space X described in the previous proof can be divided into a finite number of compact LOTS (overlapping only at the point x), each of which is closed in X. □

The following corollary says that $\mathfrak{N}^-$ not only fails to be additive under products, but for arbitrary finite powers it fails as badly as Lemma 1.1(iii) allows.

Corollary 3.5. *There is a compact LOTS X such that, for every $n \in \omega$, $\mathfrak{N}^-(X^n)$ has the maximum possible value of $2n - 1$.*

Proof. Let $X = \bigsqcup_{n\in\omega} (\omega_{2n} + 1)^\frown \omega^*_{2n+1}$ and apply Theorem 3.2 in the natural way. □

4. Spaces that are not FUN

Theorems 2.8 and 3.2 give two different ways for finding lower bounds for $\mathfrak{N}(X)$ when X is a FUN space. In this section we will prove a theorem that relates the neight of a space to its weight. This theorem will allow us to get a lower bound on $\mathfrak{N}$ for spaces with infinite neight and to prove fairly easily that certain spaces are not FUN. Recall that $w(X)$ denotes the **weight** of a space X, the smallest cardinal κ such that X has a base of size κ, and $\chi(x)$ denotes the **character** of a point x, the smallest cardinal κ such that x has a local base of size κ.

Lemma 4.1 (Yurovetskiĭ). $\mathfrak{N}(X) \leq w(X)$.

Proof. If $\mathcal{B}$ is a basis for X then $\{\{B\} \colon B \in \mathcal{B}\}$ is a collection of nests on X whose union is a subbasis for X. $\square$

Theorem 4.2. *If* $|X| < w(X)$ *then* $\mathfrak{N}(X) = w(X)$.

Proof. Let $x \in X$. We will begin by showing that $\chi(x) \leq \mathfrak{N}(X) \cdot |X|$.

By assumption, there is a collection $\mathcal{C} = \{\mathcal{N}_\alpha \colon \alpha < \mathfrak{N}(X)\}$ of nests that generates the topology on X. For each $\mathcal{N}_\alpha \in \mathcal{C}$, let $\mathcal{N}_\alpha^x$ be the restriction of $\mathcal{N}_\alpha$ to sets containing x: $\mathcal{N}_\alpha^x = \{U \colon x \in U \in \mathcal{N}_\alpha\}$. Each $\mathcal{N}_\alpha^x$ is totally ordered by $\supseteq$, and the cofinality of $\mathcal{N}_\alpha^x$ is at most $|X|$. To see this, note that there must be a well-ordered, strictly decreasing sequence $\langle U_\beta \colon \beta < \mathrm{cf}(\mathcal{N}_\alpha^x)\rangle$ of subsets of X, and (choosing $x_\beta \in U_\beta \setminus U_{\beta+1}$) we may use this sequence to find a subset of X of size $\mathrm{cf}(\mathcal{N}_\alpha^x)$.

For each α, let $\mathcal{K}_\alpha \subseteq \mathcal{N}_\alpha^x$ be cofinal in $\mathcal{N}_\alpha^x$ with $|\mathcal{K}_\alpha| = \mathrm{cf}(\mathcal{N}_\alpha^x) \leq |X|$. Because each $\mathcal{K}_\alpha$ is cofinal in $\mathcal{N}_\alpha^x$,

$$\left\{ \bigcap_{i=0}^{n} U_i \colon n \in \omega, \alpha_0, \alpha_1, ..., \alpha_n < \mathfrak{N}(X), \text{ and } U_i \in \mathcal{K}_\alpha \text{ for each } i \right\}$$

is a neighborhood basis for x. Since the cardinality of each $\mathcal{K}_\alpha$ is at most $|X|$, the cardinality of this neighborhood basis for x is at most $\mathfrak{N}(X) \cdot |X|$.

For every $x \in X$, suppose N_x is a neighborhood basis for x of size at most $\mathfrak{N}(X) \cdot |X|$. Then $\bigcup_{x \in X} N_x$ is a basis for X of size at most $\mathfrak{N}(X) \cdot |X|$. Hence $w(X) \leq \mathfrak{N}(X) \cdot |X|$. As we are assuming $|X| < w(X)$, this proves $w(X) \leq \mathfrak{N}(X)$.

Lemma 4.1 provides the opposite inequality and completes the proof of the theorem. $\square$

Combining this result with Lemma 1.1(ii), we have:

Corollary 4.3. *If* $Y \subseteq X$ *and* $|Y| < w(Y)$ *then* $\mathfrak{N}(X) \geq w(Y)$.

Corollary 4.4. *Neither* $\beta\omega$ *nor* ω^* *is FUN: both have nested weight* $\mathfrak{c}$.

Proof. Since $\omega^* \subseteq \beta\omega$ and $\beta\omega$ embeds in ω^*, these two spaces have the same nested weight.

Let $p \in \beta\omega \setminus \omega$ be a point with character $\mathfrak{c}$ (it was proved in [11] that some such point exists). Consider the space $X = \omega \cup \{p\} \subseteq \beta\omega$. X is countable, but the fact that $\chi(p) = \mathfrak{c}$ implies that $w(X) = \mathfrak{c}$. By Corollary 4.3, $\mathfrak{N}(\beta\omega) \geq \mathfrak{c}$. Since $w(\beta\omega) = \mathfrak{c}$, the result now follows from Lemma 4.1. $\square$

A similar argument shows that, for any discrete space A, βA has neight $2^{|A|}$. A more general form of this argument will be given in Corollary 4.6.

Lemma 4.5. *Let X be regular and let D be dense in X. If $x \in X$ then the character of x in X is the same as the character of x in $D \cup \{x\}$.*

Proof. See [6], Exercise 2.1.C(a). □

Corollary 4.6. *If X is regular and separable but not first countable, then $\mathfrak{N}(X)$ is uncountable. More generally, if X is regular, D is dense in X, and $x \in X$ with $\chi(x) > |D|$, then $\mathfrak{N}(X) \geq \chi(x)$.*

Proof. By Lemma 4.5, the character of x in $D \cup \{x\}$ is the same as its character in X. Applying Corollary 4.3 to the space $D \cup \{x\}$, we have $\mathfrak{N}(X) \geq \chi(x)$. □

The following lemma generalizes Exercise 3 on p. 86 of [9]:

Lemma 4.7. *Let κ, λ be infinite cardinals with $\kappa \leq 2^\lambda$. Then $[0,1]^\kappa$ has a dense subset of size $2^{<\lambda}$.*

The following result complements Corollary 2.9:

Corollary 4.8. $\mathfrak{N}([0,1]^\kappa) = \kappa$ *for every infinite κ.*

Proof. That $\mathfrak{N}([0,1]^\kappa) \leq \kappa$ follows from Lemma 1.1(iv), so we must show that $\mathfrak{N}([0,1]^\kappa) \geq \kappa$.

First suppose κ is a successor cardinal, say $\kappa = \mu^+$. Let λ be the least cardinal such that $2^\lambda \geq \kappa$; then $2^{<\lambda} \leq \mu < \kappa$. By Lemma 4.7, $[0,1]^\kappa$ has a dense subset D of size $2^{<\lambda}$. However, every point of $[0,1]^\kappa$ has character κ and, by Lemma 4.5, every point of D has character κ. That $\mathfrak{N}([0,1]^\kappa) \geq \kappa$ now follows from Corollary 4.6.

Next suppose κ is a limit cardinal. Let μ be any successor cardinal with $\mu < \kappa$. $[0,1]^\mu$ embeds naturally in $[0,1]^\kappa$. By Lemma 1.1(ii), $\mathfrak{N}([0,1]^\kappa) \geq \mathfrak{N}([0,1]^\mu) = \mu$ (the equality follows from Corollary 2.9 for finite μ and from the previous paragraph for infinite μ). Because every limit cardinal is a limit of successor cardinals, this proves $\mathfrak{N}([0,1]^\kappa) \geq \kappa$. □

Recall that X is a **Toronto space** if X is homeomorphic to every $Y \subseteq X$ with $|Y| = |X|$. It is a stubborn open problem whether any infinite, Hausdorff, non-discrete Toronto spaces can exist (it is known to be consistent with ZFC that no such spaces exist, e.g. under GCH; see [1], Proposition 2.6). The following corollary tells us about the neight of certain Toronto spaces, should they ever be constructed.

Corollary 4.9. *If X is a non-discrete Hausdorff Toronto space of size $\aleph_1$ then $\mathfrak{N}(X)$ is uncountable.*

Proof. It is shown in [1] that, if X is such a space, then every limit point x of X has a countable neighborhood and uncountable character. Corollary 4.3 implies $\mathfrak{N}(X) \geq \chi(x)$. □

5. The nested weight of a hedgehog

In this section we present a proof of Theorem 2.5, which we recall here for the reader:

Theorem 2.5. *There is a metric space X such that $\mathfrak{N}^-(X) = 2$ and $\mathrm{Dim}(X) = \infty$.*

We will prove this theorem through a sequence of lemmas. The space X that we will use to prove the theorem is the metric hedgehog with $\aleph_1$ spines. That is, $X = ((0,1] \times \omega_1) \cup \{*\}$ with the topology generated by the following metric: for every x, y, α, and β, we take $d((x,\alpha),(y,\beta)) = x+y$ if $\alpha \neq \beta$, $d((x,\alpha),(y,\alpha)) = |x-y|$ for any fixed α, and $d((x,\alpha),*) = x$.

Lemma 5.1. $\mathfrak{N}^-(X) = 2$.

Proof. The proof that $\mathfrak{N}^-(X) \leq 2$, i.e. that there are three nests that together generate the topology on X, relies on the fact that $X \setminus \{*\}$ is open in X and is a GO space. This allows us to generate the topology of $X \setminus \{*\}$ with two nests, and we may take as our third nest any nested neighborhood basis for $\{*\}$. More explicitly, consider the following three nests in X:

$$\mathcal{N}_0 = \left\{ L_{(x,\alpha)} = \{(y,\beta) \colon \beta < \alpha \text{ or } \beta = \alpha \wedge y < x\} : x \in (0,1], \alpha \in \omega_1 \right\}$$

$$\mathcal{N}_1 = \left\{ R_{(x,\alpha)} = \{(y,\beta) \colon \beta > \alpha \text{ or } \beta = \alpha \wedge y > x\} : x \in (0,1], \alpha \in \omega_1 \right\}$$

$$\mathcal{N}_2 = \left\{ B_{\frac{1}{n}}(*) \colon n \in \omega \right\}$$

It is straightforward to check that $\mathcal{N}_0 \cup \mathcal{N}_1$ is a subbasis for $X \setminus \{*\}$, and it follows that $\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2$ is a subbasis for X. This proves that $\mathfrak{N}^-(X) \leq 2$.

For the opposite inequality, recall the theorem of van Dalen and Wattel mentioned in the introduction ([2], Theorem 2.2): a T_1 space X is a GO space if and only if it has neight at most 2. Thus it is enough to show that X is not a GO space. This follows from the well-known fact that the letter Y (i.e., the space obtained from three copies of $[0,1]$ by identifying the three left-hand endpoints) is not a GO space, together with the observation that the letter Y embeds in X.

Alternatively, that $\mathfrak{N}^-(X) \geq 2$ follows from Proposition 2.3 and the fact (whose proof we are about to undertake) that $\mathrm{Dim}(X) = \infty$. □

Halfdirections, or nests more generally, are totally ordered by $\subseteq$, and we begin the next part of our proof with a general fact about total orders.

A total order $(X, \leq)$ is separable if there is some countable $Q \subseteq X$ such that every nonempty interval of the form (a, b) contains a point of Q. We will say that $(X, \leq)$ is **strongly separable** if there is some countable $Q \subseteq X$ such that every nonempty interval of the form $(a, b]$ or $[a, b)$ contains a point of Q; such a Q will be called **strongly dense**. Note that the notion of strong separability is strictly stronger than the notion of separability (for strictness, consider the "double arrow" space).

Lemma 5.2. *The following are equivalent for a total order $(X, \leq)$:*
(i) $(X, \leq)$ is strongly separable.
(ii) $(X, \leq)$ is order isomorphic to a subset of $\mathbb{R}$.
(iii) When endowed with the order topology, $(X, \leq)$ is second countable.

Proof.
$(i) \Rightarrow (ii)$: Let X be strongly separable and let Q be a countable strongly dense subset of X which, without loss of generality, contains the least and greatest points of X (if they exist). Since Q is countable, Q is order isomorphic to a subset of $\mathbb{R}$. Let $\phi : Q \to \mathbb{R}$ be an order embedding, and extend ϕ to all of X by taking suprema: for any $x \in X$, let $\phi(x) = \sup \{\phi(q) \colon q \in Q \text{ and } q \leq x\}$. Suppose $a, b \in X$ with $a < b$. Since Q is strongly dense, there is some $q_1 \in [a, b) \cap Q$ and some $q_2 \in (q_1, b] \cap Q$. We then have $\phi(a) \leq \phi(q_1) < \phi(q_2) \leq \phi(b)$, so $\phi(a) < \phi(b)$. This shows that ϕ is injective and order preserving. Thus ϕ is an order isomorphism onto its image.
$(ii) \Rightarrow (iii)$: If $X \subseteq \mathbb{R}$, then X, endowed with the subspace topology, has a countable subbasis $\mathcal{S} = \{(q, \infty) \cap X \colon q \in \mathbb{Q}\} \cup \{(-\infty, q) \cap X \colon q \in \mathbb{Q}\}$. The order topology on X may be coarser than this subspace topology, but it will have some subset of $\mathcal{S}$ as a subbasis (namely those elements of $\mathcal{S}$ that are open in X).
$(iii) \Rightarrow (i)$: If X is second countable, then for any subbasis $\mathcal{S}$ for X there is a countable $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}'$ is a subbasis for X (this follows from Theorem 1.1.15 in [6], pp. 17-18). Thus there is some countable $Q \subseteq X$ such that $\{(q, \infty) \colon q \in Q\} \cup \{(-\infty, q) \colon q \in Q\}$ is a subbasis for the order topology on X. Let Q' consist of the points of Q together with any points in X that have an immediate successor or an immediate predecessor. If there were uncountably many points of X with an immediate successor or predecessor then the order topology on X would not be second countable, so Q' is countable. It is easy to check that Q' is strongly dense in X. $\square$

The equivalence of (i) and (ii) in the above lemma makes it clear that strong separability is preserved by taking subspaces and quotients:

Lemma 5.3. *If X is a strongly separable linear order then so is any subset of X.*

Let us call a nest **complete** if it is closed under arbitrary unions. By definition, every halfdirection is complete. Notice that every nest $\mathcal{N}$ has a completion $\tilde{\mathcal{N}} = \{\bigcup \mathcal{A} \colon \mathcal{A} \subseteq \mathcal{N}\}$ and that, if $\mathcal{C}$ is a collection of nests in some space Y, then $\bigcup \mathcal{C}$ is a subbasis for Y if and only if $\bigcup \tilde{\mathcal{C}}$ is, where $\tilde{\mathcal{C}} = \left\{\tilde{\mathcal{N}} \colon \mathcal{N} \in \mathcal{C}\right\}$.

Lemma 5.4. *Let Y be a topological space and suppose that $\mathcal{C}$ is an at most countable collection of complete, strongly separable nests in Y. If $\mathcal{C}$ generates the topology on Y, then Y is second countable.*

Proof. For each $\mathcal{N} \in \mathcal{C}$ fix some $\mathcal{Q}_{\mathcal{N}} \subseteq \mathcal{N}$ such that $\mathcal{Q}_{\mathcal{N}}$ is countable and strongly dense in $\mathcal{N}$. We claim that $\mathcal{S} = \bigcup \{\mathcal{Q}_{\mathcal{N}} \colon \mathcal{N} \in \mathcal{C}\}$ is a subbasis for Y. The lemma follows directly from this claim: $\mathcal{S}$ is countable, so the set of all finite intersections of elements of $\mathcal{S}$ is also countable and is a basis for Y if $\mathcal{S}$ is a subbasis for Y.

Let $y \in Y$ and let $N \subseteq Y$ be an arbitrary neighborhood of y. As $\bigcup \mathcal{C}$ is a subbasis for Y, we may find $\mathcal{N}_0, ..., \mathcal{N}_n \in \mathcal{C}$ and $U_0 \in \mathcal{N}_0, ..., U_n \in \mathcal{N}_n$ such that $y \in \bigcap_{i \leq n} U_i \subseteq N$. For each $i \leq n$, let $V_i = \bigcup \{U \in \mathcal{N}_i \colon y \notin U\}$. Since $y \in U_i \setminus V_i$ and $\mathcal{N}_i$ is a nest, $V_i \subseteq U_i$. As $\mathcal{Q}_{\mathcal{N}_i}$ is strongly dense in $\mathcal{N}_i$, there is some $W_i \in \mathcal{Q}_{\mathcal{N}_i}$ such that $V_i \subsetneq W_i \subseteq U_i$ (order theoretically, $W_i \in (V_i, U_i])$. By our choice of V_i and W_i, $y \in W_i$, $W_i \in \mathcal{Q}_{\mathcal{N}_i}$, and $W_i \subseteq U_i$. This holds for all $i \leq n$, so $y \in \bigcap_{i \leq n} W_i \subseteq \bigcap_{i \leq n} U_i \subseteq N$. This proves that $\mathcal{S}$ is a subbasis for Y. □

Lemma 5.5. *Every complete nest of open sets in a second countable space is strongly separable.*

Proof. Let Y be a second countable space and let $\mathcal{B} = \{B_n \colon n \in \omega\}$ be a countable basis for Y. Let $\mathcal{N}$ be a nest of open subsets of Y. For each $U \in \mathcal{N}$, let $A_U = \{n \in \omega \colon B_n \subseteq U\}$. Clearly $U = \bigcup \{B_n \colon n \in A_U\}$ for every $U \in \mathcal{N}$, so the map $U \mapsto A_U$ is an injection from $\mathcal{N}$ into 2^ω. This map is order preserving: $U \subseteq V$ implies $A_U \subseteq A_V$. In particular, this map is order preserving when we consider 2^ω to have the lexicographic order.

Note that 2^ω with the lexicographic order is order isomorphic to a subset of $\mathbb{R}$, namely to the Cantor set. Thus, by the previous paragraph, $\mathcal{N}$ is order isomorphic to a subset of $\mathbb{R}$. Its strong separability now follows from Lemma 5.2. □

If $Y \subseteq X$ and $\mathcal{N}$ is a nest in X, define

$$\mathcal{N} \upharpoonright Y = \{U \cap Y \colon U \in \mathcal{N} \text{ and } U \cap Y \neq \emptyset\}.$$

Lemma 5.6. *If $\mathcal{N}$ is a strongly separable halfdirection on Y and $Z \subseteq Y$ then $\mathcal{N} \restriction Z$ is a strongly separable halfdirection on Z.*

Proof. It is obvious that if $\mathcal{N}$ is a halfdirection then so is $\mathcal{N} \restriction Z$. It suffices, then, to show that $\mathcal{N} \restriction Z$ is strongly separable if $\mathcal{N}$ is. For every $U \in \mathcal{N} \restriction Z$, pick some $f(U) \in \mathcal{N}$ such that $f(U) \cap Z = U$. Then f is injective and $\{f(U) \colon U \in \mathcal{N} \restriction Z\}$ is a subset of $\mathcal{N}$ that is clearly order isomorphic to $\mathcal{N} \restriction Z$. By Corollary 5.3, $\mathcal{N} \restriction Z$ is strongly separable. $\square$

Lemma 5.7. *If $\mathcal{N}$ is a halfdirection on X then there is an open neighborhood V of $*$ such that $\mathcal{N} \restriction V$ is a strongly separable halfdirection on V.*

Proof. Let $\mathcal{N}$ be a halfdirection in X. If V is any subspace of X then clearly $\mathcal{N} \restriction V$ is a halfdirection in V. Thus we must find an open $V \ni *$ for which $\mathcal{N} \restriction V$ is strongly separable. We consider three cases:

Case 1: Suppose $* \notin U$ for every $U \in \mathcal{N}$. The cases $|\mathcal{N}| = 0, 1, 2$ are trivial, so assume there is some nonempty, nonmaximal $U \in \mathcal{N}$. Since $\mathcal{N}$ is a halfdirection and U is not maximal, there is some $U' \in \mathcal{N}$ such that $\overline{U} \subseteq U'$. Since $* \notin U'$, $* \notin \overline{U}$. Take $V = X \setminus \overline{U}$.

Let $\alpha \in \omega_1$ such that $U \cap ((0,1] \times \{\alpha\}) \neq \emptyset$ (some such α must exist as U is nonempty and $* \notin U$). Let $Y = \{*\} \cup ((0,1] \times \{\alpha\})$ and note that Y is homeomorphic to $[0,1]$. In particular, $\mathcal{N} \restriction Y$ is strongly separable by Lemma 5.5. Consider the map $\pi : \mathcal{N} \to \mathcal{N} \restriction Y$ given by $\pi(W) = W \cap Y$. This map is clearly an order preserving surjection.

If π is not injective then there exist $W_1, W_2 \in \mathcal{N}$ such that $\overline{W_1}^X \subseteq W_2$ and $W_1 \cap Y = W_2 \cap Y$. Then $W_1 \cap Y \subseteq \overline{W_1 \cap Y}^Y \subseteq \overline{W}_1^X \cap Y \subseteq W_2 \cap Y = W_1 \cap Y$. Thus $W_1 \cap Y = \overline{W_1 \cap Y}^Y$. Since Y is connected and W_1 is open, this is only possible if $W_1 \cap Y = \emptyset$ or $W_1 \cap Y = Y$. The latter is impossible by the assumption $* \notin W_1$, so we have $W_1 \cap Y = \emptyset$, which implies $W_2 \cap Y = \emptyset$ as well. Thus $\pi(W_1) = \pi(W_2)$ if and only if $W_1 \cap Y = W_2 \cap Y = \emptyset$. That is, π fails to be injective only on those pairs of sets in $\mathcal{N}$ that miss Y.

Thus π provides an order isomorphism from a final segment of $\mathcal{N}$, namely $\mathcal{N}' = \{W \in \mathcal{N} \colon Y \cap W \neq \emptyset\}$, onto $\mathcal{N} \restriction Y$.

We claim next that $\mathcal{N} \restriction V$ is isomorphic to a subset of $\mathcal{N}'$. If $W \in \mathcal{N}$ and $W \cap V = W \setminus \overline{U} \neq \emptyset$, then $W \not\subseteq U$ and, using the fact $\mathcal{N}$ is a nest, $U \subseteq W$. Because $U \cap Y \neq \emptyset$, we have $W \cap Y \neq \emptyset$, which implies $W \cap Y \in \mathcal{N} \restriction Y$. Thus we obtain a natural map from $\mathcal{N} \restriction V$ to $\mathcal{N} \restriction Y$, namely $W \cap V \mapsto W \cap Y$. This map is clearly order preserving, and it is injective for the same reason that π is injective (if $W_1 \cap V = W_2 \cap V \neq \emptyset$, then supposing $W_1 \neq W_2$ ultimately contradicts the connectedness of Y).

Therefore $\mathcal{N}\restriction V$ is order isomorphic to a subset of $\mathcal{N}\restriction Y$. By Lemmas 5.6 and 5.3, $\mathcal{N}\restriction V$ is strongly separable.

Case 2: Suppose $* \in U$ for every $U \in \mathcal{N}$. This is similar to Case 1. Assume there is some $V \in \mathcal{N}$ such that $V \neq X$, and pick some $(x,\alpha) \in X \setminus \{*\}$ such that $(x,\alpha) \notin V$.

Let $Y = \{*\} \cup ((0,1] \times \{\alpha\})$, and consider the map $\pi : \mathcal{N} \to \mathcal{N}\restriction Y$ defined by $\pi(U) = U \cap Y$. As in Case 1, if $U, W \in \mathcal{N}$ then $\pi(U) = \pi(W)$ if and only if $U \cap Y = W \cap Y = \emptyset$ or $U \cap Y = W \cap Y = Y$. The former case is impossible since $* \in U \cap Y$ for any $U \in \mathcal{N}$. Thus π is an injection from $\{U \in \mathcal{N} \colon U \cap Y \neq Y\}$ into $\mathcal{N}\restriction Y$. This proves, as above, that $\{U \in \mathcal{N} \colon U \cap Y \neq Y\}$ is strongly separable. $\mathcal{N}\restriction V$ is isomorphic to a subset of this order, so that it too must be strongly separable.

Case 3: Suppose $* \in U$ for some $U \in \mathcal{N}$ and $* \notin U'$ for some $U' \in \mathcal{N}$. Let $\mathcal{N}_0 = \{U \in \mathcal{N} \colon * \notin U\}$ and $\mathcal{N}_1 = \{U \in \mathcal{N} \colon * \in U\}$. $\mathcal{N}_0$ and $\mathcal{N}_1$ are both complete halfdirections in X and, by Cases 1 and 2, each is strongly separable. Therefore there are open neighborhoods V_0 and V_1 of $*$ such that $\mathcal{N}_0\restriction V_0$ and $\mathcal{N}_1\restriction V_1$ are both strongly separable. Let $V = V_0 \cap V_1$. Since $\mathcal{N}_0 \restriction V$ and $\mathcal{N}_1 \restriction V$ are both strongly separable, it follows from Lemma 5.2 that $\mathcal{N}\restriction V = \mathcal{N}_0\restriction V \cup \mathcal{N}_1\restriction V$ is strongly separable. □

Suppose now that $\mathrm{D{\scriptstyle IM}}(X)$ is finite and let $\mathcal{N}_0, ..., \mathcal{N}_n$ be a collection of halfdirections on X whose union is a subbasis for X. By Lemma 5.7 we may, for each i, find some open neighborhood V_i of $*$ such that $\mathcal{N}_i\restriction V_i$ is a strongly separable halfdirection on V_i. Moreover, setting $V = \bigcap_{i\leq n} V_i$, V is an open neighborhood of $*$ such that, for each $i \leq n$, $\mathcal{N}_i\restriction V$ is a strongly separable halfdirection on V by Lemma 5.6. Recalling that halfdirection is a special kind of nest, it follows from Lemma 5.4 that V is second countable. This is a contradiction: it is clear that no neighborhood of $*$ in X can be second countable.

This completes the proof of Theorem 2.5. □

In conclusion, we point out that it is possible to pinpoint exactly the number of halfdirections needed to generate the topology of the $\aleph_1$-spined metric hedgehog:

Corollary 5.8. *If X denotes the metric hedgehog with $\aleph_1$ spines as above, then* $\mathrm{D{\scriptstyle IM}}(X) = \mathrm{Dim}(X) = \aleph_0$.

Proof. Gerlits has given an upper bound of $\aleph_0$ for $\mathrm{Dim}(X)$, the directional dimension of X (see [7], Lemma 4.3). Recalling that $\mathrm{D{\scriptstyle IM}}(X) \leq \mathrm{Dim}(X)$ for any space X, $\mathrm{D{\scriptstyle IM}}(X) \leq \aleph_0$. Our Theorem 2.5 provides the opposite inequality. □

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