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ANOTHER CONSTRUCTION OF SEMI-TOPOLOGICAL GROUPS

AKIO KATO

Abstract. For a nowhere compact, metrizable topological group G we use Stone-Čech compactifications once or twice to get an extremally disconnected semi-topological group $\breve{G}$ admitting a semi-open isomorphism onto G.

1. Introduction

Recall that for every space X there exists an extremally disconnected space $\mathbf{E}(X)$ called the "absolute", with a perfect irreducible map onto X. It has been well known (cf.[7, 9]) that given a topological group G one can find an extremally disconnected semi-topological group in the absolute $\mathbf{E}(G)$ admitting a semi-open isomorphism onto G. In this paper we will construct such a semi-topological group using Stone-Čech compactifications once or twice rather than the absolute, and this construction has an advantage in investigating the properties of resultant spaces. The idea of repeating Stone-Čech compactifications stems from [12, 13].

2. Basic Tools

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. βX denotes the Stone-Čech compactification of X. A space is *nowhere compact* (or nowhere locally compact) if it has no compact neighborhood, which is equivalent to say that the remainder $c\,X\setminus X$ of any or some compactification $c\,X$ of X is dense in $c\,X$. A collection of nonempty open sets of X is called a π-*base* for X if every nonempty open set in X contains some member of the collection. The minimal cardinality of such a π-base is called the π-*weight* of X. Observe that any dense subspace of X has

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the same π-weight as X, and that any space of countable π-weight is separable. So, for example, let $\mathbb{Q}$ be the space of rationals; then all of $\mathbb{Q}$, $\beta\mathbb{Q}$, $\mathbb{Q}^* = \beta\mathbb{Q}\backslash\mathbb{Q}$, $\beta\mathbb{Q}^*$, $\mathbb{Q}^{**} = \beta\mathbb{Q}^*\backslash\mathbb{Q}^*$ are of countable π-weight and separable.

As a basic tool we use perfect irreducible maps. Let g be a map from X onto Y. For a subset $U \subseteq X$ define $g^\circ(U) \subseteq Y$ by

$$y \in g^\circ(U) \text{ if and only if } g^{-1}(y) \subseteq U,$$

i.e., $g^\circ(U) = Y\backslash g(X\backslash U) \subseteq g(U)$. Note an obvious, but useful, formula

$$g^\circ(U \cap V) = g^\circ(U) \cap g^\circ(V)$$

for any sets $U, V \subseteq X$, which especially implies that $g^\circ(U) \cap g^\circ(V) = \emptyset$ whenever $U \cap V = \emptyset$. An onto map g is called *irreducible* if $g^\circ(U) \neq \emptyset$ for every non-empty open set U, and *semi-open* if $g^\circ(U)$ has nonempty interior for every non-empty open set U. So, closed irreducible implies semi-open, and semi-open implies irreducible. A closed map with compact fibers are called *perfect*. We assume a perfect map is always onto.

Fact 2.1. Let $g : X \to Y$ be any closed irreducible map. Then
(1) $g^\circ(U)$ is non-empty and open whenever U is. Moreover,

$$\mathrm{cl}_Y\, g^\circ(U) = \mathrm{cl}_Y\, g(U) = g(\mathrm{cl}_X U)$$

for every open subset $U \subseteq X$.
(2) g preserves density, i.e., for any dense subset D of Y its inverse $g^{-1}(D)$ is also dense in X, and its restriction to $g^{-1}(D) \to D$ is closed irreducible.
(3) Let $E \subseteq X$ be any subset such that $g(E) = Y$. Then E is dense in X, and the restriction map $g \restriction E : E \to Y$ has the property that for any nonempty open subset U of E, there exists a nonempty open subset W of Y such that $(g \restriction E)^{-1}(W) \subseteq U$. In particular, $g \restriction E$ is semi-open, though need not be closed.
(4) g preserves π-weight, i.e., a π-base $\mathcal{B}$ of X induces a π-base $\{g^\circ(U) : U \in \mathcal{B}\}$ of Y, and a π-base $\mathcal{C}$ of Y induces a π-base $\{g^{-1}(V) : V \in \mathcal{C}\}$ of X.
(5) If g is perfect irreducible, it preserves nowhere compactness.

Proof. (1) Though this is well known (cf. Ch.6, §2 in [14] or 10.49 in [15]), for completeness we give a proof of the equality $\mathrm{cl}_Y\, g^\circ(U) = \mathrm{cl}_Y\, g(U)$. It suffices to show $g(U) \subseteq \mathrm{cl}_Y\, g^\circ(U)$. Let $y \in g(U)$, and take any open neighborhood W of y. Then $U \cap g^{-1}(W) \neq \emptyset$ implies

$$\emptyset \neq g^\circ(U \cap g^{-1}(W)) \subseteq g^\circ(U) \cap W,$$

hence $\emptyset \neq g^\circ(U) \cap W$, proving $y \in \mathrm{cl}_Y\, g^\circ(U)$.
Other assertions (2), (3), (4) and (5) are easy to see. $\square$

Lemma 2.2. *Let $\phi : X \to Y$ be a perfect map and let $\Phi : bX \to cY$ be its extension where bX and cY are some compactifications of X and Y respectively. Then Φ maps the remainder of X onto that of Y, i.e., $\Phi(bX\backslash X) = cY\backslash Y$. Moreover,*
(1) ϕ is perfect irreducible if and only if Φ is.
(2) If ϕ is perfect irreducible and X (hence Y also) is nowhere compact, then the restriction of Φ to the remainders

$$bX\backslash X \to cY\backslash Y$$

is also perfect irreducible.

Proof. The equality $\Phi(bX\backslash X) = cY\backslash Y$ follows from the characteristic property of a perfect map which states that "a perfect map $\phi : X \to Y$ can not be extended to $\widetilde{X} \to Y$ for any (Hausdorff) space $\widetilde{X}$ containing X as a dense proper subspace" (see Lemma 3.7.14 in [8]). Then, (1) is easy, and (2) follows from Fact 2.1 (2), (5). $\square$

Perfect irreducible maps we encounter frequently in this paper are those induced by some homeomorphisms, e.g., when the above ϕ is an identity map.

3. Construction

Let $(G, \cdot\,)$ be a nowhere compact, dense-in-itself, metrizable topological group with the identity element e. For example, $(G, \cdot\,)$ can be the group $(\mathbb{Q}, +)$ of the rationals, the group $(\mathbb{Z}^\omega, +) \approx \mathbb{P}$ of the irrationals, or the direct sum $\bigoplus_\omega \mathbb{Z}(2)$ of the countable copies of $\mathbb{Z}(2) = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$. For a space X we denote by $\mathbf{H}(X)$ the collection of all homeomorphisms $h : X \approx X$. Let us fix some nonempty collection $\mathcal{H} \subseteq \mathbf{H}(G)$, and choose a compactification $c\,G$ of G such that

($\star$) every $h \in \mathcal{H}$ extends to $c(h) \in \mathbf{H}(c\,G)$.

In case we can not find such $c\,G$ at hand, we can take $c\,G = \beta\,G$. Let $G^{(1)} = c\,G\backslash G$ be the remainder, and define $h^{(1)} \in \mathbf{H}(G^{(1)})$ to be the restriction of $c(h)$ to the remainder $G^{(1)}$. Next consider the Stone-Čech compactification $\beta G^{(1)}$ of $G^{(1)}$ and the Stone extension $\beta h^{(1)} \in \mathbf{H}(\beta G^{(1)})$ of $h^{(1)}$. Let $G^{(2)} = \beta G^{(1)}\backslash G^{(1)}$ be the remainder, and define $h^{(2)} \in \mathbf{H}(G^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $G^{(2)}$; so that

$$h : G \approx G,\ \ h^{(1)} : G^{(1)} \approx G^{(1)},\ \ h^{(2)} : G^{(2)} \approx G^{(2)}.$$

Note that $G^{(1)}$ is dense in $c\,G$, and $G^{(2)}$ is dense in $\beta G^{(1)}$, since we assume that G is nowhere compact. Viewing that $c\,G$ is a compactification of $G^{(1)}$, we can consider the Stone extension $\Phi : \beta G^{(1)} \to c\,G$ of the identity map $id_{G^{(1)}} : G^{(1)} = G^{(1)}$. Let $\phi : G^{(2)} \to G$ be the restriction of Φ. Then it follows from Lemma 2.2 that both Φ and ϕ are perfect irreducible maps.

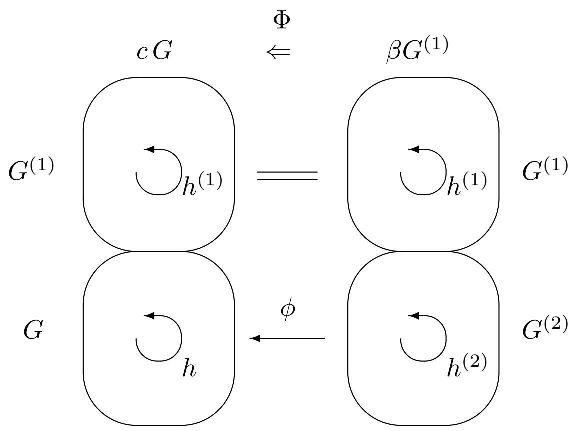

FIG. 1

We can show that the correspondence $\mathbf{H}(G) \supseteq \mathcal{H} \ni h \mapsto h^{(2)} \in \mathbf{H}(G^{(2)})$ is compatible with the perfect irreducible map ϕ, i.e.,

Lemma 3.1. $h \circ \phi = \phi \circ h^{(2)} : G^{(2)} \to G.$

Proof. To show this equality, it suffices to prove

$$c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta G^{(1)} \to cG,$$

which follows from the clear equality

$$h^{(1)} \circ id_{G^{(1)}} = id_{G^{(1)}} \circ h^{(1)} : G^{(1)} \to G^{(1)}$$

on the dense subset $G^{(1)}$ of $\beta G^{(1)}$. $\square$

Corollary 3.2. *If $h(x) = y$ for $x, y \in G$, then $h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y)$.*

Proof. The inclusion $h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y)$ follows from Lemma 3.1. Since h is a homeomorphism, we can replace h by h^{-1} to get the reverse inclusion. $\square$

We need to point out here that the map $h^{(2)}$ satisfying the equality $h \circ \phi = \phi \circ h^{(2)}$ is uniquely determined by h and ϕ. This follows from the next fact called the "cancellation law", peculiar to irreducible maps (see [10]).

Fact 3.3. Let $f, g : X \to Y$, $\varphi : Y \to Z$ be any maps such that $\varphi \circ f = \varphi \circ g$, and suppose that f, g are semi-open, and $\varphi \circ f = \varphi \circ g$ is irreducible. Then we get $f = g$.

Proof. For completeness we give a proof of this fact. Note first that the irreducibility of $\varphi \circ f = \varphi \circ g$ implies that of φ. Suppose $f \neq g$, and take $x \in X$ such that $f(x) \neq g(x)$ in Y. Then, since Y is Hausdorff (recall our tacit assumption that all spaces are Tychonoff), we can choose disjoint open sets U_1, U_2 in Y such that $f(x) \in U_1$ and $g(x) \in U_2$. Put $W = f^{-1}(U_1) \cap g^{-1}(U_2)$. Then W is an open neighborhood of x, and hence nonempty. Therefore $f^\circ(W)$, $g^\circ(W)$ are nonempty because of the irreducibility of f, g. On the other hand, since $f(W) \cap g(W) \subseteq U_1 \cap U_2 = \emptyset$, the condition $\varphi \circ f(W) = \varphi \circ g(W)$ implies $\varphi(Y \backslash g(W)) = Z$, and consequently $\varphi(Y \backslash g^\circ(W)) = Z$. This contradicts the irreducibility of φ, because the interior of $g^\circ(W)$ is nonempty by our assumption that g is semi-open. $\square$

Now let us choose $\mathcal{H} \subseteq \mathbf{H}(G)$ such that

$$\mathcal{H} = \{T_x : x \in G\} \cup \{J\}$$

where T_x is a left multiplication $T_x(y) = x \cdot y$ by x and J is the inverse operation $J(x) = x^{-1}$, and suppose $\mathcal{H}$ satisfies the above condition $(\star)$. Then we get $T_x^{(2)}, J^{(2)} \in \mathbf{H}(G^{(2)})$ and a perfect irreducible map $\phi : G^{(2)} \to G$ which satisfy by Lemma 3.1

$$T_x \circ \phi = \phi \circ T_x^{(2)}, \quad J \circ \phi = \phi \circ J^{(2)}$$

for every $x \in G$. Since $T_0 = id_G$ and $\phi \circ T_0^{(2)} = T_0 \circ \phi = \phi = \phi \circ id_{G^{(2)}}$, Fact 3.3 implies $T_0^{(2)} = id_{G^{(2)}}$. In a similar way we can see that the relations
$T_x \circ T_y = T_{x \cdot y}, \;\; J \circ J = id_Q, \;\; J \circ T_{x^{-1}} = T_x \circ J$ imply

$$T_x^{(2)} \circ T_y^{(2)} = T_{x \cdot y}^{(2)}, \quad J^{(2)} \circ J^{(2)} = id_{G^{(2)}}, \quad J^{(2)} \circ T_{x^{-1}}^{(2)} = T_x^{(2)} \circ J^{(2)}$$

respectively. Hence $J^{(2)}$ is an involution, and it follows from $T_x^{(2)} \circ T_{-x}^{(2)} = T_0^{(2)} = id_{G^{(2)}}$ that $T_{x^{-1}}^{(2)} = (T_x^{(2)})^{-1}$.

Choose one point $\breve{e}$ of the fiber $\phi^{-1}(e)$ of the identity $e \in G$, which we fix from now on, and put $\breve{x} = T_x^{(2)}(\breve{e})$ for $x \in G$. Define $G(\breve{e})$ by

$$G(\breve{e}) = \{\breve{x} : x \in G\} \subseteq G^{(2)}$$

which is an orbit of $\breve{e}$ by $\{T_x^{(2)} : x \in G\}$. Note that Corollary 3.2 implies that $\breve{x} \in \phi^{-1}(x)$ for each $x \in G$, and hence, $G(\breve{e})$ is dense in $G^{(2)}$ since ϕ is perfect irreducible. Define the multiplication $\otimes$ in $G(\breve{e})$ by

$$\breve{x} \otimes \breve{y} = (x \cdot y)\breve{}.$$

Then it is easy to see that $(G(\breve{e}), \otimes)$ is a group with the identity $\breve{e}$ and the inverse operation $\breve{x} \to (x^{-1})\breve{}$. Denote the restriction $\phi \restriction G(\breve{e})$ by

$$\breve{\phi} : \; (G(\breve{e}), \otimes) \to (G, \cdot).$$

Then $\breve{\phi}$ is algebraically an isomorphism, while topologically, a semi-open map by Fact 2.1 (3). The equality

$$\breve{x} \otimes \breve{y} = (x \cdot y)\breve{} = T^{(2)}_{x \cdot y}(\breve{e}) = T^{(2)}_x \circ T^{(2)}_y(\breve{e}) = T^{(2)}_x(\breve{y})$$

shows that if we fix $x \in G$, the "left action" $\breve{y} \to \breve{x} \otimes \breve{y}$ is continuous w.r.t. $\breve{y}$. Thus, using the terminologies in [1] we can conclude that $(G(\breve{e}), \otimes)$ is a "left-topological group". Since $J(e) = e$, we have $J^{(2)}(\phi^{-1}(e)) = \phi^{-1}(e)$, hence both $\breve{e}$ and $J^{(2)}(\breve{e})$ belong to the same fiber $\phi^{-1}(e)$. In general, as we see later in §6, $J^{(2)}$ does not fix $\breve{e}$. An obvious exception is the case $J = id_G$, for example if G is a Boolean $\bigoplus_\omega \mathbb{Z}(2)$, we have $J^{(2)} = id_{G^{(2)}}$ so that $J^{(2)}(\breve{e}) = \breve{e}$.

Lemma 3.4. *Suppose $x_n (n \in \omega) \to x$ is a convergent sequence in G. Then the countable discrete set $\{\breve{x}_n : n \in \omega\}$ is C^*-embedded in $G(\breve{e})$. (Note that it is not necessarily true that $\breve{x} \in \mathrm{cl}\,\{\breve{x}_n : n \in \omega\}$ in $G(\breve{e})$.)*

Proof. Put $F = \{\breve{x}_n : n \in \omega\}$ and $W = \beta G^{(1)} \backslash \phi^{-1}(x)$. Then F is a closed subset of W. Notice that W is a cozero-set in $\beta G^{(1)}$, i.e., $\phi^{-1}(x) = \Phi^{-1}(x)$ is a zero-set in $G^{(2)}$, because we assume G is first countable. Hence W is Lindelöf (σ-compact) so that F is a closed subset of the Lindelöf, hence normal, space W. Therefore F is C^*-embedded in W. On the other hand, the condition $G^{(1)} \subseteq W \subseteq \beta G^{(1)}$ implies that W is C^*-embedded in $\beta G^{(1)}$. Thus we can conclude that F is C^*-embedded in $\beta G^{(1)}$. □

Corollary 3.5. *$G(\breve{e})$ does not contain any convergent sequence, hence, $G(\breve{e})$ is not first countable. Consequently, $(G(\breve{e}), \otimes)$ is not a topological group if G is separable.*

Proof. Suppose $G(\breve{e})$ contains a convergent sequence $\breve{x}_n (n \in \omega) \to \breve{x}$. Then $x_n (n \in \omega) \to x$ in G. Hence Lemma 3.4 implies that $\{\breve{x}_n : n \in \omega\}$ is C^*-embedded in $G(\breve{e})$, which contradicts with the fact that $\breve{x}_n (n \in \omega)$ converges to $\breve{x}$. Now suppose G is separable, i.e., second countable as we assume G is metrizable. Hence Fact 2.1 (4) implies that $G^{(2)}$ has countable π-weight, and so does its dense subset $G(\breve{e})$. As is well known, any topological group of countable π-weight must be first countable. Therefore we can conclude $(G(\breve{e}), \otimes)$ is not a topological group. □

Since $G(\breve{e})$ is automatically determined once we choose the point $\breve{e} \in \phi^{-1}(e)$, we next investigate what kind of point we can select from the fiber $\phi^{-1}(e)$.

4. Remote Points and Extremally Disconnected Points

Let X be a dense subset of Y. A point $p \in Y \backslash X$ is called *remote from X*, if $p \notin \mathrm{cl}_Y F$ for every nowhere dense closed subset F of X.

In case $Y = \beta X$ we simply call such a point p as a *remote point of* X. The following is known about the existence of remote points.

Fact 4.1. (cf. [4, 5, 11]) Every non-pseudocompact dense-in-itself space X has remote points if X has a σ-locally finite π-base.

In particular, it follows from this Fact that if a non-pseudocompact dense-in-itself space X is metrizable or of countable π-weight, then the set of all remote points of X is dense in the remainder $\beta X \backslash X$.

A space Y is said to be *extremally disconnected at a point* $p \in Y$ (see [5]) if $p \notin \mathrm{cl}_Y\, U_1 \cap \mathrm{cl}_Y\, U_2$ for every pair of disjoint open sets U_1, U_2 in Y. We call such a point p an *extremally disconnected point* of Y, or simply, an *e.d.* point of Y. Obviously a space Y is extremally disconnected if every point of Y is an e.d. point. If S is dense in Y, we always have $\mathrm{cl}_Y U = \mathrm{cl}_Y (U \cap S)$ for every open set U of Y. So, an equivalent definition of an e.d. point is given using only open subsets of any dense subset $S \subseteq Y$:

$p \in Y$ is an e.d. point if and only if $p \notin \mathrm{cl}_Y\, V_1 \cap \mathrm{cl}_Y\, V_2$ for every pair of disjoint open sets V_1, V_2 in S.

Note that this definition does not depend on the choice of the dense subset S, while it is clear that the notion of remote points depends on the choice of the dense subset S. We denote by $\mathrm{Ed}(Y)$ the set of all e.d. points of Y. The next fact proved by van Douwen [5] tells that "remote" implies "e.d".

Fact 4.2. If $p \in \beta X \backslash X$ is remote from X, then p is an e.d. point of βX.

This fact follows from the formula in [5]

$$\mathrm{Bd}_{\beta X} \mathrm{Ex}(U) = \mathrm{cl}_{\beta X} \mathrm{Bd}_X(U)$$

which holds for any space X, where $\mathrm{Ex}(U) = \beta X \backslash \mathrm{cl}_{\beta X}(X \backslash U)$ is the maximal open extension of U, and $\mathrm{Bd}_Y(W) = \mathrm{cl}_Y W \backslash W$ denotes the boundary of an open set W of Y.

Lemma 4.3. *Suppose A is a closed subset of a normal space X. Then $A \subseteq \mathrm{Ed}(X)$ implies $\mathrm{cl}_{\beta X} A \subseteq \mathrm{Ed}(\beta X)$.*

Proof. Let A be a closed subset of a normal space X, and that $A \subseteq \mathrm{Ed}(X)$. Suppose $p \in \beta X \backslash \mathrm{Ed}(\beta X)$, then there exist open disjoint sets U, V in X such that $p \in \mathrm{Bd}_{\beta X} \mathrm{Ex}(U) \cap \mathrm{Bd}_{\beta X} \mathrm{Ex}(V)$. Using the above formula and the normality of X, we get

$$\begin{aligned} \mathrm{Bd}_{\beta X} \mathrm{Ex}(U) \cap \mathrm{Bd}_{\beta X} \mathrm{Ex}(V) &= \mathrm{cl}_{\beta X} \mathrm{Bd}_X(U) \cap \mathrm{cl}_{\beta X} \mathrm{Bd}_X(V) \\ &= \mathrm{cl}_{\beta X}(\mathrm{Bd}_X(U) \cap \mathrm{Bd}_X(V)). \end{aligned}$$

Put $F = \mathrm{Bd}_X(U) \cap \mathrm{Bd}_X(V)$. Then $F \cap \mathrm{Ed}(X) = \emptyset$. Hence F and A are disjoint closed sets in the normal space X, so that $\mathrm{cl}_{\beta X} F \cap \mathrm{cl}_{\beta X} A = \emptyset$. This proves $p \in \beta X \backslash \mathrm{cl}_{\beta X} A$. $\square$

5. Extremally Disconnected Semi-Topological Groups

Now, using the results in §4 we continue the construction in §3 to make $G(\breve{e})$ extremally disconnected. Recall that G is metrizable. Our construction depends on whether or not G is separable. Put $W = \beta G^{(1)} \backslash \phi^{-1}(e)$; then

$$G^{(1)} \subseteq W \subseteq \beta G^{(1)} = \beta W.$$

Case 1 : G is separable.
Since G is of countable π-weight, so are $\beta G, \beta G^{(1)}$ and their dense subspaces. In particular, W is of countable π-weight. Hence, by Fact 4.1, $\phi^{-1}(e) = \beta W \backslash W$ contains points remote from W. Select $\breve{e} \in \phi^{-1}(e)$ as one of such remote points of W. Then

$$\breve{e} \in \mathrm{Ed}(G^{(2)}),$$

and consequently $G(\breve{e}) \subseteq \mathrm{Ed}(G^{(2)})$ because each $T_x^{(2)}$ for $x \in G$ is a homeomorphism of $G^{(2)}$.

Case 2 : G is not separable.
Note that in this case W is not of countable π-weight, so we can not use the same argument as Case 1. Choose $cG = \beta G$ as a compactification of G. Since G is metrizable, Fact 4.1 implies that the set $\rho(G)$ of remote points of G is dense in $G^{(1)}$, hence dense also in W. Since $\phi^{-1}(e)$ is a zero-set of $\beta G^{(1)}$, we can choose a countable discrete closed subset A of W such that $A \subseteq \rho(G) \subseteq W$. Fact 4.2 implies that

$$A \subseteq \rho(G) \subseteq \mathrm{Ed}(G^{(1)}) \subseteq \mathrm{Ed}(\beta G^{(1)}) = \mathrm{Ed}(\beta W).$$

Since the cozero-set W of $\beta G^{(1)}$ is Lindelöf, hence normal, by Lemma 4.3 we get $A^* = \mathrm{cl}_{\beta W} A \backslash A \subseteq \mathrm{Ed}(\beta W)$. Now select $\breve{e}$ as

$$\breve{e} \in A^* \cap \phi^{-1}(e).$$

Then

$$\breve{e} \in G(\breve{e}) \subseteq \mathrm{Ed}(G^{(2)}).$$

Thus, in either case we have succeeded in constructing an extremally disconnected $G(\breve{e})$. Note that $\breve{e}$ in Case 1 is a remote point of W, but $\breve{e}$ in Case 2 is not, being accessible by the discrete closed set A of W. Nevertheless, we can show

Property 5.1. *For every nowhere dense subset F of $G \backslash \{e\}$ we have*

$$\breve{e} \notin \mathrm{cl}\, \breve{\phi}^{-1}(F) \ \text{ in } G(\breve{e}).$$

Proof. Case 1 is easy. Indeed, since ϕ is perfect irreducible, $\phi^{-1}(F)$ is nowhere dense in $G^{(2)}\backslash\phi^{-1}(e)$, hence also in $\beta G^{(1)}\backslash\phi^{-1}(e) = W$. Since $\breve{e}$ is chosen to be remote from W, we get $\breve{e} \notin \mathrm{cl}\ \phi^{-1}(F)$ in $\beta W = \beta G^{(1)}$, which obviously implies $\breve{e} \notin \mathrm{cl}\ \breve{\phi}^{-1}(F)$ in $\breve{G}$. Next let us consider Case 2. Since every point of A in Case 2 is remote from G, we have $A \cap \mathrm{cl}\, F = \emptyset$ in βG, which obviously implies $A \cap\ \mathrm{cl}\, \phi^{-1}(F) = \emptyset$ in $\beta G^{(1)}$. Since $A,\ \phi^{-1}(F) \subseteq W \subseteq \beta G^{(1)}$, we get

$$A \cap \mathrm{cl}_W\, \phi^{-1}(F) = \emptyset \quad \text{in } W.$$

This implies $A^* \cap \mathrm{cl}\, \phi^{-1}(F) = \emptyset$ in $\beta W = \beta G^{(1)}$, because of the normality of W. This completes the proof since $\breve{e} \in A^*$. $\square$

Summarizing the hitherto results, we get

Theorem 5.2. *Let $(G, \cdot\,)$ be a nowhere compact, dense-in-itself, metrizable topological group. Then there exist a left-topological group $(G(\breve{e}), \otimes)$ with no convergent sequence, and a semi-open isomorphism $\breve{\phi} :\ (G(\breve{e}), \otimes) \to (G, \cdot\,)$. We can find this $G(\breve{e})$ as a dense subset of $G^{(2)} = \beta G^{(1)}\backslash G^{(1)}$ where $G^{(1)} = \beta G\backslash G$. Moreover, we can make $G(\breve{e})$ to be an extremally disconnected space with the property that*

$$\breve{x} \notin \mathrm{cl}\ \breve{\phi}^{-1}(F) \ \text{ in } G(\breve{e}).$$

for every $x \in G$ and every nowhere dense set F of $G\backslash\{x\}$, where $\breve{\phi}(\breve{x}) = x$.

Recall that a group is called a *semitopological group* if it has a topology such that left and right multiplications are separately continuous. When G is Abelian, our example $G(\breve{e})$ in the above theorem, being Abelian, is a semitopological group.

6. The case $G = (\mathbb{Q}/\mathbb{Z}, +)$

Here we examine Theorem 5.2 for the special case that $(G, \cdot\,)$ is the countable dense subgroup $Q = (\mathbb{Q}/\mathbb{Z}, +)$ of the circle group $\mathbb{T} = (\mathbb{R}/\mathbb{Z}, +)$ where "+" is the addition modulo 1. As this addition "+" is commutative, let us express the corresponding $(G(\breve{e}), \otimes)$ by $(Q(\breve{0}), \oplus)$. We can take $\mathbb{T}$ as a compactification cQ of Q satisfying the condition $(\star)$ in §3. Put $P = \mathbb{T}\backslash Q$. Then $Q^{(1)} = P$, and $(Q(\breve{0}), \oplus)$ is such that

$$Q(\breve{0}) = \{\breve{r} : r \in Q\} \subseteq Q^{(2)} = \beta P\backslash P, \qquad \breve{r} \oplus \breve{s} = (r+s)\breve{}.$$

We express each element of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ using $0 \leqslant t \leqslant 1$ identifying 0=1; then the inverse operation J of $\mathbb{T}$ will be expressed as $J(t) = 1 - t$. Put $\mathbb{T}_+ = (0, 1/2),\ \mathbb{T}_- = (1/2, 1)$ and

$$Q_+ = \mathbb{T}_+ \cap Q,\ \ Q_- = \mathbb{T}_- \cap Q,\ \ P_+ = \mathbb{T}_+ \cap P,\ \ P_- = \mathbb{T}_- \cap P.$$

Then the clopen partition $P = P_+ \cup P_-$ of P induces that of βP

$$\beta P = \mathrm{Ex}(P_+) \cup \mathrm{Ex}(P_-)$$

where $\Phi^{-1}(\mathbb{T}_+) \subseteq \mathrm{cl}_{\beta P} P_+ = \mathrm{Ex}(P_+)$ and $\Phi^{-1}(\mathbb{T}_-) \subseteq \mathrm{cl}_{\beta P} P_- = \mathrm{Ex}(P_-)$. Since $0 \in [0, 1/2] = \mathrm{cl}_{\mathbb{T}} P_+ = \Phi(\mathrm{Ex}(P_+))$ and $0\,(=1) \in [1/2, 1] = \mathrm{cl}_{\mathbb{T}} P_- = \Phi(\mathrm{Ex}(P_-))$, both $\phi^{-1}(0) \cap \mathrm{Ex}(P_+)$ and $\phi^{-1}(0) \cap \mathrm{Ex}(P_-)$ are nonempty. Putting $U_+ = \mathrm{Ex}(P_+) \cap Q^{(2)}$, $U_- = \mathrm{Ex}(P_-) \cap Q^{(2)}$, we can conclude that $Q^{(2)}$ is partitioned into two clopen sets $Q^{(2)} = U_+ \cup U_-$ such that $\phi^{-1}(Q_+) \subseteq U_+$, $\phi^{-1}(Q_-) \subseteq U_-$, and

$$\phi^{-1}(0) = (\phi^{-1}(0) \cap U_+) \cup (\phi^{-1}(0) \cap U_-)$$

is a partition into two nonempty clopen sets of $\phi^{-1}(0)$. The identity element $\breve{0}$ of $Q(\breve{0})$, chosen from the fiber $\phi^{-1}(0)$, must belong to either U_+ or U_-. Taking account of symmetry, let us assume that $\breve{0} \in U_+$. Note that both the translation $T_{1/2}$ and the inverse operation J exchange Q_+ with Q_-. Therefore $T^{(2)}_{1/2}$ and $J^{(2)}$ exchange $\phi^{-1}(Q_+)$ with $\phi^{-1}(Q_-)$, and hence, exchange U_+ with U_-. So the condition $\breve{0} \in U_+$ implies that both $(1/2)\breve{}$ and $J^{(2)}(\breve{0})$ belong to U_-. Put

$$\breve{Q}_+ = \{\breve{r} : r \in Q_+\} \subseteq \phi^{-1}(Q_+) \ \text{ and } \ \breve{Q}_- = \{\breve{r} : r \in Q_-\} \subseteq \phi^{-1}(Q_-).$$

Then $\breve{Q}_+, \breve{Q}_-$ are dense in U_+, U_-, respectively. Since U_+, U_- are clopen in $Q^{(2)}$, we can conclude that $\breve{0} \in \mathrm{cl}\, \breve{Q}_+$, $(1/2)\breve{} \in \mathrm{cl}\, \breve{Q}_-$ in $Q(\breve{0})$, and that $Q(\breve{0})$ is partitioned into two clopen sets

$$Q(\breve{0}) = (\{\breve{0}\} \cup \breve{Q}_+) \ \cup \ (\{(1/2)\breve{}\} \cup \breve{Q}_-). \tag{6−0}$$

Property 6.1. *The inverse operation of* $(Q(\breve{0}), \oplus)$ *is not continuous.*

Proof. The inverse operation $\zeta(\breve{r}) = (1-r)\breve{}$ of $(Q(\breve{0}), \oplus)$ has the property that $\zeta(\breve{Q}_+) = \breve{Q}_-$ and $\zeta(\breve{0}) = \breve{0}$, $\zeta((1/2)\breve{}) = (1/2)\breve{}$. If ζ were continuous, we would have $\zeta(\mathrm{cl}\, \breve{Q}_+) = \mathrm{cl}\, \breve{Q}_-$ in $Q(\breve{0})$, i.e.,

$$\zeta(\{\breve{0}\} \cup \breve{Q}_+) = \{(1/2)\breve{}\} \cup \breve{Q}_-, \quad \text{i.e.,} \quad \zeta(\breve{0}) = (1/2)\breve{},$$

contradicting with $\zeta(\breve{0}) = \breve{0}$. □

Put $0^* = J^{(2)}(\breve{0})$. Then, since $0^* \in \phi^{-1}(0)$, we can consider $Q(0^*) \subseteq Q^{(2)}$. Define r^* for $r \in Q$ by

$$0^* = J^{(2)}(\breve{0}) \ \text{ and } \ r^* = T^{(2)}_r(0^*).$$

Then it is easy to see that both $\breve{r} \neq r^*$ belong to the fiber $\phi^{-1}(r)$. Put

$$Q^*_+ = \{r^* : r \in Q_+\} \subseteq \phi^{-1}(Q_+) \ \text{ and } \ Q^*_- = \{r^* : r \in Q_-\} \subseteq \phi^{-1}(Q_-).$$

Since $J^{(2)}(\breve{r}) = J^{(2)} \circ T_r^{(2)}(\breve{0}) = T_{1-r}^{(2)} \circ J^{(2)}(\breve{0}) = T_{1-r}^{(2)}(0^*) = (1-r)^*$, the homeomorphism $J^{(2)}$ carries the clopen partition (6-0) of $Q(\breve{0})$ to that of $Q(0^*)$

$$(6-1) \qquad Q(0^*) = (\{0^*\} \cup Q_-^*) \ \cup \ (\{(1/2)^*\} \cup Q_+^*)$$

where $\{0^*\} \cup Q_-^* \subseteq U_- = J^{(2)}(U_+)$ and $\{(1/2)^*\} \cup Q_+^* \subseteq U_+ = J^{(2)}(U_-)$. Define the operation $\oplus$ on $Q(0^*)$ by $r^* \oplus s^* = (r+s)^*$, then the homeomorphism $J^{(2)}$ of $Q^{(2)}$ induces an isomorphism $(Q(\breve{0}), \oplus) \approx (Q(0^*), \oplus)$. Let us consider the subspace $Q(\breve{0}) \cup Q(0^*) \subseteq Q^{(2)}$, and define on it a semigroup operation $\uplus$ by

$$s^* \uplus \breve{r} = \breve{s} \uplus \breve{r} = \breve{s} \oplus \breve{r} \in Q(\breve{0}) \ \text{ and } \ \breve{s} \uplus r^* = s^* \uplus r^* = s^* \oplus r^* \in Q(0^*)$$

for any $r, s \in Q$, which is obviously left-topological. Put

$$Q(\breve{0}, 0^*) = Q(\breve{0}) \cup Q(0^*) \subseteq Q^{(2)}.$$

This left-topological semigroup $(Q(\breve{0}, 0^*), \uplus)$ has the following properties:

(1) Both $Q(\breve{0})$ and $Q(0^*)$ are minimal left ideals.
(2) $\breve{0}$, 0^* are idempotents.
(3) $J^{(2)}$ is an involution exchanging $Q(\breve{0})$ and $Q(0^*)$. Due to the existence of this involution, $Q(\breve{0}, 0^*)$ is topologically homogeneous.

Note also that both $Q(\breve{0})$ and $Q(0^*)$ are semitopological groups as we remarked before at the end of §5.

Next we will show that $(Q(\breve{0}, 0^*), \uplus)$ has a close connection with an example described in [2]. Let $\mathbb{A} = \mathbb{T}_0 \cup \mathbb{T}_1$ be the union of two copies of the circle group $\mathbb{T}$, and let $\alpha_i : \mathbb{T} \to \mathbb{T}_i$ $(i = 0, 1)$ be isomorphisms. We assume $\mathbb{A}$ has the topology of the "Alexandroff double arrow" space, i.e., the sets

$$\alpha_0([t, s)) \cup \alpha_1((t, s)) \ \text{ for } \ 0 \leqslant t < s < 1$$

and

$$\alpha_0((t, s)) \cup \alpha_1((t, s]) \ \text{ for } \ 0 < t < s \leqslant 1$$

are the neighborhood base of $\mathbb{A}$. The multiplication on $\mathbb{A}$ is defined by

$$\alpha_j(s) \cdot \alpha_i(t) = \alpha_i(s+t) \ \text{ for } \ 0 \leqslant t, s \leqslant 1 \ \text{ and } \ i, j = 0, 1.$$

Then it is easy to see that $(\mathbb{A}, \cdot)$ is a compact left-topological semigroup. Let $T_r^{\mathbb{A}}$, $J^{\mathbb{A}}$ denote translation and involution on $\mathbb{A}$, respectively, i.e.,

$$T_r^{\mathbb{A}}(\alpha_i(t)) = \alpha_i(t+r), \ \ J^{\mathbb{A}}(\alpha_i(t)) = \alpha_{1-i}(1-t)$$

for $0 \leqslant t, r \leqslant 1$ and $i = 0, 1$. Let $\mathbb{A}(Q) = Q_0 \cup Q_1$ be the subsemigroup of $\mathbb{A}$ such that $Q_i = \alpha_i(Q)$ for $i = 0, 1$. Define a correspondence

$$\xi : (Q(\breve{0}, 0^*), \uplus) \to (\mathbb{A}(Q), \cdot)$$

by $\xi(\check{r}) = \alpha_0(r)$, $\xi(r^*) = \alpha_1(r)$ for any $r \in Q$. It is clear that this ξ is an isomorphism, algebraically, and that ξ commutes with $T_r^{\mathbb{A}}$, $J^{\mathbb{A}}$, i.e.,

$$\xi \circ T_r^{(2)} = T_r^{\mathbb{A}} \circ \xi, \quad \xi \circ J^{(2)} = J^{\mathbb{A}} \circ \xi \quad \text{on } Q(\check{0}, 0^*).$$

Property 6.2. *ξ is continuous.*

Proof. Taking account of the above commutativity with $T_r^{\mathbb{A}}$, $J^{\mathbb{A}}$, it suffices to show the continuity of ξ at only one point $\check{0}$ of $Q(\check{0}, 0^*)$. Take any $0 < \varepsilon < 1/2$, and consider an open neighborhood $\alpha_0([0,\varepsilon)) \cup \alpha_1((0,\varepsilon))$ of $\xi(\check{0}) = \alpha_0(0) \in \mathbb{A}(Q)$. Note that

$$\check{0} \in Q(\check{0}, 0^*) \cap U_+ = \check{Q}_+ \cup Q_+^* \cup \{\check{0}, (1/2)^*\},$$

hence

$$\check{0} \in Q(\check{0}, 0^*) \cap U_+ \cap \phi^{-1}((-\varepsilon, +\varepsilon)) = [0,\varepsilon)\check{} \cup (0,\varepsilon)^*,$$

where $[0,\varepsilon)\check{}$ is the set of all points $\check{r}$ for $r \in [0,\varepsilon) \cap Q$, and $(0,\varepsilon)^*$ is the set of all points r^* for $r \in (0,\varepsilon) \cap Q$. Of course, $(-\varepsilon, +\varepsilon)$ is identified with $[0,\varepsilon) \cup (1-\varepsilon, 1)$ modulo 1. Therefore the neighborhood $Q(\check{0}, 0^*) \cap U_+ \cap \phi^{-1}((-\varepsilon, +\varepsilon))$ of $\check{0}$ is carried by ξ onto $\alpha_0([0,\varepsilon)) \cup \alpha_1((0,\varepsilon))$, and this proves the continuity of ξ. □

Let $\pi : (\mathbb{A}(Q), \cdot\,) \to (Q, +)$ be the natural 2-1 projection, i.e., $\pi(\alpha_i(r)) = r$ $(i = 0, 1)$. Then we have $\pi \circ \xi = \phi \restriction Q(\check{0}, 0^*)$.

Property 6.3. *ξ is semi-open.*

Proof. Let U be any nonempty open set in $Q(\check{0}, 0^*)$. Since ϕ is perfect irreducible, by Property 2.1 (3) we can find a nonempty open set in W in Q such that $(\pi \circ \xi)^{-1}(W) = \xi^{-1}(\pi^{-1}(W)) \subseteq U$. Hence we get a nonempty open subset $\pi^{-1}(W)$ of $\mathbb{A}(Q)$ contained in $\xi(U)$. □

Thus we can summarize that the map $\phi \restriction Q(\check{0}, 0^*)$ is factorized into a semi-open map ξ and a perfect irreducible map π

$$Q(\check{0}, 0^*) \xrightarrow{\xi} (\mathbb{A}(Q), \cdot\,) \xrightarrow{\pi} (Q, +)$$

where ξ is an isomorphism and π is a 2-1 homomorphism w.r.t. semi-group structure. Of course, the space $Q(\check{0}, 0^*)$ can be extremally disconnected if the point $\check{0} \in Q^{(2)} = \beta P \backslash P$ is chosen to be remote from P.

7. The case $G = \bigoplus_\omega \mathbb{Z}(2)$

Let us consider the case G is a topological group $(\bigoplus_\omega \mathbb{Z}(2), +)$ with the Boolean operation $x + x = 0$. Then, since

$$\check{x} \oplus \check{x} = (x + x)\check{} = \check{0},$$

$(G(\check{0}), \oplus)$ is also a Boolean group. Since the inverse operation of a Boolean group is the identity map, the inverse operation is obviously continuous.

But $(G(\breve{0}), \oplus)$ fails to be a topological group as pointed out in Corollary 3.5. Using the notation in §3 we get $J = id_G,\ J^{(2)} = id_{G^{(2)}},\ J^{(2)}(\breve{0}) = \breve{0}$. So, the present situation is quite different from §6. Since $(G(\breve{0}), \oplus)$ is Abelian, we can conclude that $(G(\breve{0}), \oplus)$ is a semitopological group with continuous inverse, which can be extremally disconnected by choosing an appropriate $\breve{0}$, by Theorem 5.2.

8. Modified Construction

Let us consider the case $G = (\mathbb{Q}, +) \subseteq (\mathbb{R}, +)$ is the group of rationals. Applying the method of §3, taking $c\,\mathbb{Q} = \beta\,\mathbb{Q}$, we can construct $(\mathbb{Q}(\breve{0}), \oplus)$ as a dense subset of $\mathbb{Q}^{(2)} = \beta\,\mathbb{Q}^{(1)} \backslash \mathbb{Q}^{(1)}$ where $\mathbb{Q}^{(1)} = \beta\,\mathbb{Q} \backslash \mathbb{Q}$. Here we will modify the construction in §3 to find such $(\mathbb{Q}(\breve{0}), \oplus)$ inside the Stone-Čech remainder $\beta S \backslash S$, where S can be the space of irrationals $\mathbb{P}$ or even be a homeomorphic copy of $\mathbb{Q}$.

Take any irrational ε and fix it. Define

$$\mathbb{Q} + \varepsilon = \{r + \varepsilon : r \in \mathbb{Q}\} \subseteq \mathbb{R}$$

and let S be either this $\mathbb{Q} + \varepsilon$ or $\mathbb{R} \backslash \mathbb{Q} = \mathbb{P}$. Consider $\mathbb{Q} \cup S$ $(\subseteq \mathbb{R})$. Then its Stone-Čech extension $\beta\,(\mathbb{Q} \cup S)$ can be seen as a compactification of S, so that we can consider the Stone extension $\Phi : \beta S \to \beta\,(\mathbb{Q} \cup S)$ of the identity map id_S of S. Let $\phi : \Phi^{-1}(\mathbb{Q}) \to \mathbb{Q}$ denote the restriction of Φ. Consider $\mathcal{H} = \{T_r : r \in \mathbb{Q}\} \subseteq \mathbf{H}(\mathbb{Q})$, the collection of translations $T_r(s) = r + s$. Then it is clear that every $h \in \mathcal{H}$ naturally extends to $\overline{h} \in \mathbf{H}(\mathbb{Q} \cup S)$ in such a way that $\overline{h}(S) = S$. Let $h^{(1)} = \overline{h} \restriction S \in \mathbf{H}(S)$ so that $\overline{h} = h \cup h^{(1)}$. Then the equality $(\beta\overline{h} \circ \Phi) \restriction S = h^{(1)} = (\Phi \circ \beta h^{(1)}) \restriction S$ implies

$$\beta\overline{h} \circ \Phi = \Phi \circ \beta h^{(1)} : \beta S \to \beta\,(\mathbb{Q} \cup S)$$

from which we see that $\beta h^{(1)}(\Phi^{-1}(\mathbb{Q})) \subseteq \Phi^{-1}(\mathbb{Q})$. Since we can consider also h^{-1} instead of h, we get

$$\beta h^{(1)}(\Phi^{-1}(\mathbb{Q})) = \Phi^{-1}(\mathbb{Q}).$$

Define $h^{(2)} \in \mathbf{H}(\Phi^{-1}(\mathbb{Q}))$ to be this restriction $\beta h^{(1)} \restriction \Phi^{-1}(\mathbb{Q})$. Then we get the equality

$$h \circ \phi = \phi \circ h^{(2)} : \Phi^{-1}(\mathbb{Q}) \to \mathbb{Q}$$

similar to that of Lemma 3.1. Therefore, hereafter, we can carry out the same construction as in §3 or §6, i.e., $(\mathbb{Q}(\breve{0}), \oplus)$ is such that

$$\mathbb{Q}(\breve{0}) = \{\breve{r} : r \in \mathbb{Q}\} \subseteq \beta S \backslash S$$

where $\breve{0} \in \phi^{-1}(0)$ and $\breve{r} = T_r^{(2)}(\breve{0}) \in \phi^{-1}(r)$. So our conclusion is:

Theorem 8.1. *We can construct* $(\mathbb{Q}(\breve{0}), \oplus)$ *inside* $\beta S \backslash S$*, where* S *is the subspace of the real line such that* $S = \mathbb{P}$ *(irrationals) or* $S = \mathbb{Q} + \varepsilon \approx \mathbb{Q}$ *for any fixed irrational* ε*, and a perfect irreducible map* ϕ *from a dense subset* D *of* $\beta S \backslash S$ *onto* $\mathbb{Q}$ *such that*

$$\mathbb{Q}(\breve{0}) = \{\breve{r} : r \in \mathbb{Q}\} \text{ where } \breve{r} = T_r^{(2)}(\breve{0}) \in \phi^{-1}(r), \text{ and } \breve{r} \oplus \breve{s} = (r+s)\breve{}$$

is an orbit of an element $\breve{0} \in \phi^{-1}(0)$ *by* $T_r^{(2)}$ $(r \in \mathbb{Q})$*, where* $T_r^{(2)}$ *is a homeomorphism of* D *induced naturally by the Stone extension of the translation* $T_r(t) = r + t$ *of the space* S*.*

Of course, by choosing an appropriate $\breve{0}$, we can make the above $\mathbb{Q}(\breve{0})$ to be an extremally disconnected space with the property stated in Theorem 5.2. If we want to construct further the space $\mathbb{Q}(\breve{0}, 0^*)$ similar to $Q(\breve{0}, 0^*)$ in §6, that will be done by enlarging $\mathcal{H}$ and S to

$$\mathcal{H} = \{T_r : r \in \mathbb{Q}\} \cup \{J\} \subseteq \mathbf{H}(\mathbb{Q}) \quad \text{and} \quad S = (\mathbb{Q} + \varepsilon) \cup (\mathbb{Q} - \varepsilon) \subseteq \mathbb{R},$$

respectively, where J is the inverse operation $J(r) = -r$. In any way, S can be the space of irrationals $\mathbb{P}$ or a homeomorphic copy of $\mathbb{Q}$.

9. Concluding Remarks

We don't know if we can apply our construction in this paper when the topological group G is not metrizable, i.e., not first countable. For example, let $G_\square = \bigoplus_\omega \mathbb{Q}$ be the direct sum of countably many copies of $\mathbb{Q}$ endowed with the box topology. This countable topological group is known to be stratifiable (see [6, 3]), though not metrizable. Is it possible to find an extremally disconnected semi-topological group, admitting a semi-open map onto $G_\square$, by using Stone-Čech compactifications once or twice?

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