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A WEAKENING OF SUBMETRIZABILITY AND PROPERTIES OF SPACES

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ABSTRACT. The notions of weakly jointly compact-metrizable space and of σ -stratifiable mapping are introduced. Corollary 2.5 affirms that a weakly jointly compact-metrizable feebly compact sequential space is metrizable. By Theorem 4.3, X is a strong Σ -space if and only if X is a σ -stratifiable image of some paracompact p -space. This fact leads to general conditions under which a Σ -space is a σ -space (Theorem 5.1 and Corollary 5.2). Some concrete corollaries of these facts are mentioned.

1. INTRODUCTION

Let X be a topological space and let \mathcal{F} be a family of subspaces of X . Following [5] and [6], we say that X is *jointly metrizable on \mathcal{F}* , or that X is *\mathcal{F} -metrizable*, if there is a metric d on X which *metrizes all members* \mathcal{F} (that is, the restriction of d to A generates the subspace topology on A , for any $A \in \mathcal{F}$).

In particular, we say that X is *compactly metrizable*, or that X is *jointly metrizable on compacta*, or is a *JCM-space*, if X is jointly metrizable on the family of all compact subspaces of X (see [5]).

A space is *countably metrizable* if it is jointly metrizable on all countable subspaces [5].

It is natural to give the following definition [5]: A space X will be called *jointly partially metrizable*, or X is a *JPM-space*, if there is a metric d on X which metrizes all metrizable subspaces of X .

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2. A GENERALIZATION OF SUBMETRIZABILITY OF SPACES

A space X will be called *weakly jointly compact-metrizable* if there is a metric d on X which metrizes all compact metrizable subspaces of X . In this case we say that d is a *wJCM-metric* on the space X .

Any *JPM*-space is a weakly jointly compact-metrizable space. Every submetrizable space is a *JCM*-space. Obviously, each *JCM*-space is a *wJCM*-space.

Example 2.1. Let X be an infinite space without non-trivial convergent sequences. Then any compact metrizable subspace of X is finite and any metrizable subspace of X is discrete. Thus, the discrete metric d , given by $d(x, x) = 0$ and $d(x, y) = 1$ for any distinct points $x, y \in X$, metrizes all metrizable subspaces of the space X . Therefore, X is a *JPM*-space. If the space X is compact or X contains an infinite compact subspace, then X is not jointly metrizable on compacta.

A space X is called *sequential* if a subset P of X is closed in X if and only if the intersection of P with any metrizable compact subspace F of X is closed in the subspace F .

Following the idea in the definition of a k -leader of a space, one comes to the definition of a k_m -leader of a space. In the class of Hausdorff spaces, the object $k_m X$ is the well-known sequential co-reflection of the space X .

Suppose that (X, \mathcal{T}) is an arbitrary topological space. Define a topology \mathcal{T}_m on X as follows: A subset V of X belongs to \mathcal{T}_m if and only if, for every compact metrizable subspace F of (X, \mathcal{T}) , the set $V \cap F$ is open in the subspace F . Endowing the set X with the topology \mathcal{T}_m so defined, we obtain a topological space $k_m X$ called the k_m -leader of the space (X, \mathcal{T}) . Clearly, $\mathcal{T} \subset \mathcal{T}_m$ and $k_m X$ is a sequential space.

Obviously, we have the following characterization of weakly jointly compact-metrizable spaces.

Theorem 2.2. *A space X is weakly jointly compact-metrizable if and only if the space $k_m X$ is submetrizable.*

Corollary 2.3. *A sequential space X is weakly jointly compact-metrizable if and only if it is submetrizable.*

A subset L of a space X is called *bounded* (in X) if, for every locally finite family γ of open subsets in X , the set $\{U \in \gamma : U \cap L \neq \emptyset\}$ is finite. A space X is called *feebly compact* if every locally finite family of open subsets in X is finite, i.e., X is bounded in X . For Tychonoff spaces, feeble compactness is equivalent to pseudocompactness. Every countably compact space is feebly compact.

Theorem 2.4. *Let X be a weakly jointly compact-metrizable regular sequential space. Then any closed bounded G_δ -subspace F of X is compact and metrizable. In particular, X is jointly metrizable on compacta.*

Proof. Fix a $wJCM$ -metric d on X . Let $\{U_n : n \in \omega\}$ be a sequence of open subsets of X such that $U_{n+1} \subset U_n$ and $F = \bigcap \{U_n : n \in \omega\}$.

Denote by Y the set X with the topology \mathcal{T}_d generated by d . The identity mapping $f : X \rightarrow Y$ is continuous.

CLAIM 1. The set $H = f(F)$ is closed in Y . Therefore, H is a compact metrizable subspace of Y .

Assume that the set H is not closed in Y . Then there exist a point $x \in X \setminus F$ and a sequence $\{x_n : n \in \omega\}$ such that $x_n \in F$ and $d(x, x_{n+1}) < 2^{-1}d(x, x_n)$ for each $n \in \omega$. There exist an open subset V and a disjoint family $\{V_n : n \in \omega\}$ of open subsets of X such that $x \in V$ and $x_n \in V_n \subset U_n \cap \{y \in X : d(x_n, y) < 2^{-n} \cap (X \setminus V)\}$ for any $n \in \omega$. Then the family $\{V_n : n \in \omega\}$ is locally finite in X and $V_n \cap F \neq \emptyset$ for each $n \in \omega$, a contradiction.

CLAIM 2. The mapping $g = f|_F : F \rightarrow H$ is a homeomorphism.

Assume that g is not a homeomorphism. Then there exist a point $x \in F$ and a sequence $L = \{x_n : n \in \omega\}$ such that $x_n \in F$ and $d(x, x_{n+1}) < 2^{-1}d(x, x_n)$ for each $n \in \omega$, and the set L is closed in X . As in Claim 1, we can construct an open subset V and a disjoint family $\{V_n : n \in \omega\}$ of open subsets of X such that $x \in V$ and $x_n \in V_n \subset U_n \cap \{y \in X : d(x_n, y) < 2^{-n} \cap (X \setminus V)\}$ for any $n \in \omega$. Then the family $\{V_n : n \in \omega\}$ is locally finite in X and $V_n \cap F \neq \emptyset$ for each $n \in \omega$, a contradiction. \square

Corollary 2.5. *Every weakly jointly compact-metrizable feebly compact sequential space is metrizable.*

3. WEAKLY JOINTLY COMPACT-METRIZABLE SPACES AND PERFECT MAPPINGS

Theorem 3.1. *Let $f : X \rightarrow Y$ be a perfect mapping of a weakly jointly compact-metrizable space X onto a JPM -space Y . If all the fibers $f^{-1}(y)$, where $y \in Y$, are metrizable, then X is a JPM -space.*

Proof. Fix a $wJCM$ -metric d_1 on X and a JPM -metric d_2 on a space Y . We put $d(x, y) = d_1(x, y) + d_2(f(x), f(y))$ for all $x, y \in X$. Obviously, d is a $wJCM$ -metric on X . Let \mathcal{T} be the original topology on X and \mathcal{T}_d be the topology on X generated by the metric d . Denote by Z the metric space (X, d) and by $g : Z \rightarrow k_m X$ the identity mapping. The spaces Z and $k_m X$ are sequential. It is sufficient to prove that $\mathcal{T} \subset \mathcal{T}_d$.

If F is a metrizable subspace of $k_m X$, then F is a metrizable subset of the space Z too. Thus, the mapping $g^{-1} : k_m X \rightarrow Z$ is continuous.

Let \mathcal{T}_1 be the topology of the space Y and \mathcal{T}_2 be the topology on Y generated by the metric d_2 . Then $\mathcal{T}_1 \subset \mathcal{T}_2$ (see [4]). The mapping f of Z onto (Y, \mathcal{T}_2) is continuous.

If H is a compact subspace of (Y, \mathcal{T}_2) , then H is a metrizable compact subspace of the space Y , and $F = f^{-1}(H)$ is a first-countable compact subspace of the space X . By Theorem 2.5, F is a metrizable compact subspace of X . Therefore, F , as the subspace of Z , is compact and the mapping f of Z onto (Y, \mathcal{T}_2) is perfect. In particular, if L is a compact subset of Z , then $f(L)$ is a metrizable compact subset of Y ; $F = f^{-1}(f(L))$ is a compact metrizable subspace of the spaces X , $k_m X$, and Z ; $L \subset F$; and $g|_F : F \rightarrow k_m X$ is an embedding. Hence, the mapping $g : Z \rightarrow k_m X$ is a homeomorphism. Then $\mathcal{T} \subset \mathcal{T}_d$. \square

Corollary 3.2. *Let $f : X \rightarrow Y$ be a perfect mapping of a weakly jointly compact-metrizable space X onto a metric space Y . If the fibers $f^{-1}(y)$, $y \in Y$, are metrizable, then X is metrizable.*

Example 3.3. Let X be the Stone-Ćech compactification of the discrete space D_1 of the uncountable cardinality, let Y be the one-point compactification of D_1 , and let $f : X \rightarrow Y$ be the continuous mapping for which $f(x) = x$ for any $x \in D$. Let $Y \setminus D_1 = \{b\}$. Then X is a *JPM*-space, Y is sequential, Y is not weakly jointly compact-metrizable, f is a perfect mapping, and only one fiber $f^{-1}(b)$ is not metrizable.

Example 3.4. Let X be the Stone-Ćech compactification of the discrete space ω , let Y be the one-point compactification of ω , and let $f : X \rightarrow Y$ be the continuous mapping for which $f(x) = x$ for each $x \in \omega$. Let $Y \setminus \omega = \{b\}$. Then X is a *JPM*-space, X is not metrizable, Y is metrizable space, f is a perfect mapping, and only one fiber $f^{-1}(b)$ of f is not metrizable.

4. Σ -SPACES AND σ -STRATIFIABLE MAPPINGS

A space X is called a Σ -space if there exist a family $\{F(x) : x \in X\}$ of countably compact subsets of X and a σ -locally finite family \mathcal{L} of subsets of the space X such that, for each $x \in X$ and each open subset U which contains $F(x)$, there exists $L \in \mathcal{L}$ such that $x \in L \cap F(x) \subset L \subset U$. The family \mathcal{L} is called a Σ -net. If each $F(x)$ is compact, then X is called a strong Σ -space and \mathcal{L} is called a strong Σ -net [17]. If every $F(x)$ is a metrizable compactum, then X is called a Σ_m -space and \mathcal{L} is said to be a Σ_m -net.

A space with a σ -locally finite net is called a σ -space (see [24]). Any σ -space is a strong Σ -space and a Σ_m -space. Any non-metrizable compact space is an example of a strong Σ -space which is not a σ -space.

A family $\mu = \{H_{n\alpha} : \alpha \in A, n \in \omega\}$ of subsets of a space X is called a σ -stratification of a family $\gamma = \{H_\alpha : \alpha \in A\}$ if $H_\alpha = \cup\{H_{n\alpha} : n \in \omega\}$ and $\mu_n = \{H_{n\alpha} : \alpha \in A\}$ is a locally finite family of subsets of the space X for all $\alpha \in A$ and $n \in \omega$.

A family $\mu = \{M_\beta : \beta \in B\}$ of subsets of X is called a *weak σ -stratification* of the family $\gamma = \{H_\alpha : \alpha \in A\}$ if $H_\alpha = \cup\{M_\beta : \beta \in B, M_\beta \subseteq H_\alpha\}$ for each $\alpha \in A$ and μ is a σ -locally finite family of subsets of the space X .

Any σ -stratifiable family of subsets is point-countable. In a σ -discrete perfect space any family is weakly σ -stratifiable. Thus, there exists a weakly σ -stratifiable not σ -stratifiable family of subsets.

A mapping $g : X \rightarrow Y$ is called (*weakly*) σ -stratifiable if g is continuous and, for any locally finite family $\gamma = \{H_\alpha : \alpha \in A\}$ of subsets of X , the family $g(\gamma) = \{g(H_\alpha) : \alpha \in A\}$ admits a (weak) σ -stratification.

Any perfect mapping is σ -stratifiable. Any continuous mapping into a perfect σ -discrete space is weakly σ -stratifiable.

Proposition 4.1. *Let $g : X \rightarrow Y$ be a σ -stratifiable mapping of a space X onto a space Y . Then*

- (1) *If X is a σ -space, then Y is a σ -space too.*
- (2) *If X is a Σ -space, then Y is a Σ -space too.*
- (3) *If X is a strong Σ -space, then Y is a strong Σ -space too.*

Proof. The proofs of these assertions are obvious. □

A space X is called a paracompact p -space if there exist a metrizable space Y and a perfect mapping $g : X \rightarrow Y$ [1]. If all the fibers $f^{-1}(y)$, $y \in Y$, are metrizable, then we say that X is a paracompact p_m -space.

Any paracompact p -space is a strong Σ -space and any paracompact p_m -space is a Σ_m -space. Each paracompact p_m -space is first-countable.

Lemma 4.2. *Let γ be a σ -locally finite family of subsets of a space X , let \mathcal{L} be a family of subsets of X , and suppose for each $x \in X$ there exists $H \in \gamma$ such that $x \in H$ and the set $s(H) = \{L \in \mathcal{L} : L \cap H \neq \emptyset\}$ is finite. Then the family \mathcal{L} admits a σ -stratification.*

Proof. There exists a sequence $\{\gamma_n = \{P_\beta : \beta \in B_n\}\}$ of locally finite families of subsets of X such that

- $\gamma_n \subset \gamma$ for each $n \in \omega$;
- $X = \cup\{P_\beta : \beta \in B_n, n \in \omega\}$;
- the set $s(H)$ is finite for all $\beta \in B_n$ and $n \in \omega$.

Assume that $\mathcal{L} = \{L_\alpha : \alpha \in A\}$. Let $C_n = \{(\alpha, \beta) \in A \times B_n : L_\alpha \cap H_\beta \neq \emptyset\}$. Clearly, the family $\{L_{(\alpha, \beta)} = L_\alpha \cap H_\beta : (\alpha, \beta) \in C_n\}$ is locally finite. Fix $\alpha \in A$ and $n \in \omega$. If $(\alpha, \beta) \notin C_n$ for each $\beta \in B_n$,

then we put $L_{n\alpha} = \emptyset$. If $(\alpha, \beta) \in C_n$ for some $\beta \in B_n$, then we put $L_{n\alpha} = \cup\{L_{(\alpha, \beta)} : \beta \in B_n\}$. Then $\mathcal{L}_n = \{L_{n\alpha} : \alpha \in A\}$ is a locally finite family of sets and $L_\alpha = \cup\{L_{n\alpha} : n \in \omega\}$. \square

Theorem 4.3. *For a regular space X , the following assertions are equivalent.*

- (1) X is a strong Σ -space.
- (2) X is an image under a σ -stratifiable mapping of a strong Σ -space.
- (3) X is an image under a σ -stratifiable mapping of a paracompact p -space.
- (4) There exist a metrizable space Y , a paracompact p -subspace S of the space $X \times Y$, and a σ -stratifiable mapping $g : S \rightarrow X$ onto X , such that $g(x, y) = x$ for each $(x, y) \in S$.

Proof. Implications $4 \rightarrow 3 \rightarrow 2$ are obvious. Implication $2 \rightarrow 1$ follows from Proposition 4.1.

Let X be a strong Σ -space. Then there exist a family $\{F(x) : x \in X\}$ of compact subsets of X and a sequence $\{\gamma_n = \{H_\alpha : \alpha \in A_n\} : n \in \omega\}$ of locally finite families \mathcal{L} of closed subsets of X such that

- for all $n, m \in \omega$, $m \leq n$, $\alpha \in A_m$, and $\beta \in A_n$ there exists $\mu \in A_n$ such that $H_\mu = H_\alpha \cap H_\beta$;
- for any $x \in X$ there exists a sequence $\alpha(x) = (\alpha_n \in A_n : n \in \omega)$ such that $x \in \cap\{H_{\alpha_n(x)} : n \in \omega\} \subseteq F(x)$, $H_{\alpha_{n+1}} \subset H_{\alpha_n}$ for each $n \in \omega$, and, for each open subset U which contains $F(x)$, there exists $m \in \omega$ such that $x \in H_{\alpha_m} \subseteq U$;
- $H_\alpha = \cup\{H_\beta : \beta \in A_{n+1}, H_\beta \subseteq H(\alpha)\}$ for all $n \in \omega$ and $\alpha \in A_n$.

A sequence $\alpha = (\alpha_n : n \in \omega)$ is called an m -sequence if $H(\alpha) = \cap\{H_n : n \in \omega\}$ is a non-empty compact subset of X , $\alpha_n \in A_n$, $H_{\alpha_{n+1}} \subset H_{\alpha_n}$ for each $n \in \omega$, and for each open subset U which contains $H(\alpha)$ there exists $m \in \omega$ such that $H_{\alpha_m} \subset U$.

Obviously, any sequence $\alpha(x)$ is an m -sequence. Moreover, for all $n \in \omega$, $\beta_1 \in A_0, \dots, \beta_n \in A_n$, with $H_{\beta_i} \subset H_{\beta_j}$ for $0 \leq j < i \leq n$, and a point $x \in H_{\beta_n}$, there exists an m -sequence $\alpha = (\alpha_n : n \in \omega)$ such that $\beta_1 = \alpha_1, \dots, \beta_n = \alpha_n$ and $x \in H(\alpha)$.

Denote by Y the set of all m -sequences with the Baire metric $d((\alpha_n : n \in \omega), (\beta_n : n \in \omega)) = \Sigma\{2^{-n} : \alpha_n \neq \beta_n\}$ (see [10]). Consider the subspace $S = \cup\{H(\alpha) \times \{\alpha\} : \alpha \in Y\}$ of the space $X \times Y$ and the natural projections $g : S \rightarrow X$ and $f : S \rightarrow Y$.

CLAIM 1. f is a perfect mapping.

Obviously, the mapping f is compact and $f^{-1}(\alpha) = H(\alpha) \times \{\alpha\}$. Fix an open subset W of the space $X \times Y$ and assume that $f^{-1}(\alpha) = H(\alpha) \times \{\alpha\} \subseteq W$ for a given $\alpha = (\alpha_n : n \in \omega) \in Y$. For each $m \in \omega$, we put $\delta_m(\alpha) = \{\beta = (\beta_n : n \in \omega) \in Y : \beta_i = \alpha_i \text{ for all } i \leq m\}$. The sequence

$\{\delta_m(\alpha) : m \in \omega\}$ is a base of the space Y at the point α . Since the set $H(\alpha)$ is compact, there exist an open subset U of X and an open subset V of Y such that $H(\alpha) \times \{\alpha\} \subseteq U \times V \subseteq W$. Since α is an m -sequence, there exists $m \in \omega$ such that $H_{\alpha_m} \subseteq U$ and $\delta_m(\alpha) \subset V$. Then $f^{-1}(\delta_m(\alpha)) \subset U \times V \subset W$. Thus, f is perfect.

CLAIM 2. S is a paracompact p -space.

This claim follows from Claim 1.

CLAIM 3. The mapping g is σ -stratifiable.

Let $\xi = \{G_\mu : \mu \in M\}$ be a locally finite family of subsets of S . Fix $x \in X$. For some $\alpha \in Y$ we have $x \in H(\alpha)$. There exists an open subset W of $X \times V$ such that $f^{-1}(\alpha) \subset W$ and the set $\{\mu \in M : W \cap G_\mu \neq \emptyset\}$ is finite. As it was established in the proof of Claim 1, there exists $m \in \omega$ such that $H_{\alpha_m} \times \delta_m(\alpha) \subset W$. Then the set $\{\mu \in M : H_{\alpha_m} \cap g(G_\mu) \neq \emptyset\}$ is finite. A reference to Lemma 4.2 completes the proof. \square

A space X is called an M -space if there exist a metrizable space Y and a quasi-perfect mapping $g : X \rightarrow Y$, i.e., a continuous closed mapping with non-empty countably compact fibers $g^{-1}(y)$, $y \in Y$ [16]. Any M -space is a Σ -space. The proofs of the following three theorems are similar.

Theorem 4.4. *For any normal space X , the following assertions are equivalent.*

- (1) X is a Σ -space.
- (2) X is an image under a σ -stratifiable mapping of a Σ -space.
- (3) X is an image under a σ -stratifiable mapping of an M -space.
- (4) There exist a metrizable space Y , an M -subspace S of the space $X \times Y$, and a σ -stratifiable mapping $g : S \rightarrow X$ onto X .

Theorem 4.5. *For a normal space X , the following assertions are equivalent.*

- (1) X is a Σ_m -space.
- (2) X is an image under a σ -stratifiable mapping of a Σ_m -space.
- (3) X is an image under a σ -stratifiable mapping of a p_m -space.
- (4) There exist a metrizable space Y , a paracompact p_m -subspace S of the space $X \times Y$, and a σ -stratifiable mapping $g : S \rightarrow X$ onto X .

Theorem 4.6. *For any space X , the following assertions are equivalent.*

- (1) X is a σ -space.
- (2) X is an image under a σ -stratifiable mapping of a σ -space.
- (3) X is an image under a σ -stratifiable mapping of a metrizable space.

- (4) *There exist a metrizable space S and a one-to-one σ -stratifiable mapping $g : S \rightarrow X$ onto X .*

5. CONDITIONS UNDER WHICH A Σ -SPACE IS A σ -SPACE

Theorem 5.1. *Let \mathcal{P} be a topological property satisfying the following conditions:*

- (ch) *Any subspace of a space with the property \mathcal{P} has the property \mathcal{P} ;*
- (cm) *if X is a space with \mathcal{P} and Y is a metrizable space, then $X \times Y$ has the property \mathcal{P} .*

Then the next assertions are equivalent.

- (1) *Any paracompact p -space with the property \mathcal{P} is metrizable.*
- (2) *Any regular strong Σ -space with \mathcal{P} is a σ -space.*

Proof. A paracompact p -space with a σ -discrete network is metrizable (see [2, Theorem 5.3]). This fact proves the implication $2 \rightarrow 1$.

Assume that every paracompact p -space with the property \mathcal{P} is metrizable. Let X be a regular strong Σ -space with the property \mathcal{P} . By Theorem 4.3, there exist a metrizable space Y , a paracompact p -subspace S of $X \times Y$, and a σ -stratifiable mapping $g : S \rightarrow X$ onto X such that $g(x, y) = x$ for each $(x, y) \in S$. Clearly, S is a paracompact p -space with the property \mathcal{P} . Hence, S is metrizable. Now it follows from Proposition 4.1 that X is a σ -space. Implication $1 \rightarrow 2$ and the theorem are proved. \square

Corollary 5.2. *Let \mathcal{P} be a topological property satisfying the following conditions:*

- (ch) *Any subspace of a space with \mathcal{P} has the property \mathcal{P} ;*
- (cr) *any regular countably compact space with the property \mathcal{P} is compact;*
- (cm) *if X is a space with \mathcal{P} and Y is a metric space, then $X \times Y$ has the property \mathcal{P} .*

Then the next assertions are equivalent.

- (1) *Any paracompact p -space with the property \mathcal{P} is metrizable.*
- (2) *Any regular Σ -space with the property \mathcal{P} is a σ -space.*

Corollary 5.2 implies a variety of new and old results on the existence of σ -discrete networks in Σ -spaces. Here are some of them.

Corollary 5.3 ([3, Corollary 2.8]). *Let X be a regular jointly metrizable on compacta strong Σ -space. Then X is a σ -space.*

Proof. Consider the paracompact p -space S and the σ -stratifiable mapping $g : S \rightarrow X$ constructed in the proof of Theorem 4.3 (see the proof of Claim 3). In this case, $X \times Y$ is jointly metrizable on compacta as the

topological product of two jointly metrizable on compacta spaces (see [4]). Therefore, the space S is jointly metrizable on compacta. However, every jointly metrizable on compacta paracompact p -space S is metrizable [6]. A reference to Proposition 4.1 completes the proof. \square

Conditions under which a regular jointly metrizable on compacta space has a network of the cardinality $\leq \tau$ were established in [3].

Corollary 5.4. *Let X be a regular weakly jointly compact-metrizable Σ_m -space. Then X is a σ -space.*

A space X is symmetrizable if there exists a distance function (symmetric) $d(x, y)$ on X with the following properties:

- (i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(y, x) = d(x, y)$;
- (iii) a non-empty subset L is closed in X if and only if $d(x, L) = \inf\{d(x, y) : y \in L\} > 0$ for each $x \in X \setminus L$ [2].

Corollary 5.5. *Let X be a regular Σ -space. If X is a symmetrizable space, then X is a σ -space.*

Proof. In a symmetrizable space any countably compact subspace is metrizable [19] [20]. Thus, X is a strong Σ -space. By Theorem 4.3, there exist a metrizable space Y , a paracompact p -subspace S of the space $X \times Y$, and a σ -stratifiable mapping $g : S \rightarrow X$ onto X , such that $g(x, y) = x$ for each $(x, y) \in S$. Any p -space is a k -space. A k -subspace of the Cartesian product of a countable family of symmetrizable spaces is symmetrizable. Thus, S is a symmetrizable space. By [2, Theorem 2.5], a symmetrizable paracompact p -space is metrizable. Hence, S is metrizable. Now it follows from Proposition 4.1 that X is a σ -space. \square

Corollary 5.6. *Let X be a regular (strong) Σ -space. If d is a symmetric on X which symmetrizes every countably compact (compact) subspace of X , then X is a σ -space. Moreover, if X is an M -space (a paracompact p -space), then X is metrizable.*

Proof. Under these assumptions about X , any countably compact subspace of X is metrizable and compact [19] [20]. Thus, X is a strong Σ -space. Assume that X is a paracompact p -space and $\varphi : X \rightarrow Y$ is a perfect mapping onto a metric space Y with a metric ρ_1 . Then $\rho(x, y) = d(x, y) + \rho_1(\varphi(x), \varphi(y))$ is a symmetric on X which symmetrizes X . From Corollary 5.5 it follows that X is a σ -space. \square

Corollary 5.7 (see [12, Corollary 7.11]). *If X is a regular Σ -space with a point-countable base, then X has a development. Moreover, if X is also collectionwise normal, then X is metrizable.*

Proof. In a space with a point-countable base any countably compact subspace is metrizable [15] [10]. Thus, X is a strong Σ -space. By Theorem 4.3, X is a σ -space. By virtue of [2, Theorem 2.8], X is symmetrizable. In [13] R. W. Heath has proved that a symmetrizable space with a point-countable base has a development. \square

Corollary 5.8 (see [12, Corollary 7.10]). *Let X be a regular strong Σ -space. If X is a space with a point-countable T_1 -separating open cover, then X is a σ -space.*

Proof. By Jun-iti Nagata's theorem [18], a paracompact p -space with a point-countable T_1 -separating open cover is metrizable. \square

A space X has a W_δ -diagonal if there exist a sequence $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ of open covers of X and a sequence $\pi = \{\pi_n : A_{n+1} \rightarrow A_n : n \in \omega\}$ of mappings such that

- (1) $\cup\{U_\beta : \beta \in A_n\} = X$ for each $n \in \omega$;
- (2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$;
- (3) if $\alpha = \{\alpha_n \in A_n : n \in \omega\}$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every n , then $H(\alpha) = \cap\{U_{\alpha_n} : n \in \omega\}$ contains at most one point of X .

Any space with a G_δ -diagonal has a W_δ -diagonal. A paracompact space with a W_δ -diagonal has a G_δ -diagonal. A paracompact p -space with a G_δ -diagonal is metrizable [23]. Moreover, a paracompact p -space with a W_δ -diagonal is metrizable (see [9] and [8]).

Hence, the next statement follows from Corollary 5.2.

Corollary 5.9 (see [12, Theorem 6.6]). *Let X be a regular strong Σ -space. Then the following assertions are equivalent.*

- (1) X is a σ -space.
- (2) X has a G_δ -diagonal.
- (3) X has a W_δ -diagonal.

Remark 5.10. In connection with Corollary 5.9, we mention the following facts.

1. Implication $2 \rightarrow 1$ for paracompact Σ -spaces was proved by Keiô Nagami [17]. In [7] J. Chaber has proved that a regular countably compact space with a G_δ -diagonal is metrizable. Thus, a regular Σ -space with a G_δ -diagonal is a strong Σ -space, and implication $2 \rightarrow 1$ holds for all Σ -spaces.

2. A space with a locally W_δ -diagonal has a W_δ -diagonal. In particular, a space with a locally G_δ -diagonal is a space with a W_δ -diagonal.

3. Implication $3 \rightarrow 1$ is not true in general for Σ -spaces. Let W be the space of all countable ordinal numbers with the topology induced by the

linear order. The space W is collectionwise normal, countably compact, locally metrizable, has a W_δ -diagonal, is a Σ -space, and is not a σ -space.

A space X is *quasi-metrizable* if there exists a distance function (quasi-metric) $d(x, y)$ on X with the following properties:

- (i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, z) \leq d(x, y) + d(y, z)$;
- (iii) a non-empty subset L is closed in X if and only if $d(x, L) = \inf\{d(x, y) : y \in L\} > 0$ for each $x \in X \setminus L$ [2] [12].

We now mention the following statement, a particular case of R. E. Hodel's more general theorem that a T_1 -space that both is a β -space and a γ -space must be developable [14] (see also [12, Theorem 10.7 and Corollary 10.8]).

Corollary 5.11 ([14]). *Let X be a regular quasi-metrizable Σ -space. Then X has a development.*

Proof. By [21, Corollary 4], any quasi-metrizable M -space is metrizable. In particular, a regular quasi-metrizable Σ -space is a strong Σ -space. Hence, X is a σ -space. Since X is first-countable as a quasi-metrizable space, by [2, Theorem 2.8], X is symmetrizable. [22, Theorem 2] affirms that any symmetrizable quasi-metrizable space has a development. \square

Corollary 5.12. *Let X be a regular (strong) Σ -space. If d is a quasi-metric on X which quasi-metrizes every countably compact (compact) subspace, then X is a σ -space. Moreover, if X is an M -space (a paracompact p -space), then X is metrizable.*

Proof. Under these assumptions about X , any countably compact subspace of X is metrizable [21]. Thus, X is a strong Σ -space. Assume that X is a paracompact p -space and $\varphi : X \rightarrow Y$ is a perfect mapping onto a metric space Y with a metric ρ_1 . Then $\rho(x, y) = d(x, y) + \rho_1(\varphi(x), \varphi(y))$ is a quasi-metric on X which quasi-metrizes X . From Corollary 5.11 it follows that X is a σ -space. \square

The next two examples are a part of the folklore.

Example 5.13. By the construction of J. Novak and Z. Frolik (see [10, Example 3.10.19]), there exists a countably compact subspace X of the compactification $\beta\omega$ of the discrete countable space ω such that $\omega \subset X$ and $|X| \leq \exp(\aleph_0)$. Then any compact subset of X is finite and any metrizable subspace of X is discrete. Thus, the discrete metric d given by $d(x, x) = 0$ and $d(x, y) = 1$ for any distinct points $x, y \in X$ metrizes all compact and all metrizable subspaces of X . Therefore, the space X has the following properties:

- X is a JPM -space;
- X is a space jointly metrizable on compacta;
- X is an M -space and a Σ -space;
- X is not a σ -space.

Example 5.14. Let $X = C_0 \cup C_1$, where $C_0 = \{(t, 0) : 0 < t \leq 1\}$ and $C_1 = \{(t, 0) : 0 \leq t < 1\}$, be the *double arrows space* of P. Alexandroff and P. Urysohn (see [10, Exercise 3.10.C]). The space X is compact, perfectly normal, hereditarily separable, hereditarily Lindelöf, and non-metrizable. Let $I = [0, 1]$ be the unit interval. Consider the mapping $g : X \rightarrow I$, where $g(t, i) = t$ for each $(t, i) \in X$. Clearly, X is a compact p_m -space. Since X is a first-countable non-metrizable compact space, X is not a JCM -space. Note that any continuous image of the space X is a compact p_m -space.

Example 5.15. Consider the *Alexandroff double circle* (see [10, Example 3.1.26],) $X = C_1 \cup C_2$, where $C_1 = \{(x, y) : x^2 + y^2 = 1\}$ and $C_2 = \{(x, y) : x^2 + y^2 = 2\}$ are two concentric circles and p is the radial projection of C_1 onto C_2 from $(0, 0)$. The space X is compact and hereditarily paracompact; it is also a paracompact p_m -space and is neither metrizable nor a JCM -space. The space X is a compactification of the discrete subspace C_2 . Let $Y = C_2 \cup \{b\}$ be the Alexandroff compactification, by the one point b , of the discrete space C_2 . Take the continuous mapping $g : X \rightarrow Y$, where $g^{-1}(b) = C_1$ and $g(z) = z$ for each $z \in C_2$. The space Y is compact and is not a paracompact p_m -space. Thus, an image under a perfect mapping with metrizable fibers of a compact p_m -space need not be a compact p_m -space.

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