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THE TORSION BOHR COMPACTIFICATION OF ABELIAN GROUPS

OMAR BECERRA-MURATALLA AND MIKHAIL TKACHENKO

Abstract. Let G be an abstract abelian group and $G^\natural$ be the underlying group G endowed with the *torsion Bohr topology*, i.e., the topology on G induced by the family $G^\circledast$ of *all* homomorphisms of G to the torsion subgroup of the circle group $\mathbb{T}$. The completion of $G^\natural$ is known as the *torsion Bohr compactification* of G and is denoted by $\mathfrak{b}G$. The main results of the article are as follows:

(1) The group $\mathfrak{b}\mathbb{Z}$ is topologically isomorphic to $\Delta_{\mathfrak{a}}$, the additive group of $\mathfrak{a}$-adic integers with $\mathfrak{a} = (2, 3, 4, 5, \dots)$, where $\mathbb{Z}$ is the group of integers. (2) If G is divisible, then $\mathfrak{b}G$ contains a closed subgroup topologically isomorphic to a power of the $\mathfrak{a}$-adic solenoid with $\mathfrak{a} = (2, 3, 4, 5, \dots)$ multiplied by a product of powers of p-adic integers, with prime p's. (3) The group G is divisible if and only if $\mathfrak{b}G$ is divisible. (4) If $\mathfrak{b}G$ is zero-dimensional, then the group G is reduced, i.e., the unique divisible subgroup of G is $\{0\}$. Furthermore, $\mathfrak{b}G$ is zero-dimensional if and only if $G^\circledast$ is torsion if and only if G is isomorphic to $\mathbb{Z}^n \oplus tor(G)$ for some integer $n \geq 0$, where $tor(G)$ is a bounded torsion group. (5) If H is a subgroup of G, then $\mathfrak{b}(G/H) \cong \mathfrak{b}G/\mathfrak{b}H$ and the same relation is valid for the Bohr compactification, i.e., $b(G/N) \cong bG/bH$.

1. Introduction

The *torsion Bohr topology* on abelian groups was defined and studied in [2]. It admits a simple description as follows. Let G be an abstract abelian group. The coarsest topological group topology on G that makes every homomorphism of G to the torsion subgroup of the circle group $\mathbb{T}$

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continuous is called the torsion Bohr topology of G. The group G endowed with this topology is denoted by $G^{\natural}$.

Let $\mathfrak{r}_G$ be the diagonal product of the family $G^{\circledast}$ of the homomorphisms of G to $tor(\mathbb{T})$, the torsion subgroup of $\mathbb{T}$. Then $\mathfrak{r}_G$ is a homomorphism of G to the product group $tor(\mathbb{T})^{G^{\circledast}}$. It is shown in [2, Theorem 2.1] that $\mathfrak{r}_G$ is a monomorphism, i.e., for every $x \in G$ distinct from the neutral element of G, there exists $h \in G^{\circledast}$ such that $h(x) \neq 1$. The closure in $\mathbb{T}^{G^{\circledast}}$ of the image $\mathfrak{r}_G(G)$, considered with the topology inherited from $\mathbb{T}^{G^{\circledast}}$, is a compact Hausdorff topological group which will be denoted by $\mathfrak{b}G$. The group $\mathfrak{b}G$ is called the *torsion Bohr compactification* of G. Further, $\mathfrak{r}_G\colon G \to \mathfrak{b}G$ is a monomorphism of G onto a dense subgroup of $\mathfrak{b}G$ which is topologically isomorphic to $G^{\natural}$ [2, Theorem 2.1 and Corollary 2.2].

Let G be an abstract or topological group. We denote by G^* the group of all homomorphisms of a discrete group G to $\mathbb{T}$. The coarsest topological group topology on G in which all homomorphisms of G to $\mathbb{T}$ are continuous is called the *Bohr topology* of G. The group G endowed with the Bohr topology is $G^{\#}$. The completion of $G^{\#}$ is known as the *Bohr compactification* of G and is denoted by bG. The group bG is characterized by the property that *every* homomorphism of G (identified with a dense subgroup of bG) to a compact topological group K extends to a continuous homomorphism of bG to K. The Bohr topology is a subject of a thorough study and it appears in different areas of mathematics (see [5], [7], [9], [11], [12], just to mention a few contributions).

It is clear from the above definitions that the Bohr topology on a group G is always finer than the torsion Bohr topology and the two topologies coincide if G is a torsion group. Since our aim is to continue the study of the torsion Bohr topology started in [2] and compare it with the Bohr topology, we will be mainly concerned with non-torsion groups.

In section 2 we show that the torsion Bohr compactification $\mathfrak{b}\mathbb{Z}$ of the group of integers is metrizable and topologically isomorphic to the group of $\mathfrak{a}$-adic integers with $\mathfrak{a} = (2, 3, 4, 5, \dots)$ (Corollary 2.3). It is worth mentioning that the topological character (i.e., the minimal cardinality of a local base at the neutral element) of the Bohr compactification of $\mathbb{Z}$ is equal to $\mathfrak{c} = 2^{\omega}$. Also we present several conditions on a group G, necessary and sufficient, in order that $\mathfrak{b}G$ be zero-dimensional (Corollary 2.12 and Theorem 2.13).

In section 3 it is shown that G is divisible if and only if bG and $\mathfrak{b}G$ are divisible (Lemma 3.2, Corollary 3.3, and Theorem 3.4).

The relation between the torsion Bohr compactification and taking quotient groups is considered in section 4, where we prove that $\mathfrak{b}(G/H)$ is topologically isomorphic to $\mathfrak{b}G/\mathfrak{b}H$ whenever H is a subgroup of a group

G (Theorem 4.6). The same argument shows that $b(G/H)$ is topologically isomorphic to bG/bH (Theorem 4.7).

The algebraic and topological structures of the torsion Bohr compactification of various classical groups is described in section 5. In particular, we present a description of the compact groups $\mathfrak{b}\mathbb{Q}$, $\mathfrak{b}\mathbb{R}$, $\mathfrak{b}\mathbb{T}$, $\mathfrak{b}tor(\mathbb{T})$, $\mathfrak{b}\mathbb{Z}(p^\infty)$, etc.

1.1. Notation and terminology.

We consider only abelian groups here, so we will use the additive notation, except for the case of the circle group $\mathbb{T}$. In the latter case the traditional multiplicative notation is adopted.

A group G is *divisible* if the equation $nx = a$ has a solution in G for each $a \in G$ and each integer $n \neq 0$. It is said that G is *reduced* if a unique divisible subgroup of G is the trivial group $\{0_G\}$, where 0_G is the identity of G. A subgroup H of G is *pure* if $nG \cap H = nH$, for each integer $n \geq 1$.

Elements $a_1, \dots, a_k$ of G are *linearly independent* if the equality $n_1a_1 + \cdots + n_ka_k = 0_G$ implies that $n_1a_1 = \cdots = n_ka_k = 0_G$, where $n_1, \dots, n_k$ are arbitrary integers. An infinite set $A \subset G$ is linearly independent if every finite subset of A is linearly independent. The order of an element $a \in G$ distinct from 0_G is denoted by $o(a)$.

The additive group $\Delta_{\mathfrak{a}}$ of $\mathfrak{a}$-adic integers with $\mathfrak{a} = (2, 3, 4, 5, \dots)$ is presented and studied in detail in [8, Definition 10.2, theorems 10.3 and 10.5, and Note 10.6]. This group is topologically isomorphic to the compact group $\prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$ when the latter carries the usual product topology, where $\mathbb{Z}/n\mathbb{Z} \cong \{0, 1, \dots, n-1\}$ (see the proof of Theorem 10.5 in [8]). We will use the symbol $\mathfrak{a}$ exclusively for the sequence $(2, 3, 4, 5, \dots)$.

The additive group of p-adic integers, Δ_p, is presented in [8, Definition 10.2 and Theorem 10.3]. It is known that the group $\Delta_{\mathfrak{a}}$ is compact and torsion-free. Further, this group is topologically isomorphic to the product $\prod_{p\in\mathbb{P}} \Delta_p$, where $\mathbb{P}$ is the set of prime numbers [8, theorems 25.8 and 25.28(a)].

The group $\mathbb{Z}(p^\infty) = \{z \in \mathbb{T} : z^{p^n} = 1 \text{ for some } n \in \omega\}$, with a prime p, is called *quasicyclic*. The additive groups of the rationals and reals are $\mathbb{Q}$ and $\mathbb{R}$, respectively. The direct sum of κ copies of a group G is denoted by $G^{(\kappa)}$.

Given a set $A \subset G$, we use $\langle A \rangle$ to denote the minimal subgroup of G containing A. The fact that D is a subgroup of G is abbreviated to $D \leq G$.

The group of continuous homomorphisms of a topological group G to the circle group $\mathbb{T}$, with the compact-open topology, is denoted by $G^\wedge$. If G is an abstract or topological group, G^* is the group of all homomorphisms of the discrete group G to $\mathbb{T}$, while $G^*_{\mathbf{p}}$ denotes the group

G^* endowed with the pointwise convergence topology. Similarly, $G^{\circledast}$ is the family of all homomorphisms of G to $tor(\mathbb{T})$.

Suppose that G and H are topological groups. We write $G \cong H$ if G is topologically isomorphic to H. If K is a topological or abstract group, K_d stands for the same group with discrete topology.

We say that $0 \to A \to B \to C \to 0$ is a *short exact sequence* if $A \to B$ is a monomorphism and $B \to C$ is an epimorphism with kernel A.

The cardinality of a set X is $|X|$.

1.2. Preliminary facts.

We collect here several classical results of the Pontryagin duality theory that will be frequently used in the article.

Theorem 1.1. *The following are valid.*

(i) *A discrete group G is divisible if and only if its dual group $G^\wedge = G^*$ is torsion-free* [8, Theorem 24.23]*.*
(ii) *A Hausdorff compact group G is divisible if and only if $G^\wedge$ is torsion-free if and only if G is connected* [8, Theorem 24.25]*.*
(iii) *If G is a compact Hausdorff group, then G is zero-dimensional if and only if $G^\wedge$ is a torsion group* [8, Theorem 24.26].
(iv) *(Pontryagin duality in the compact-discrete case) If G is discrete, then $G^*_{\mathbf{p}}$ is compact and $(G^*_{\mathbf{p}})^\wedge \cong G$; if G is compact, then $G^\wedge$ is discrete and $(G^\wedge)^*_{\mathbf{p}} \cong G$* [8, Theorem 24.8].

We also note that the groups Δ_p with $p \in \mathbb{P}$ and $\Delta_{\mathfrak{a}}$ are reduced. Indeed, by [10, Theorem 18, p. 46], the additive group of p-adic integers is *indecomposable*, i.e., Δ_p cannot be represented as a direct sum of two non-trivial subgroups. Since every abelian group is a direct sum of a divisible subgroup and a reduced subgroup [6, Theorem 21.3] and Δ_p is not divisible, we see that Δ_p is reduced. A similar argument applies to the group $\Delta_{\mathfrak{a}}$.

2. Zero-Dimensionality of $\mathfrak{b}G$

In this section we study algebraic properties of G and $G^{\circledast}$ which are necessary or sufficient for the zero-dimensionality of $\mathfrak{b}G$ (Proposition 2.10, Corollary 2.12, and Theorem 2.13). First we need a couple of definitions.

Let G be an abstract abelian group and Γ a nonempty subset of G^*. The *diagonal product* $\triangle_\Gamma$ of the family Γ is the mapping of G to $\mathbb{T}^\Gamma$ defined by the formula $\triangle_\Gamma(x) = (\chi(x))_{\chi\in\Gamma} \in \mathbb{T}^\Gamma$, where $x \in G$. It is clear that $\triangle_\Gamma$ is a homomorphism and that $\triangle_\Gamma$ is one-to-one if and only if Γ separates points of G. Let $b_\Gamma G = cl_{\mathbb{T}^\Gamma}(\triangle_\Gamma(G))$. Using this terminology we can say that $b_{G^*}G$ and $b_{G^{\circledast}}G$ are bG and $\mathfrak{b}G$, respectively.

Lemma 2.1. *If* $\Gamma \subseteq G^*$, *then* $(b_\Gamma G)^\wedge \cong \langle \Gamma \rangle_d$ *and, in particular,* $(bG)^\wedge \cong G^*_d$. *Therefore,* $b_\Gamma G \cong (\langle \Gamma \rangle)^*_{\mathbf{p}}$ *and* $bG \cong (G^*)^*_{\mathbf{p}}$.

Proof. Given $\chi \in \Gamma$, let π_χ be the projection of $\mathbb{T}^\Gamma$ onto the factor $\mathbb{T}_{(\chi)} = \mathbb{T}$. Then $\pi_\chi(\triangle_\Gamma(x)) = \chi(x)$, for each $x \in G$. The projections π_χ are continuous homomorphisms of $\mathbb{T}^\Gamma$ and when restricted to $b_\Gamma G$, they become continuous homomorphisms of $b_\Gamma G$ to $\mathbb{T}$. The family $\{\pi_\chi : \chi \in \Gamma\}$ separates points of $b_\Gamma G$. Indeed, take $a \in b_\Gamma G$ such that $a \neq e_{\mathbb{T}^\Gamma}$. Then there exists $\chi \in \Gamma$ such that $\pi_\chi(a) = a_\chi \neq 1$.

Hence, [8, Theorem 23.20] (see also [3, Theorem 1.3]) implies that $(b_\Gamma G)^\wedge$ consists of the homomorphisms $\pi_{\chi_1}^{\alpha_1}\pi_{\chi_2}^{\alpha_2}\cdots\pi_{\chi_m}^{\alpha_m}$, where $\alpha_1 \dots \alpha_m$ are integers and $\chi_1, \dots, \chi_m \in \Gamma$. It is clear that $\pi_{\chi_1}^{\alpha_1}\pi_{\chi_2}^{\alpha_2}\cdots\pi_{\chi_m}^{\alpha_m} \equiv 1$ in $b_\Gamma G$ if and only if $\chi_1^{\alpha_1}\chi_2^{\alpha_2}\cdots\chi_m^{\alpha_m} \equiv 1$ in G. Hence, $(b_\Gamma G)^\wedge \cong \langle \Gamma \rangle_d$. By the Pontryagin duality (see Theorem 1.1 (iv)), we have that $b_\Gamma G \cong ((b_\Gamma G)^\wedge)^*_{\mathbf{p}} \cong (\langle \Gamma \rangle)^*_{\mathbf{p}}$. □

Corollary 2.2. $(\mathfrak{b}G)^\wedge \cong G^{\circledast}_d$ *and* $\mathfrak{b}G \cong (G^{\circledast})^*_{\mathbf{p}}$.

Corollary 2.3. *The group* $\mathfrak{b}\mathbb{Z}$ *is topologically isomorphic to* $\Delta_{\mathfrak{a}}$, *the additive group of* $\mathfrak{a}$-*adic integers.*

Proof. It is easy to see that $\mathbb{Z}^{\circledast} \cong tor(\mathbb{T}) \cong \mathbb{Q}/\mathbb{Z}$. Indeed, the mapping $\chi \to \chi(1)$ gives an isomorphism of $\mathbb{Z}^{\circledast}$ onto $tor(\mathbb{T})$, while $tor(\mathbb{T})$ is evidently isomorphic to $\mathbb{Q}/\mathbb{Z}$ since an element e^{ix} is torsion if and only if x is rational.

The group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$, by [8, A.14]. Hence, $(b_{\mathbb{Z}^{\circledast}}\mathbb{Z})^\wedge = (\mathfrak{b}\mathbb{Z})^\wedge \cong \mathbb{Z}^{\circledast}_d \cong (\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty))_d$ and [8, Theorem 23.22, Theorem 25.2, and Theorem 24.8] imply that $\mathfrak{b}\mathbb{Z} \cong (\mathbb{Z}^{\circledast})^*_{\mathbf{p}} \cong \prod_{p\in\mathbb{P}}(\mathbb{Z}(p^\infty))^*_{\mathbf{p}} \cong \prod_{p\in\mathbb{P}} \Delta_p \cong \Delta_{\mathfrak{a}}$. This completes the proof. □

[2, Corollary 2.10] shows that $\mathfrak{b}(G_1 \times G_2) \cong \mathfrak{b}G_1 \times \mathfrak{b}G_2$, for arbitrary abstract groups G_1 and G_2. In the following result we describe the torsion Bohr compactification of an infinite finitely generated group. Its proof is omitted since it suffices to combine the aforementioned equivalence and Corollary 2.3.

Corollary 2.4. *Let* G *be an infinite finitely generated group. Then* G *is isomorphic to the group* $\mathbb{Z}^m \times K$, *where* K *is a finite group and* $m > 0$ *is an integer. Therefore,*

$$\mathfrak{b}G \cong \Delta_{\mathfrak{a}}^m \times K_d.$$

Let us denote by $\Sigma_{\mathfrak{a}}$ the $\mathfrak{a}$-adic solenoid (see Definition 10.12 and Theorem 10.13 in [8]). We know, by Corollary 2.3, that $\mathfrak{b}\mathbb{Z} \cong \Delta_{\mathfrak{a}}$. Making use of the Pontryagin duality in the compact-discrete case (see Theorem 1.1(iv)), [8, Theorem 23.22], and the equality $\mathbb{Q}_{\mathbf{p}}^* \cong \Sigma_{\mathfrak{a}}$ (see [8, Theorem 25.4]), we see that

$$\begin{aligned} b\mathbb{Z} &\cong (\mathbb{Z}^*)_{\mathbf{p}}^* \cong \mathbb{T}_{\mathbf{p}}^* \cong (\mathbb{Q}^{(\mathfrak{c})} \times \mathbb{Z}^{\circledast})_{\mathbf{p}}^* \cong (\mathbb{Q}^{(\mathfrak{c})} \times (\Delta_{\mathfrak{a}})^\wedge)_{\mathbf{p}}^* \\ &\cong (\mathbb{Q}^{(\mathfrak{c})})_{\mathbf{p}}^* \times ((\Delta_{\mathfrak{a}})^\wedge)_{\mathbf{p}}^* \cong (\mathbb{Q}_{\mathbf{p}}^*)^{\mathfrak{c}} \times \Delta_{\mathfrak{a}} \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \mathfrak{b}\mathbb{Z}. \end{aligned} \tag{2.1}$$

It also follows from [1, Corollary 9.9.12] that $b(G_1 \times G_2) \cong bG_1 \times bG_2$, for arbitrary groups G_1 and G_2, and the same equivalence holds for the torsion Bohr compactification [2, Corollary 2.10]. The following result is a simple combination of the latter fact and the equality (2.1).

Corollary 2.5. *Let $G = \mathbb{Z}^m$, where $m \geq 1$. Then $bG \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$.*

The next fact is a slightly more general form of Corollary 2.5.

Corollary 2.6. *Suppose that $G = \mathbb{Z}^m \oplus K$, where m is a positive integer and K is a torsion group. Then $bG \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$.*

Proof. Since K is a torsion group, it follows that $bG = \mathfrak{b}G$. Therefore, we apply Corollary 2.5 (jointly with [1, Corollary 9.9.12] and [2, Corollary 2.10]) to deduce that

$$bG \cong (b\mathbb{Z})^m \times bK \cong (\mathfrak{b}\mathbb{Z})^m \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \mathfrak{b}K \cong \mathfrak{b}(\mathbb{Z}^m \oplus K) \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}}.$$

This finishes the proof. □

It is quite natural to ask, after corollaries 2.5 and 2.6, whether a torsion-free group G satisfies the equality $bG \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$. More generally, it would be interesting to characterize the groups G satisfying the equality in Corollary 2.6; i.e., we wonder when the formula $bG \cong \mathfrak{b}G \oplus \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$ is valid.

Our answer to these questions requires the following useful fact.

Proposition 2.7. *Let G be an abelian group. Then $G^* = \hom(G, \mathbb{T}) = G^{\circledast} \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})})$ and $bG \cong \mathfrak{b}G \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})})_{\mathbf{p}}^*$. Further, the summand $\hom(G, \mathbb{Q}^{(\mathfrak{c})})_{\mathbf{p}}^*$ is divisible provided that the group $\hom(G, \mathbb{Q}^{(\mathfrak{c})})$ is trivial or torsion-free.*

Proof. Let us note that $\mathbb{T}_d = tor(\mathbb{T}) \oplus \mathbb{Q}^{(\mathfrak{c})}$, by [8, Theorem 25.13], where the group $\mathbb{Q}^{(\mathfrak{c})}$ is torsion-free. Hence,

$$\begin{aligned} G^* &\cong (bG)^\wedge \cong \hom(G, \mathbb{T}) \cong \hom(G, tor(\mathbb{T}) \oplus \mathbb{Q}^{(\mathfrak{c})}) \\ &\cong \hom(G, tor(\mathbb{T})) \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})}) \cong (\mathfrak{b}G)^\wedge \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})}) \\ &\cong G^{\circledast} \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})}). \end{aligned} \tag{2.2}$$

To finish the proof it suffices to apply our Lemma 2.1 and [8, Theorem 26.12]. □

Theorem 2.8. *The equality* $bG \cong \mathfrak{b}G \oplus \hom(G, \mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}}$ *is valid for every torsion-free abelian group* G.

Proof. It is clear that $\hom(G, \mathbb{Q}^{(\mathfrak{c})})$ is an uncountable, torsion-free, divisible abelian group. Since maximal independent subsets of G constitute a $\mathbb{Q}$-basis of the divisible hull of G, $\hom(G, \mathbb{Q}^{(\mathfrak{c})})$ is in bijective correspondence with $(\mathbb{Q}^{(\mathfrak{c})})^A$, for any maximal independent subset A of G. Therefore,

$$\hom(G, \mathbb{Q}^{(\mathfrak{c})}) \cong (\mathbb{Q}^{(\mathfrak{c})})^{r_0(G)} \cong \mathbb{Q}^{(\mathfrak{c}^{r_0(G)})}$$

and

$$bG \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}^{r_0(G)}}.$$

This completes the proof. □

The following corollary is immediate from Theorem 2.8; it answers the question preceding Proposition 2.7

Corollary 2.9. *A torsion-free abelian group* G *with* $r_0(G) \leq \omega$ *satisfies* $bG \cong \mathfrak{b}G \times \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$. *Further, under* CH, *the condition* $r_0(G) \leq \omega$ *is necessary.*

We know that the groups $G^\natural$ and $G^\#$ coincide, for every torsion group G. Hence, $bG = \mathfrak{b}G$ in this case. Theorem 2.8 and Corollary 2.9 present a simple relation between the groups bG and $\mathfrak{b}G$ in the case of a torsion-free group G.

Let us recall that a space X with a base of clopen sets is called *zero-dimensional.* The following result will be substantially refined in Theorem 2.13.

Proposition 2.10. *If* $\mathfrak{b}G$ *is zero-dimensional, then the group* G *is reduced.*

Proof. By (ii) and (iii) of Theorem 1.1, the group $\mathfrak{b}G$ is not divisible. Hence, G is not divisible according to [1, Exercise 9.11.f] (see also Proposition 3.1). Therefore, G has the form $G \cong D \times R$, where $D \neq G$ is divisible and $R \neq \{0_G\}$ is reduced. [8, Theorem 24.26] implies that $\chi(D)$ is a finite subgroup of $\mathbb{T}$ for each $\chi \in (\mathfrak{b}G)^\wedge$. Hence, D is trivial and G is reduced. □

Observation 2.11. The converse to Proposition 2.10 is false. Indeed, the torsion group $G = \bigoplus_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$ is reduced. Hence, $G^{\circledast} = G^*$, for G is torsion. Applying [8, theorems 23.22 and 23.27 (c)], we see that $G^* = (\bigoplus_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z})^* \cong \prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z} \cong \Delta_{\mathfrak{a}}$. It also follows from [8, theorems 25.8 and 25.28(a)] that $\Delta_{\mathfrak{a}} \cong \prod_{p \in \mathbb{P}} \Delta_p$ is a compact torsion-free group. If H is a compact Hausdorff group, then H is divisible (zero-dimensional) if and only if $H^\wedge$ is torsion-free (torsion), by [8, theorems 24.25 and 24.26]. Therefore, $\mathfrak{b}G \cong (G^*)^*_{\mathbf{p}} = (\Delta_{\mathfrak{a}})^*_{\mathbf{p}}$ is divisible but not zero-dimensional since $(\mathfrak{b}G)^\wedge \cong G^*_d \cong (\Delta_{\mathfrak{a}})_d$ is torsion-free.

As a consequence of Corollary 2.2 and Theorem 1.1(iii), we obtain a characterization of the groups whose torsion Bohr compactification is zero-dimensional.

Corollary 2.12. *For an abelian group G, $\mathfrak{b}G$ is zero-dimensional if and only if $G^{\circledast}$ is a torsion group.*

The following theorem is the main result of this section.

Theorem 2.13. *Let G be an abelian group. The torsion Bohr compactification $\mathfrak{b}G$ of G is zero-dimensional if and only if the torsion part of G, $tor(G)$, is bounded torsion and $G \cong \mathbb{Z}^m \oplus tor(G)$, where m is a non-negative integer.*

Proof. By Corollary 2.12 it suffices to show that the group $G^{\circledast}$ is torsion if and only if G is isomorphic to the group $\mathbb{Z}^m \oplus tor(G)$, for some $m \geq 0$, and $tor(G)$ is bounded torsion.

Suppose that $G^{\circledast}$ is a torsion group. First we show that $r_0(G) < \omega$; i.e., every linearly independent system of elements of infinite order in G is finite. Suppose for a contradiction that $\{x_1, x_2, \ldots\}$ is an infinite system of linearly independent elements of infinite order in G. Let $t_k = e^{\pi i/k}$, for each integer $k \geq 1$. Clearly, $t_k \in tor(\mathbb{T})$. Denote by H the subgroup of G generated by the set $\{x_k : k \in \mathbb{N}\}$. Let also χ be a homomorphism of H to $tor(\mathbb{T})$ such that $\chi(x_k) = t_k$ for each $k \in \mathbb{N}$. It is clear that χ is of infinite order in $H^{\circledast}$. Denote by $\bar{\chi}$ an extension of χ to a homomorphism of G to $tor(\mathbb{T})$. Then $\bar{\chi}$ has infinite order in $G^{\circledast}$, which is a contradiction. We have thus proved that $m = r_0(G)$ is finite.

Let $a_1, \ldots, a_m$ be a maximal linearly independent system of elements of infinite order in G. Denote by L the subgroup of G generated by the set $\{a_1, \ldots, a_m\}$. Let us note that the group L is torsion-free and the quotient group $K = G/L$ is torsion. We claim that K is bounded torsion. First we verify that the group $K^{\circledast}$ is torsion. Indeed, let $\pi\colon G \to G/L$ be the quotient homomorphism. Denote by $\pi^{\circledast}$ the dual homomorphism of $K^{\circledast}$ to $G^{\circledast}$ defined by $\pi^{\wedge}(\phi) = \phi \circ \pi$ for each $\phi \in K^{\circledast}$. It clear that $\pi^{\circledast}$ is a monomorphism. Thus, $K^{\circledast}$ is isomorphic to a subgroup of $G^{\circledast}$, and hence it is a torsion group as well.

Suppose for a contradiction that K is an unbounded torsion group and choose a sequence $\{b_k : k \in \mathbb{N}\}$ of elements of K such that $o(b_k) < o(b_{k+1})$ for each $k \in \mathbb{N}$. We define a sequence $\{c_k : k \in \mathbb{N}\} \subset K$ such that $sc_{k+1} \notin \langle c_1, \ldots c_k\rangle$ if $|s| \leq k$ and $s \neq 0$. Let $c_1 = b_1$ and suppose that we have defined elements $c_1, \ldots, c_k$ in K for some $k \geq 1$. Since K is a torsion group, the subgroup C_k of K generated by the elements $c_1, \ldots, c_k$ is finite. Take an element b_n such that $o(b_n) > |C_k| \cdot k$. Then $sb_n \notin C_k$ if $|s| \leq k$ and $s \neq 0$. Indeed, if $sb_n \in C_k$ and $s \neq 0$, then $sMb_n = 0_K$, where $M = |C_k|$. Hence, $Mk < o(b_n) \le |s| \cdot M$, whence $k < |s|$. It remains to put $c_{k+1} = b_n$. This finishes our definition of the sequence $\{c_k : k \in \mathbb{N}\} \subset K$.

Since $sc_{k+1} \notin C_k$ if $0 < s \le k$, for each $k \in \mathbb{N}$, we can define by induction a homomorphism ϕ of K to $tor(\mathbb{T})$ such that $\phi(c_k) = e^{2\pi i/n_k}$, where $n_k \in \mathbb{N}$ and $n_k > k$ for each $k \in \mathbb{N}$. Again, this implies that ϕ has infinite order in $K^{\circledast}$. This contradiction proves that K is a bounded torsion group.

We show now that G is isomorphic with $\mathbb{Z}^m \oplus tor(G)$, where $m = r_0(G)$. It is easy to see that the subgroup $tor(G)$ is bounded torsion. Indeed, let N be the exponent of the quotient group $K = G/L$. Then $Ny = 0_K$, for each $y \in K$ and, hence, $Nx \in L$ for each $x \in G$. Suppose that $x \in tor(G)$. Then $Nx \in tor(G) \cap L$ and since L is torsion-free, we conclude that $Nx = 0_K$. We have thus shown that $tor(G)$ is bounded torsion. Clearly, the torsion part of G is a pure subgroup of G. Since $tor(G)$ is bounded torsion, it follows from [13, 4.3.8] that G is isomorphic with the group $tor(G) \oplus G/tor(G)$. Further, our definition of the subgroup L of G implies that $G/tor(G) \cong L \cong \mathbb{Z}^m$. Therefore, $G \cong \mathbb{Z}^m \oplus tor(G)$.

Conversely, suppose that $G \cong \mathbb{Z}^m \oplus K$, where m is a non-negative integer and K is a bounded torsion group. Then $tor(G) \cong K$ and $G^{\circledast} \cong (\mathbb{Z}^{\circledast})^m \oplus K^{\circledast} \cong (tor(\mathbb{T}))^m \oplus K^{\circledast}$. Since K is bounded torsion, so is $K^{\circledast}$ and, therefore, $G^{\circledast}$ is a torsion group. This finishes the proof of the theorem. □

3. Divisibility of $\mathfrak{b}G$

In this section we study the question of when the torsion Bohr compactification of an abelian group is divisible. The following result is well known (see [1, Exercise 9.11.f]).

Proposition 3.1. *Let H be a dense subgroup of a compact group G. If H is divisible, so is G.*

Proof. For every positive integer n, let M_n be the mapping of G to itself defined by $M_n(x) = x^n$ for each $x \in G$. If H is divisible, then $M_n(H) = H$ for each $n \ge 1$. Since the mapping M_n is continuous and G is compact, $M_n(G)$ is closed in G. It is clear that $H = M_n(H) \subseteq M_n(G)$ is dense in G, so $M_n(G) = G$ for each $n \ge 1$. Hence, G is divisible. □

For the (torsion) Bohr compactification, Proposition 3.1 can be given a more precise form (see also Theorem 3.4).

Lemma 3.2. *The group G is divisible if and only if $\mathfrak{b}G$ is divisible.*

Proof. By Theorem 1.1(i), the group G is divisible if and only if G^* is torsion-free, and by (ii) of the same theorem, G^* is torsion-free if and only if bG is divisible. □

The relation between bG and $\mathfrak{b}G$ established in Proposition 2.7 enables us to deduce the following.

Corollary 3.3. *Let G be an abelian group. Then bG is divisible if and only if $\mathfrak{b}G$ is as well.*

Theorem 3.4. *A group G is divisible if and only if $\mathfrak{b}G$ is divisible.*

Proof. It suffices to combine Lemma 3.2 and Corollary 3.3. □

In the next proposition we describe the algebraic structure of the group $\mathfrak{b}G$ in the case when G is divisible. Our argument is based on [8, Theorem 25.23].

Proposition 3.5. *If G is divisible and non-trivial, then $\mathfrak{b}G$ is algebraically isomorphic to*

$$\mathbb{Q}^{(2^{2^{|G|}})} \times \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{(\mathfrak{b}_p)},$$

where $\mathfrak{b}_p$ is finite or equals 2^{e_p} for some infinite cardinal $e_p \leq 2^{|G|}$.

Proof. Suppose that the group G is divisible. It is clear that G is infinite. Theorem 3.5 implies that the compact group $\mathfrak{b}G$ is divisible, while [8, Theorem 25.23] says that a compact divisible group of weight κ is algebraically isomorphic to

$$\mathbb{Q}^{(2^\kappa)} \times \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{(\mathfrak{b}_p)},$$

where $\mathfrak{b}_p$ is finite or equals 2^{e_p} for some infinite cardinal $e_p \leq \kappa$. [2, Lemma 6.1 and Theorem 6.5] imply that $\kappa = 2^{|G|}$. This finishes the proof. □

4. Torsion Bohr Compactification of Quotient Groups

We will show in this section that the torsion Bohr compactification of the quotient group G/H is topologically isomorphic to quotient group $\mathfrak{b}G/\mathfrak{b}H$. This requires several auxiliary results.

Let $\psi\colon A \to B$ be a homomorphism of discrete groups. For $\chi \in B^{\circledast}$, we define $\psi^{\circledast}(\chi)\colon A \to tor(\mathbb{T})$ by

$$\psi^{\circledast}(\chi)(a) = \chi(\psi(a))$$

for each $a \in A$. It is clear that $\psi^{\circledast}$ is a homomorphism of $B^{\circledast}$ to $A^{\circledast}$.

The following lemmas (4.1–4.5) are quite elementary.

Lemma 4.1. *Let $\psi\colon A \to B$ be a monomorphism of discrete groups. Then $\psi^{\circledast}$ is an epimorphism of $B^{\circledast}$ to $A^{\circledast}$.*

Proof. Since $tor(\mathbb{T})$ is divisible, every homomorphism $\chi\colon \psi(A) \to tor(\mathbb{T})$ extends to a homomorphism of B to $tor(\mathbb{T})$. Therefore, $\psi^{\circledast}$ is an epimorphism. □

Lemma 4.2. *Let $\psi\colon A \to B$ be an epimorphism of discrete groups. Then $\psi^{\circledast}$ is a monomorphism of $B^{\circledast}$ to $A^{\circledast}$.*

Proof. Suppose that $\chi \in B^{\circledast}$ satisfies $\psi^{\circledast}(\chi) \equiv 1$. Then $\psi^{\circledast}(\chi)(A) = \{1\}$, i.e., $\chi(B) = \chi(\psi(A)) = \{1\}$. Hence, $\chi \equiv 1$ and so $\psi^{\circledast}$ is a monomorphism. □

Lemma 4.3. *Let A be a subgroup of a discrete group B and let $R\colon B^{\circledast} \to A^{\circledast}$ be the restriction mapping, $R(\chi) = \chi\restriction A$ for each $\chi \in B^{\circledast}$. Then the kernel of R is isomorphic to $(B/A)^{\circledast}$.*

Proof. Given a homomorphism $\psi\colon B/A \to tor(\mathbb{T})$, we define $\chi \in B^{\circledast}$ by $\chi(b) = \psi(\pi(b))$, where $\pi\colon B \to B/A$ is the quotient homomorphism. Then $\chi(A) = \psi(\pi(A)) = \{1\}$ and $\psi \equiv 1$ if and only if $\chi \equiv 1$. Let us note that if $g \in B^{\circledast}$ is an element of the kernel of R, then $g \circ \pi^{-1} \in (B/A)^{\circledast}$ and $g = (g \circ \pi^{-1}) \circ \pi$. Therefore, the kernel of R is isomorphic to $(B/A)^{\circledast}$. □

Lemma 4.4. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of discrete groups. Then the sequence*

$$0 \to C^{\circledast} \xrightarrow{g^{\circledast}} B^{\circledast} \xrightarrow{f^{\circledast}} A^{\circledast} \to 0$$

is also exact. Furthermore, $B^{\circledast}/C^{\circledast}$ is isomorphic to $A^{\circledast}$.

Proof. Lemma 4.2 implies that $g^{\circledast}\colon C^{\circledast} \to B^{\circledast}$ is a monomorphism, so we can identify $C^{\circledast}$ with the subgroup $g^{\circledast}(C^{\circledast})$ of $B^{\circledast}$. It follows from Lemma 4.1 that $f^{\circledast}\colon B^{\circledast} \to A^{\circledast}$ is an epimorphism with kernel $C^{\circledast}$, by [4, Exercise 1.6.15]. Then the first isomorphism theorem for groups guarantees that $B^{\circledast}/C^{\circledast}$ is isomorphic to $A^{\circledast}$. □

The functor * also satisfies lemmas 4.1, 4.2, 4.3, and 4.4. In fact, we have the following lemma.

Lemma 4.5. *Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an exact sequence of discrete groups. Then there exists an exact sequence with continuous homomorphisms*

$$0 \to C^*_{\mathbf{p}} \to B^*_{\mathbf{p}} \xrightarrow{f^*} A^*_{\mathbf{p}} \to 0.$$

Furthermore, $B^*_{\mathbf{p}}/C^*_{\mathbf{p}}$ is topologically isomorphic to $A^*_{\mathbf{p}}$.

Proof. The groups $B_{\mathbf{p}}^*$ and $A_{\mathbf{p}}^*$ are compact when endowed with the pointwise convergence topology. The continuous epimorphism f^* of $B_{\mathbf{p}}^*$ to $A_{\mathbf{p}}^*$ is closed and has the kernel $C_{\mathbf{p}}^*$. The first isomorphism theorem for topological groups implies that $B_{\mathbf{p}}^*/C_{\mathbf{p}}^* \cong A_{\mathbf{p}}^*$. □

If H is a subgroup of a group G, then the identity monomorphism $i\colon H \to G$ extends to a topological monomorphism $\varphi\colon \mathfrak{b}H \to \mathfrak{b}G$; i.e., $\mathfrak{b}H$ is topologically isomorphic to the subgroup $cl_{\mathfrak{b}G}H$ of the group $\mathfrak{b}G$ [2, Corollary 2.8]. We use this fact in the proof of the next result.

Theorem 4.6. *Let G/H be the quotient group of an abelian group G with respect to its subgroup H. Then $\mathfrak{b}(G/H) \cong \mathfrak{b}G/\mathfrak{b}H$.*

Proof. Given the short exact sequence

$$0 \to H \to G \to G/H \to 0,$$

it follows from lemmas 4.4 and 4.5 that there exist short exact sequences

$$0 \to (G/H)_d^{\circledast} \to G_d^{\circledast} \to H_d^{\circledast} \to 0$$

and

$$0 \to (H^{\circledast})_{\mathbf{p}}^* \to (G^{\circledast})_{\mathbf{p}}^* \to ((G/H)^{\circledast})_{\mathbf{p}}^* \to 0$$

with $(G^{\circledast})_{\mathbf{p}}^*/(H^{\circledast})_{\mathbf{p}}^* \cong ((G/H)^{\circledast})_{\mathbf{p}}^*$. By Corollary 2.2 we have that $(G^{\circledast})_{\mathbf{p}}^* \cong \mathfrak{b}G$, $(H^{\circledast})_{\mathbf{p}}^* \cong \mathfrak{b}H$, and $((G/H)^{\circledast})_{\mathbf{p}}^* \cong \mathfrak{b}(G/H)$. This implies the conclusion of the theorem. □

The following theorem has a proof similar to the proof of Theorem 4.6; one has only to replace the functor $^{\circledast}$ with *.

Theorem 4.7. *Let G/H be the quotient group of G with respect to its subgroup H. Then $b(G/H) \cong bG/bH$.*

5. Several Structure Theorems

In this section we describe the torsion Bohr compactifications of various classic groups such as $\mathbb{Z}(p^\infty)$, $tor(\mathbb{T})$, $\mathbb{Q}$, $\mathbb{R}$, etc.

Theorem 5.1. *The following are valid:*

1. $\mathbb{Z}(p^\infty)^* = \mathbb{Z}(p^\infty)^{\circledast} \cong \Delta_p$ *and* $b\mathbb{Z}(p^\infty) = \mathfrak{b}\mathbb{Z}(p^\infty) \cong (\Delta_p)_{\mathbf{p}}^*$ *for each* $p \in \mathbb{P}$.
2. $tor(\mathbb{T})^* = tor(\mathbb{T})^{\circledast} \cong \Delta_{\mathfrak{a}}$ *and* $btor(\mathbb{T}) = \mathfrak{b}tor(\mathbb{T}) \cong (\Delta_{\mathfrak{a}})_{\mathbf{p}}^*$.
3. $\mathbb{Q}^{\circledast} \cong \mathbb{Q} \times \Delta_{\mathfrak{a}}$ *and* $\mathfrak{b}\mathbb{Q} \cong \Sigma_{\mathfrak{a}} \times (\Delta_{\mathfrak{a}})_{\mathbf{p}}^*$.

Proof. (1) Since $\mathbb{Z}(p^\infty)$ is a torsion group, we see that $\mathbb{Z}(p^\infty)^* = \mathbb{Z}(p^\infty)^{\circledast}$. [8, Theorem 24.8 and Theorem 25.2] imply that $\mathbb{Z}(p^\infty)^* = \Delta_p$. Finally, we use Corollary 2.2.

(2) Once again, $tor(\mathbb{T})$ is a torsion group; hence, $tor(\mathbb{T})^* = tor(\mathbb{T})^{\circledast}$. [8, Theorem 25.13] implies that $tor(\mathbb{T}) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$. Further, [8, Theorem 23.22] implies that $tor(\mathbb{T})^* = (\bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty))^* = \prod_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^* \cong \prod_{p\in\mathbb{P}} \Delta_p \cong \Delta_{\mathfrak{a}}$. It remains to apply Corollary 2.2.

(3) By [8, theorems 25.4 and 25.7], $\mathbb{Q}^{\circledast} \cong \mathbb{Q} \times \prod_{p\in\mathbb{P}} \Delta_p \cong \mathbb{Q} \times \Delta_{\mathfrak{a}}$. The required equality follows from the fact that $\mathbb{Q}^*_{\mathbf{p}} \cong \Sigma_{\mathfrak{a}}$ (see [8, Theorem 25.4] and Corollary 2.2). □

Theorem 5.2. *The following equalities are valid:*

(1) $\mathbb{R}^{\circledast} \cong (\mathbb{Q}^{\circledast})^{\mathfrak{c}} \cong (\mathbb{Q} \times \Delta_{\mathfrak{a}})^{\mathfrak{c}}$ *and* $\mathfrak{b}\mathbb{R} \cong (\mathbb{Q}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}})^*_{\mathbf{p}} \cong \Sigma_{\mathfrak{a}}^{2^{\mathfrak{c}}} \times (\Delta_{\mathfrak{a}}^{\mathfrak{c}})^*_{\mathbf{p}}$.

(2) $(\Delta_{\mathfrak{a}})^{\circledast}_{\mathbf{p}} \cong \mathfrak{b}tor(\mathbb{T})/\hom(\Delta_{\mathfrak{a}}, \mathbb{Q}^{(\mathfrak{c})})_{\mathbf{p}}$ *and hence,* $\mathfrak{b}\Delta_{\mathfrak{a}}$ *is topologically isomorphic to the group* $(\mathfrak{b}tor(\mathbb{T}))^*_{\mathbf{p}}/\hom(\Delta_{\mathfrak{a}}, \mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}}$.

Proof. (1) By [8, Theorem A.14, Theorem 23.22, and Theorem 25.4], we conclude that $\mathbb{R}^*_{\mathbf{p}} = (\mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}} \cong (\mathbb{Q}^*_{\mathbf{p}})^{\mathfrak{c}} \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}}$. [8, Theorem 23.22 and Theorem 25.4] imply that the homomorphisms of $(\mathbb{Q}^{(\mathfrak{c})})_d = \mathbb{R}_d$ to $tor(\mathbb{T}) \subset \mathbb{T}$ correspond to the subgroup $(\mathbb{Q} \times \prod_{p\in\mathbb{P}} \Delta_p)^{\mathfrak{c}} \cong (\mathbb{Q} \times \Delta_{\mathfrak{a}})^{\mathfrak{c}}$ of $\Sigma_{\mathfrak{a}}^{\mathfrak{c}}$. Then we use Corollary 2.2. Finally, we know that $\mathbb{Q}^{\mathfrak{c}}$ is a linear space over $\mathbb{Q}$ which has dimension $2^{\mathfrak{c}}$. Hence, the Pontryagin dual $(\mathbb{Q}^{\mathfrak{c}})^*_{\mathbf{p}}$ is the compact group $\Sigma_{\mathfrak{a}}^{2^{\mathfrak{c}}} \cong (\mathbb{Q}^*_{\mathbf{p}})^{2^{\mathfrak{c}}}$.

(2) By Theorem 5.1(2) and Proposition 2.7, we have that

$$\mathfrak{b}tor(\mathbb{T}) \cong (\Delta_{\mathfrak{a}})^*_{\mathbf{p}} \cong (\Delta_{\mathfrak{a}})^{\circledast}_{\mathbf{p}} \times \hom(\Delta_{\mathfrak{a}}, \mathbb{Q}^{(\mathfrak{c})})_{\mathbf{p}}$$

and

$$(\mathfrak{b}tor(\mathbb{T}))^*_{\mathbf{p}} \cong (((\Delta_{\mathfrak{a}})^*)^*_{\mathbf{p}} \cong b\Delta_{\mathfrak{a}} \cong \mathfrak{b}\Delta_{\mathfrak{a}} \times \hom(\Delta_{\mathfrak{a}}, \mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}}.$$

This implies the required conclusion. □

Since $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ and $tor(\mathbb{T}) \cong \mathbb{Q}/\mathbb{Z}$, the following corollary is immediate from Theorem 4.6, and theorems 5.1 and 5.2.

Corollary 5.3. *The following are valid:*

(1) $\mathfrak{b}\mathbb{T} \cong \mathfrak{b}\mathbb{R}/\Delta_{\mathfrak{a}}$, *where* $\mathfrak{b}\mathbb{R} \cong \Sigma_{\mathfrak{a}}^{2^{\mathfrak{c}}} \times (\Delta_{\mathfrak{a}}^{\mathfrak{c}})^*_{\mathbf{p}}$ *and* $\Delta_{\mathfrak{a}} \cong cl_{\mathfrak{b}\mathbb{R}}\mathbb{Z}$.

(2) $\mathfrak{b}(tor(\mathbb{T})) \cong \mathfrak{b}\mathbb{Q}/\Delta_{\mathfrak{a}}$, *where* $\mathfrak{b}\mathbb{Q} \cong \Sigma_{\mathfrak{a}} \times (\Delta_{\mathfrak{a}})^*_{\mathbf{p}}$ *and* $\Delta_{\mathfrak{a}} \cong cl_{\mathfrak{b}\mathbb{Q}}\mathbb{Z}$.

For our further calculations we need various auxiliary results.

Lemma 5.4. *Let* Δ_p *be the group of* p*-adic integers with a prime* p *and let* $u = (1, 0, 0, 0, \ldots) \in \Delta_p$. *Let also* $C_p = \langle u \rangle$ *and let* $\pi\colon \Delta_p \to \Delta_p/C_p = X_p$ *be the canonical homomorphism. Then the following hold:*

(a) *if* p *does not divide* n, *then the equation* $nx = b$ *has a solution in* Δ_p *for each* $b \in \Delta_p$, *i.e.,* $n\Delta_p = \Delta_p$;

(b) $p^k\Delta_p \cap C_p = \langle p^k u \rangle$, *for each* $k \in \mathbb{N}$;

(c) *the group* $X_p = \Delta_p/C_p$ *is divisible;*

(d) *the group X_p is algebraically isomorphic with*

$$\mathbb{Q}^{(\mathfrak{c})} \oplus \bigoplus_{q\in\mathbb{P}\wedge q\neq p} \mathbb{Z}(q^\infty).$$

Proof. Item (a) follows from [1, Problem 1.1.F]. To verify (b), note first that $\langle p^k u\rangle \subseteq p^k\Delta_p \cap C_p$, for each $k \in \mathbb{N}$. Conversely, suppose that $p^k x = mu$, for some $x \in \Delta_p$, and $k, m \in \mathbb{N}$. We have to show that p^k divides m. Suppose not, then $x \notin C_p$. Since the group Δ_p is torsion-free, the equality $p^k x = mu$ can be rewritten after dividing it by an appropriate power of p in the equivalent form $p^l x = nu$, where $1 \le l \le k$ and p does not divide n. By (a) of the lemma, there exists $y \in \Delta_p$ such that $ny = x$, whence $p^l y = u$. Since $C_p = \langle u\rangle$ is dense in the compact group Δ_p, the latter equality implies that $p^l\Delta_p = \Delta_p$, which is a contradiction.

Let us deduce (c). Let $k \in \mathbb{N}$. Since C_p is dense in Δ_p and $p^k\Delta_p$ is open in Δ_p, we have that $\Delta_p = C_p + p^k\Delta_p$. Therefore, $p^k\pi(\Delta_p) = \pi(p^k\Delta_p) = \pi(\Delta_p)$, i.e., $p^k X_p = X_p$ for each $k \in \mathbb{N}$, where $X_p = \Delta_p/C_p$. Further, since $n\Delta_p = \Delta_p$ for each $n \in \mathbb{N}$ which is not a multiple of p, we have that $nX_p = X_p$ for such an integer n. This proves that the group X_p is divisible.

It remains to deduce (d). The groups $\mathbb{Q}^{(\mathfrak{c})} \times \bigoplus_{q\in\mathbb{P}\wedge q\neq p} \mathbb{Z}(q^\infty)$ and $X_p = \Delta_p/C_p$ are isomorphic. Indeed, the group X_p is divisible by (c) of the lemma. We note that the equality $r_0(\Delta_p) = |\Delta_p| = \mathfrak{c}$ follows from the fact that $r_0(H) = |H|$ for each uncountable torsion-free group H. Let $\pi\colon \Delta_p \to \Delta_p/C_p$ be the quotient homomorphism. Since the kernel of π is countable, we infer that $r_0(X_p) = \mathfrak{c}$.

We claim that the group X_p does not contain elements of order p. If $x^* \in X_p$ is distinct from zero and $px^* = 0$, take $x \in \Delta_p$ with $\pi(x) = x^*$. Then $px \in C_p$ and, hence, $px = mu$ for some $m \in \mathbb{Z}$. By item (b) of the lemma this implies that $m = pn$ for $n \in \mathbb{Z}$, whence $px = mu = pnu$ and $x = nu \in C_p$. We conclude that $x^* = \pi(x) = 0$ in X_p, which contradicts our choice of the element x^*. In particular, $r_p(X_p) = 0$.

Let $q \in \mathbb{P}$, $q \neq p$. Let us show that $r_q(X_p) = 1$. By (a) of the lemma, there exists an element $z \in \Delta_p$ such that $qz = u$. Then the elements $z, 2z, \ldots, (q-1)z$ are not in C_p; otherwise, $kz = nu$ for some $k \in \{1, 2, \ldots, q-1\}$ and $n \in \mathbb{Z}$. Hence, $kz = nqz$ and $k = nq$, which is impossible since $1 \le k < q$. Since $qz = u$, the element $z^* = \pi(z) \in X_p$ is distinct from zero and satisfies $qz^* = 0$. So the order of z^* in X_p is equal to q. If $t^* \in X_p$ and $qt^* = 0$, take $t \in \Delta_p$ with $\pi(t) = t^*$. Then $q(z-t) \in C_p$, whence it follows that $q(z-t) = mu$ for some integer m. Since $qz = u$, we infer that $(1-m)u = qt$ or, equivalently, $(1-m)qz = qt$. Therefore, $t = (1-m)z$; i.e., $t \in \langle z\rangle$ and $t^* \in \langle z^*\rangle$. We conclude that all the elements of X_p of order q are in $\langle z^*\rangle$. Thus, $r_q(X_p) = 1$.

Summing up, the divisibility of X_p implies that the group X_p is isomorphic to $\mathbb{Q}^{(\mathfrak{c})} \times \bigoplus_{q\in\mathbb{P},\, q\neq p} \mathbb{Z}(q^\infty)$. □

Our next step is to calculate several algebraic invariants of a group that will appear in the further proofs.

Lemma 5.5. *Let $X_p = \Delta_p/C_p$, where C_p is the cyclic subgroup of Δ_p generated by the element $u = (1,0,0,\ldots)$. Then the group $X = \prod_{p\in\mathbb{P}} X_p$ satisfies $r_0(X) = \mathfrak{c}$ and $r_p(X) = \mathfrak{c}$ for each $p \in \mathbb{P}$.*

Proof. By Lemma 5.4(d) we have that $r_q(X_p) = 1$ for each prime $q \neq p$. In particular, for each $p \in \mathbb{P}$ distinct from q, X_p contains a subgroup isomorphic to the group $\mathbb{Z}/q\mathbb{Z}$. Hence, the group $X = \prod_{p\in\mathbb{P}} X_p$ contains a subgroup isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\omega$. Since $r_q((\mathbb{Z}/q\mathbb{Z})^\omega) = |(\mathbb{Z}/q\mathbb{Z})^\omega| = \mathfrak{c}$, we deduce that $r_q(X) = \mathfrak{c}$ for each $q \in \mathbb{P}$. It is also easy to see that $r_0(X) = |X| = \mathfrak{c}$ since X contains a subgroup isomorphic to X_p, for each prime p, and $r_0(X_p) = \mathfrak{c}$. □

We are now in the position to describe the torsion Bohr compactification of the group $\mathbb{Q}$.

Proposition 5.6. *The group $\mathfrak{b}\mathbb{Q} \cong (\mathbb{Q}^\circledast)^*_{\mathbf{p}} = (\mathbb{Q}\times\Delta_{\mathfrak{a}})^*_{\mathbf{p}}$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}}$ such that $\mathfrak{b}\mathbb{Q}/N \cong \Sigma_{\mathfrak{a}} \times (\mathbb{Z}^\omega)^*_{\mathbf{p}} \cong (\mathbb{Q}\times\mathbb{Z}^\omega)^*_{\mathbf{p}}$. Therefore, there exists a short exact sequence*

$$0 \to \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}} \to \mathfrak{b}\mathbb{Q} \to \Sigma_{\mathfrak{a}} \times (\mathbb{Z}^\omega)^*_{\mathbf{p}} \to 0.$$

Proof. Let C_p and X_p, with $p \in \mathbb{P}$, be as in Lemma 5.4, and let $X = \prod_{p\in\mathbb{P}} X_p$. We know that $(\mathbb{Q}\times\prod_{p\in\mathbb{P}}\Delta_p)/(\mathbb{Q}\times\prod_{p\in\mathbb{P}} C_p)$ is isomorphic to X. By lemmas 5.4(c) and 5.5, the group X is divisible and isomorphic to $\mathbb{Q}^{(\mathfrak{c})} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(\mathfrak{c})}$. Therefore, there exists a short exact sequence

$$0 \to \mathbb{Q} \times \prod_{p\in\mathbb{P}} C_p \to \mathbb{Q}_d^\circledast \to \mathbb{Q}^{(\mathfrak{c})} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(\mathfrak{c})} \to 0.$$

So Lemma 4.5 implies that $\mathfrak{b}\mathbb{Q} \cong (\mathbb{Q}^\circledast)^*_{\mathbf{p}}$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}} \cong X^*_{\mathbf{p}}$ such that $\mathfrak{b}\mathbb{Q}/N \cong \Sigma_{\mathfrak{a}} \times (\mathbb{Z}^\omega)^*_{\mathbf{p}} \cong (\mathbb{Q}\times\mathbb{Z}^\omega)^*_{\mathbf{p}}$, and

$$0 \to \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}} \to \mathfrak{b}\mathbb{Q} \to \Sigma_{\mathfrak{a}} \times (\mathbb{Z}^\omega)^*_{\mathbf{p}} \to 0. \qquad \square$$

Lemma 5.7. *Let $G = \mathbb{Q}^{(\kappa)} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(\lambda_p)}$, where κ and λ_p with $p \in \mathbb{P}$ are cardinal numbers. Then $G^*_{\mathbf{p}} \cong \Sigma_{\mathfrak{a}}^\kappa \times \prod_{p\in\mathbb{P}} \Delta_p^{\lambda_p}$ and $G^\circledast \cong \mathbb{Q}^\kappa \times \prod_{p\in\mathbb{P}} \Delta_p^{\kappa+\lambda_p}$.*

Proof. By [8, Theorem 23.22, Theorem 25.2, and Theorem 24.8] we have that

$$\Big(\bigoplus_{p\in\mathbb{P}}\mathbb{Z}(p^\infty)^{(\lambda_p)}\Big)^*_{\mathbf{p}} \cong \prod_{p\in\mathbb{P}}((\mathbb{Z}(p^\infty))^*_{\mathbf{p}})^{\lambda_p} \cong \prod_{p\in\mathbb{P}}\Delta_p^{\lambda_p}.$$

[8, Theorem 23.22 and Theorem 25.4] imply that $(\mathbb{Q}^{(\kappa)})^*_{\mathbf{p}} \cong (\mathbb{Q}^*_{\mathbf{p}})^\kappa \cong \Sigma_{\mathfrak{a}}^\kappa$. Hence, $G^*_{\mathbf{p}} \cong \Sigma_{\mathfrak{a}}^\kappa \times \prod_{p\in\mathbb{P}}\Delta_p^{\lambda_p}$. Applying [8, Theorem 23.22 and Theorem 25.4], we see that the homomorphisms of $\mathbb{Q}^{(\kappa)}$ to $tor(\mathbb{T}) \subset \mathbb{T}$ correspond to the subgroup $(\mathbb{Q}\times\prod_{p\in\mathbb{P}}\Delta_p)^\kappa$ of $\Sigma_{\mathfrak{a}}^\kappa$. Therefore, $G^{\circledast} \cong (\mathbb{Q}\times\prod_{p\in\mathbb{P}}\Delta_p)^\kappa \times \prod_{p\in\mathbb{P}}\Delta_p^{\lambda_p} \cong \mathbb{Q}^\kappa \times \prod_{p\in\mathbb{P}}\Delta_p^{\kappa+\lambda_p}$. □

The following lemma will be used in the proof of Proposition 5.9.

Lemma 5.8. *If* $G = \mathbb{Q}^{(\kappa)} \oplus \bigoplus_{p\in\mathbb{P}}\mathbb{Z}(p^\infty)^{(\lambda_p)}$, *then*

$$\mathfrak{b}G \cong \Sigma_{\mathfrak{a}}^{2^\kappa} \times \Big(\prod_{p\in\mathbb{P}}\Delta_p^{\kappa+\lambda_p}\Big)^*_{\mathbf{p}} \ \textit{and}\ bG \cong \mathfrak{b}G \oplus \hom(G,\mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}}.$$

Proof. Corollary 2.2 and Lemma 5.7 imply that

$$\mathfrak{b}G \cong (\mathbb{Q}^\kappa)^*_{\mathbf{p}} \times (\prod_{p\in\mathbb{P}}\Delta_p^{\kappa+\lambda_p})^*_{\mathbf{p}}$$

and by Proposition 2.7, $bG \cong \mathfrak{b}G \oplus \hom(G,\mathbb{Q}^{(\mathfrak{c})})^*_{\mathbf{p}}$. We know that $\mathbb{Q}^\kappa$ is a linear space over $\mathbb{Q}$ which has dimension 2^κ. Hence, its Pontryagin dual group $(\mathbb{Q}^\kappa)^*_{\mathbf{p}}$ is the compact group $\Sigma_{\mathfrak{a}}^{2^\kappa} \cong (\mathbb{Q}^*_{\mathbf{p}})^{2^\kappa}$. □

Proposition 5.9. *Let* $G = \mathbb{Q}^{(\lambda)}$, *where* $\lambda \geq \omega$. *Then* $\mathfrak{b}G$ *contains a closed subgroup* $N \cong \Sigma_{\mathfrak{a}}^{2^\lambda} \times \Delta_{\mathfrak{a}}^{2^\lambda}$ *such that*

$$\mathfrak{b}G/N \cong \Sigma_{\mathfrak{a}}^{2^\lambda} \times \Big(\mathbb{Z}^\lambda\Big)^*_{\mathbf{p}} \cong \Big(\mathbb{Q}^\lambda \times \mathbb{Z}^\lambda\Big)^*_{\mathbf{p}}.$$

Therefore, there is a short exact sequence

$$0 \to \Sigma_{\mathfrak{a}}^{2^\lambda} \times \Delta_{\mathfrak{a}}^{2^\lambda} \to \mathfrak{b}G \to \Sigma_{\mathfrak{a}}^{2^\kappa} \times \big(\mathbb{Z}^\lambda\big)^*_{\mathbf{p}} \to 0.$$

Proof. We use notation of Lemma 5.4. Evidently, the quotient group

$$\Big(\mathbb{Q}\times\prod_{p\in\mathbb{P}}\Delta_p\Big)^\lambda / \Big(\mathbb{Q}\times\prod_{p\in\mathbb{P}}C_p\Big)^\lambda$$

is isomorphic to $\prod_{p\in\mathbb{P}}X_p^\lambda = X^\lambda$, where $X_p = \Delta_p/C_p$ and $X = \prod_{p\in\mathbb{P}}X_p$. Note that X^λ is divisible and isomorphic to $\mathbb{Q}^{(r_0(X^\lambda))} \oplus \bigoplus_{p\in\mathbb{P}}\mathbb{Z}(p^\infty)^{(r_p(X^\lambda))}$. Since $C_p \cong \mathbb{Z}$ for each $p\in\mathbb{P}$, Lemma 5.8 implies that $\mathfrak{b}G \cong \Sigma_{\mathfrak{a}}^{2^\lambda} \times$

$((\prod_{p\in\mathbb{P}}\Delta_p^{\lambda})_d)^*_{\mathbf{p}}$ and by Lemma 5.8 and Lemma 4.5, the group $\mathfrak{b}G$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{r_0(X^\lambda)} \times \prod_{p\in\mathbb{P}}\Delta_p^{r_p(X^\lambda)} \cong (X^\lambda)^*_{\mathbf{p}}$ such that

$$\mathfrak{b}G/N \cong \Sigma_{\mathfrak{a}}^{2^\lambda} \times \left(\mathbb{Z}^\lambda\right)^*_{\mathbf{p}} \cong \left(\mathbb{Q}^\lambda\right)^*_{\mathbf{p}} \times \left(\left(\prod_{p\in\mathbb{P}} C_p\right)^\lambda\right)^*_{\mathbf{p}}.$$

Since $\mathbb{Q}^\lambda$ is a linear space over $\mathbb{Q}$ which has dimension 2^λ, its Pontryagin dual $(\mathbb{Q}^\lambda)^*_{\mathbf{p}}$ is the compact group $\Sigma_{\mathfrak{a}}^{2^\lambda} \cong (\mathbb{Q}^*_{\mathbf{p}})^{2^\lambda}$. The group $\mathbb{Z}^\lambda$ is not free, but it contains a subgroup F isomorphic to a free abelian group of rank $r_0(F) = |F| = |\mathbb{Z}^\lambda| = 2^\lambda$. Hence, F is a subgroup of the minimal divisible extension $D(\mathbb{Z}^\lambda)$ of $\mathbb{Z}^\lambda$. The linear space $D(\mathbb{Z}^\lambda)$ over $\mathbb{Q}$ has dimension 2^λ, so its Pontryagin dual $(D(\mathbb{Z}^\lambda))^*_{\mathbf{p}}$ is $\Sigma_{\mathfrak{a}}^{2^\lambda} \cong (\mathbb{Q}^*_{\mathbf{p}})^{2^\lambda}$. Therefore, the dual $(\mathbb{Z}^\lambda)^*_{\mathbf{p}}$ is in the sandwich

$$\Sigma_{\mathfrak{a}}^{2^\lambda} \to \left(\mathbb{Z}^\lambda\right)^*_{\mathbf{p}} \to \mathbb{T}^{2^\lambda}.$$

Finally, since $r_0(X) = \mathfrak{c}$ and $r_p(X) = \mathfrak{c}$ for each $p \in \mathbb{P}$ (see Lemma 5.5), we see that $r_0(X^\lambda) = 2^\lambda$ and $r_p(X^\lambda) = \mathfrak{c}^\lambda = 2^\lambda$ for each $p \in \mathbb{P}$. □

Since the groups $\mathbb{R}$ and $\mathbb{Q}^{(\mathfrak{c})}$ are algebraically isomorphic, the next result follows from Proposition 5.9 if we take $\lambda = \mathfrak{c}$.

Corollary 5.10. *The group $\mathfrak{b}\mathbb{R} \cong ((\mathbb{Q}^{\circledast})^{\mathfrak{c}})^*_{\mathbf{p}}$ with $\mathbb{Q}^{\circledast} \cong \mathbb{Q} \times \Delta_{\mathfrak{a}}$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{2^\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{2^\mathfrak{c}}$ such that $\mathfrak{b}\mathbb{R}/N \cong \Sigma_{\mathfrak{a}}^{2^\mathfrak{c}} \times (\mathbb{Z}^\mathfrak{c})^*_{\mathbf{p}} \cong (\mathbb{Q}^\mathfrak{c} \times \mathbb{Z}^\mathfrak{c})^*_{\mathbf{p}}$. Hence, there is a short exact sequence*

$$0 \to \Sigma_{\mathfrak{a}}^{2^\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{2^\mathfrak{c}} \to \mathfrak{b}\mathbb{R} \to \Sigma_{\mathfrak{a}}^{2^\mathfrak{c}} \times \left(\mathbb{Z}^\mathfrak{c}\right)^*_{\mathbf{p}} \to 0.$$

Proposition 5.11. *Let $G = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(\lambda_p)}$, where λ_p is an arbitrary cardinal number for each $p \in \mathbb{P}$. Let also $\kappa = \sum_{p\in\mathbb{P}} \lambda_p$, $\kappa_p = \sum_{q\in\mathbb{P},\, q\neq p} \lambda_q$, and $\mu_p = \kappa_p$ if $\kappa_p < \omega$ or $\mu_p = 2^{\kappa_p}$ if $\kappa_p \geq \omega$, for each $p \in \mathbb{P}$. If $|G| \geq \omega$, then $\mathfrak{b}G$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{2^{\omega\cdot\kappa}} \times \prod_{p\in\mathbb{P}} \Delta_p^{\mu_p}$ such that*

$$\mathfrak{b}G/N \cong \left(\mathbb{Z}^\kappa\right)^*_{\mathbf{p}}.$$

Therefore, there is an exact sequence

$$0 \to \Sigma_{\mathfrak{a}}^{2^{\omega\cdot\kappa}} \times \prod_{p\in\mathbb{P}} \Delta_p^{\mu_p} \to \mathfrak{b}G \to \left(\mathbb{Z}^\kappa\right)^*_{\mathbf{p}} \to 0.$$

Proof. Once again we use notation of Lemma 5.4. Evidently, the quotient group

$$\prod_{p\in\mathbb{P}} \Delta_p^{\lambda_p} / \prod_{p\in\mathbb{P}} C_p^{\lambda_p}$$

is isomorphic to $\prod_{p\in\mathbb{P}} X_p^{\lambda_p}$. Notice that the group $\prod_{p\in\mathbb{P}} X_p^{\lambda_p}$ is divisible and isomorphic to $\mathbb{Q}^{(r_0(\prod_{p\in\mathbb{P}} X_p^{\lambda_p}))} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(\prod_{p\in\mathbb{P}} X_p^{\lambda_p}))}$. Since

$C_p \cong \mathbb{Z}$ for each $p \in \mathbb{P}$, Lemma 5.8 and Lemma 4.5 imply that $\mathfrak{b}G \cong (\prod_{p\in\mathbb{P}} \Delta_p^{\lambda_p})^*_{\mathbf{p}}$ contains a closed subgroup

$$N \cong \Sigma_{\mathfrak{a}}^{r_0(\prod_{p\in\mathbb{P}} X_p^{\lambda_p})} \times \prod_{p\in\mathbb{P}} \Delta_p^{r_p(\prod_{p\in\mathbb{P}} X_p^{\lambda_p})} \cong (\prod_{p\in\mathbb{P}} X_p^{\lambda_p})^*_{\mathbf{p}}$$

such that

$$\mathfrak{b}G/N \cong \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}^{\lambda_p}\Big)^*_{\mathbf{p}} \cong \Big(\mathbb{Z}^{\kappa}\Big)^*_{\mathbf{p}}.$$

If $\kappa = \sum_{p\in\mathbb{P}} \lambda_p$ is finite, then [8, Theorem 23.27(b)] implies that $(\mathbb{Z}^\kappa)^*_{\mathbf{p}} \cong (\mathbb{Z}^*_{\mathbf{p}})^\kappa \cong \mathbb{T}^\kappa$. Otherwise, we argue as in the proof of Proposition 5.9. Finally, since $r_0(X_p) = \mathfrak{c}$ and $r_p(X_p) = 0$ for each $p \in \mathbb{P}$, and $r_q(X_p) = 1$ if $q \neq p$ (we apply Lemma 5.4(d) here), we have that $r_0(\prod_{p\in\mathbb{P}} X_p^{\lambda_p}) = \prod_{p\in\mathbb{P}} \mathfrak{c}^{\lambda_p} = \mathfrak{c}^\kappa = 2^{\omega\cdot\kappa}$ and $r_p(\prod_{q\in\mathbb{P}} X_q^{\lambda_q}) = r_p(\prod_{q\in\mathbb{P},\, q\neq p} \mathbb{Z}_p^{\lambda_p})$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Hence, $r_p(\prod_{q\in\mathbb{P}} X_q^{\lambda_q})$ is equal to κ_p if κ_p is finite and is 2^{κ_p} otherwise. □

Taking $\lambda_p = 1$ for a given $p \in \mathbb{P}$ and $\lambda_q = 0$ for each $q \neq p$ in Proposition 5.11, we obtain the following corollary.

Corollary 5.12. *The compact group $K_p = \mathfrak{b}\mathbb{Z}(p^\infty) \cong (\Delta_p)^*_{\mathbf{p}}$, with a prime p, contains a closed subgroup $N_p \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \prod_{q\in\mathbb{P},\, q\neq p} \Delta_q \cong (X_p)^*_{\mathbf{p}}$ such that $K_p/N_p \cong \mathbb{T}$, where the group $X_p = \Delta_p/C_p$ is defined in Lemma 5.4. In other words, there is a short exact sequence*

$$0 \to \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \prod_{q\in\mathbb{P},\, q\neq p} \Delta_q \to \mathfrak{b}\mathbb{Z}(p^\infty) \to \mathbb{T} \to 0.$$

Taking $\lambda_p = 1$ for each $p \in \mathbb{P}$ in Proposition 5.11, we get the following result as a corollary.

Corollary 5.13. *Let $K = \mathfrak{b}tor(\mathbb{T}) \cong (\Delta_{\mathfrak{a}})^*_{\mathbf{p}}$. Then K contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}}$ such that $K/N \cong (\mathbb{Z}^\omega)^*_{\mathbf{p}}$. Hence, there is a short exact sequence*

$$0 \to \Sigma_{\mathfrak{a}}^{\mathfrak{c}} \times \Delta_{\mathfrak{a}}^{\mathfrak{c}} \to \mathfrak{b}tor(\mathbb{T}) \to (\mathbb{Z}^\omega)^*_{\mathbf{p}} \to 0.$$

Combining propositions 5.9 and 5.11, we obtain the following theorem.

Theorem 5.14. *Let $G = \mathbb{Q}^{(\lambda)} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(\lambda_p)}$, where $\lambda \geq \omega$ and λ_p is an arbitrary cardinal number for each $p \in \mathbb{P}$. Then $\mathfrak{b}G$ contains a closed subgroup $N \cong \Sigma_{\mathfrak{a}}^{2^{\lambda+\kappa}} \times \prod_{p\in\mathbb{P}} \Delta_p^{2^{\lambda+\kappa_p}}$ such that*

$$\mathfrak{b}G/N \cong \Sigma_{\mathfrak{a}}^{2^\lambda} \times \Big(\mathbb{Z}^{\lambda+\kappa}\Big)^*_{\mathbf{p}},$$

where $\kappa = \sum_{p\in\mathbb{P}} \lambda_p$ *and* $\kappa_p = \sum_{q\in\mathbb{P},\, q\neq p} \lambda_q$. *Therefore, there is a short exact sequence*

$$0 \to \Sigma_{\mathfrak{a}}^{2^{\lambda+\kappa}} \times \prod_{p\in\mathbb{P}} \Delta_p^{2^{\lambda+\kappa_p}} \to \mathfrak{b}G \to \Sigma_{\mathfrak{a}}^{2^{\lambda}} \times \left(\mathbb{Z}^{\lambda+\kappa}\right)^*_{\mathbf{p}} \to 0.$$

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References

[1] Alexander Arhangel'skii and Mikhail Tkachenko, *Topological Groups and Related Structures.* Atlantis Studies in Mathematics, 1. Paris: Atlantis Press; Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd., 2008.

[2] Omar Becerra-Muratalla, *The torsion Bohr topology for Abelian groups.* Topology Appl. **163** (2014), 25–38.

[3] W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters*, Fund. Math. **55** (1964), 283–291.

[4] Dikran N. Dikranjan, Ivan R. Prodanov, and Luchezar N. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies.* Monographs and Textbooks in Pure and Applied Mathematics, 130. New York, Marcel Dekker, Inc., 1990.

[5] Dikran Dikranjan and Stephen Watson, *A solution to van Douwen's problem on Bohr topologies*, J. Pure Appl. Algebra **163** (2001), no. 2, 147–158.

[6] László Fuchs, *Infinite Abelian Groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. New York-London: Academic Press, 1970.

[7] Jorge Galindo and Salvador Hernández, *Interpolation sets and the Bohr topology of locally compact groups*, Adv. Math. **188** (2004), no. 1, 51–68.

[8] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis. Vol. I: Structure of Topological Groups, Integration Theory, Group Representations.* 2nd ed. Grundlehren der Mathematischen Wissenschaften, 115. Berlin-New York: Springer-Verlag, 1979.

[9] Per Holm, *On the Bohr compactification*, Math. Ann. **156** (1964), 34–46.

[10] Irving Kaplansky, *Infinite Abelian Groups.* Rev. ed. Ann Arbor, Mich.: The University of Michigan Press, 1969.

[11] Kenneth Kunen and Walter Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 1, 117–137.

[12] Horst Leptin, *Abelsche gruppen mit kompakten charaktergruppen und dualitätstheorie gewisser linear topologischer abelscher gruppen*, Abh. Math. Sem. Univ. Hamburg **19** (1955), 244–263.

[13] Derek John Scott Robinson, *A Course in the Theory of Groups.* Graduate Texts in Mathematics, 80. New York-Berlin: Springer-Verlag, 1982.

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