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UNIFORM BOX PRODUCTS II

JOCELYN R. BELL AND SCOTT W. WILLIAMS

Abstract. The uniform box topology is defined on the product set of infinitely many copies of a completely regular space. It is finer than the Tychonoff topology but coarser than the box topology. Here we study connectedness and separation properties.

1. Introduction

The uniform box product, introduced by Scott W. Williams in 2001, is a generalization of the sup metric topology on the product of metric spaces to powers of uniform spaces. Its topology is finer than the Tychonoff but coarser than the box topology. Many questions open for box products [9] are open for uniform box products as well.

On a compact space, there is a unique uniformity which generates the topology. However, when a uniform space is not compact, there may be many uniformities which generate the topology. Even if two uniformities generate the same topology on a space, their respective uniform box products may differ. For example, in section 4 we show that the uniform box product of copies of the real line is not connected when we use its canonical uniformity, yet it is connected when we use the uniformity of a homeomorph. The situation is somewhat less complex if we restrict our attention to uniformities induced by a compactification (section 3) and which are totally bounded. One consequence of our approach is that, by restricting to totally bounded uniformities, the uniform box product of a connected uniform space is connected.

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A well-known and difficult problem is determining whether a box product is normal or paracompact, and many results rely on additional set-theoretic axioms [12]. However, there are ZFC pseudonormal results. The box product of compact spaces, no matter how many, is pseudonormal [2]. A box product of countably many spaces is pseudonormal if the spaces are σ-compact [13]. Are similar results true for uniform box products? This paper details partial positive results.

The countable uniform box product of a σ-compact, locally compact space K is pseudonormal if every countable subset of K has countable closure (section 5). From this it follows that the countable uniform box product of a σ-compact ordinal space is pseudonormal. In section 6 we extend to all ordinal spaces. This is of particular interest, since it is not known whether the uniform box product of $[0, \omega_1]$ is normal [8].

2. Preliminaries

This section contains background material on uniformities. All definitions may be found in [3].

Suppose X is a set and $D \subseteq X \times X$. We will use the following notation throughout.

(1) The diagonal is the set $\triangle = \{(x,x) : x \in X\}$.
(2) $D^{-1} = \{(y,x) : (x,y) \in D\}$.
(3) $D \circ D = \{(x,z) : \exists\, y \in X \text{ such that } (x,y) \in D \text{ and } (y,z) \in D\}$.
(4) For $A \subseteq X$ $D[A] = \{y : (x,y) \in D \text{ for some } x \in A\}$.

The *D-ball* about $x \in X$ is $D(\{x\}) = D(x) = \{y : (x,y) \in D\}$. D is an *entourage of the diagonal* if $\triangle \subseteq D$.

Definition 2.1. A *uniformity* $\mathbb{D}$ on a set X is a collection of entourages of the diagonal such that

(1) if $E \in \mathbb{D}$ and $E \subseteq D$, then $D \in \mathbb{D}$;
(2) if $D_1, D_2 \in \mathbb{D}$, then $D_1 \cap D_2 \in \mathbb{D}$;
(3) for every $E \in \mathbb{D}$, there is $D \in \mathbb{D}$ such that $D^{-1} \subseteq E$;
(4) for every $E \in \mathbb{D}$, there is $D \in \mathbb{D}$ such that $D \circ D \subseteq E$;
(5) $\bigcap \mathbb{D} = \triangle$.

This last condition ensures the topology generated by $\mathbb{D}$ is Hausdorff and some authors do not require this condition [4]. A collection of entourages of the diagonal satisfying conditions (3) and (4) of Definition 2.1 is a *uniformity subbase*.

A *uniform space* is a pair $(X, \mathbb{D})$, a set X together with a uniformity $\mathbb{D}$ on X. If $\mathbb{D}$ is a uniformity base, we will refer to $(X, \mathbb{D})$ as a uniform space as well. The *uniform topology* on X generated by $\mathbb{D}$ is as follows: A set $G \subseteq X$ is open if, for each $x \in X$, there is $D \in \mathbb{D}$ with $D(x) \subseteq$

G. If X is a topological space, then the uniform topology is *compatible* with the topology on X if the uniformity generates the topology on X. The topological spaces which admit a compatible uniform topology are precisely the completely regular spaces.

The following notion will be used frequently.

Definition 2.2. A uniformity $\mathbb{D}$ on a set X is *totally bounded* if, for every $D \in \mathbb{D}$, there is a finite open cover $\mathcal{C}$ of X such that for each $A \in \mathcal{C}$ we have $A \times A \subseteq D$, or, equivalently, for each $D \in \mathbb{D}$ there is a finite set $F \subseteq X$ such that $D[F] = X$.

Every compact space has exactly one uniformity generating its topology and that uniformity is totally bounded. Every compactification of a space generates a compatible totally bounded uniformity [3]. Explicitly, let $X \subseteq Y$ where Y is a compactification of X and $\mathbb{U}$ is the (unique, totally bounded) uniformity on Y compatible with the topology on Y. We may form the subspace uniformity in the obvious way: Let $\mathbb{U}_X = \{U \cap (X \times X) : U \in \mathbb{U}\}$. Then $(X, \mathbb{U}_X)$ is a uniform space. Each finite open cover of Y defines a finite open cover of X. Let $F = \{A_1, \ldots, A_n\}$ be such a finite open cover of Y. Define the *entourage determined by F*

$$D_F = \bigcup_{i \leq n} (A_i \times A_i)\,.$$

Then $\{D_F : F \text{ is a finite open cover of } Y\}$ is a base for the uniformity $\mathbb{U}$ on Y and so also for the uniformity $\mathbb{U}_X$ on X. Then D-ball about x is

$$D_F(x) = \bigcup_{x \in A_i} A_i.$$

The collection of all compatible totally bounded uniformities on a space form a partial order. However, this partial order need not have a minimal element; in fact, a minimal element exists if and only if the space is locally compact [11]. When it exists, this coarsest totally bounded uniformity corresponds to the Alexandroff, or one point, compactification of the space.

3. Uniform Box Product

In this section we consider uniform box products of locally compact spaces. The general definition may also be found in [1].

Let $(X, \mathbb{D})$ be a uniform space. For any index set I, let $\prod^I X$ denote the product set. For each $D \in \mathbb{D}$, define $\bar{D} \subseteq \prod^I X \times \prod^I X$ as

$$\bar{D} = \{(x, y) : \text{ for all } \lambda \in I \ (x(\lambda), y(\lambda)) \in D\}\,.$$

Then $\bar{\mathbb{D}} = \left\{\bar{D} : D \in \mathbb{D}\right\}$ is a uniformity on $\prod^I X$. The topology generated on $\prod^I X$ by $\bar{\mathbb{D}}$ is the $\mathbb{D}$-*uniform box topology*, and $\prod^I X$ with this topology is the $\mathbb{D}$-*uniform box product*. The uniform box topology is finer than the Tychonoff topology on $\prod^I X$ but coarser than the box topology. If $\mathbb{D}$ is clear from context, we refer to $\prod^I X$ as the uniform box product. If $I = \omega$, we say $\prod^\omega X$ is a *countable uniform box product*. If $\mathbb{D}$ has a countable base, then X is metrizable, and the uniform box product is the sup metric product, where the metric is induced by $\mathbb{D}$.

If $\mathbb{D}$ is a uniformity base, rather than a uniformity, then $\bar{\mathbb{D}}$ is also uniformity base. It is easy to see that if uniformity bases $\mathbb{D}$ and $\mathbb{E}$ generate the same uniformity on a set X, then $\bar{\mathbb{D}}$ and $\bar{\mathbb{E}}$ generate the same uniformity on $\prod^I X$.

For metric spaces there may be several metrics which generate the topology. Similarly, for a uniform space there may be several uniformities compatible with its topology. In section 4, we show this may lead to uniform box products of the same space with different properties. These examples motivate the following restriction: For locally compact spaces, we consider only the coarsest compatible totally bounded uniformity.

Definition 3.1. The $\mathbb{U}$-uniform box product of a locally compact space, where $\mathbb{U}$ is the coarsest compatible totally bounded uniformity, is the *uniform box product* of a locally compact space.

A more explicit description is as follows. Let X be a locally compact space and let $Y = X \cup \{\infty\}$ be its one-point compactification. Let $C \subseteq X$ be any compact set and let $\{A_1, \ldots, A_k\}$ be any finite open cover of C. Then $F = X \backslash C \cup \bigcup_{i=1}^k A_i$ is a finite open cover of X. It can be shown that the entourages determined by such sets F form a basis for the coarsest uniformity on X compatible with the topology. Its completion is Y.

If Y is also zero dimensional, we can require $P = \{A_1, \ldots, A_k\}$ to be a clopen (both closed and open) partition of Y. In this case, $D_P(x)$ is simply the unique element of P which contains x. If H_1 and H_2 are collections of sets, we say H_2 *refines* H_1 provided, for all $A \in H_2$, there is $B \in H_1$ such that A is contained in B. Notice that $\left\{\bar{D}_P(x) : x \in \prod^I X\right\}$ is a clopen partition of $\prod^I X$, and if P_1 and P_2 are partitions of X such that P_2 refines P_1, then $\left\{\bar{D}_{P_2}(x) : x \in \prod^I X\right\}$ refines $\left\{\bar{D}_{P_1}(x) : x \in \prod^I X\right\}$.

4. Connectedness in Uniform Box Products

For a connected uniform space X, its uniform box product will also be connected provided the uniformity is totally bounded.

Theorem 4.1. *Suppose $\mathbb{D}$ is a totally bounded uniformity compatible with the topology on a connected space X. Then, for any index set I, the uniform box product $\prod^I X$ is connected.*

Proof. For each finite partition $P = \{A_1, A_2, \ldots, A_n\}$ of the index set I, we let

$$\prod\nolimits_P = \left\{x \in \prod\nolimits^I X : \forall\, i \le n \;\forall\, j_1, j_2 \in A_i \;\; x(j_1) = x(j_2)\right\}$$

that is, the set of all points whose restriction to an element of the partition is constant. As P is finite, $\prod_P$ is homeomorphic to the finite product $X^{|P|}$ of connected spaces, and hence is also connected [3]. For each $y \in \prod_{\{I\}}$ (a constant point) and for every partition P, $y \in \prod_P$. Thus, $Z = \bigcup\{\prod_P : P \text{ is a finite partition of } I\}$ is connected in $\prod^I X$.

For each $D \in \mathbb{D}$, choose a finite set $F_D \subseteq X$ such that $D[F_D] = X$. Then for any $x \in \prod^I X$ and for any $i \in I$, we have $F_D \cap D(x(i)) \ne \emptyset$. Hence, $\prod^I F_D \cap D(x) \ne \emptyset$. Thus, there is a finite partition P of I such that some element of $\prod^I F_D \cap D(x)$ is constant on each member of P. So $\prod_P \cap D(x) \ne \emptyset$. Therefore, Z is dense in the uniform box product $\prod^I X$. As Z is connected, $\prod^I X$ is connected. □

Example 4.2. Note that $\prod^\omega \mathbb{R}$ with the canonical uniformity derived from the Euclidean metric on $\mathbb{R}$ is not connected. To see this, let $C \subseteq \prod^\omega \mathbb{R}$

$$C = \{x : \exists\, a, b \in \mathbb{R} \text{ such that } \forall\, n, x(n) \in [a, b]\}.$$

Then C and its complement are both open in $\prod^\omega \mathbb{R}$. However, the canonical uniformity on the open unit interval $(0, 1)$, a homeomorph of $\mathbb{R}$, is totally bounded. This illustrates the strength of total boundedness.

However, the assumption of total boundedness is not necessary.

Example 4.3 (The metric hedgehog with uncountably many spines)**.** Let I be an uncountable set and for all $i \in I$, set

$$X(i) = \left\{x \in \prod\nolimits^I [0, 1] : \forall\, j \ne i \;\; x(j) = 0\right\}.$$

Let $X = \bigcup_{i \in I} X(i)$. On X we define a metric d by $d(x, y) = |x(i) - y(i)|$ if there exists $i \in I$ such that $x, y \in X(i)$. Otherwise, find $j \ne i$ such that $x \in X(i)$ and $y \in X(j)$, and define $d(x, y) = x(i) + y(j)$. Each $(X(i), d)$ is homeomorphic to $[0, 1]$ and contains the constant 0 function 0; thus, each $X(i)$ and hence, the metric space (X, d) is connected. As X is not separable, it is not totally bounded. Therefore, the uniformity D induced by d is not totally bounded.

Now consider the uniform box product $\prod^{\omega} X$. For all $f : \omega \to I$, the uniform space $\left(\prod_{i\in I} X(f(i)), D\right)$ is homeomorphic to $\prod^{\omega}[0,1]$ with the sup-metric, and is therefore connected by Theorem 4.1 above. Since the constant 0 function belongs to each $\prod_{i\in I} X(f(i))$, $\prod^{\omega} X = \bigcup_{f\in^{\omega} I} X(f(i))$ is connected.

5. Pseudonormality in Uniform Box Products

As it is a uniform space, a uniform box product is completely regular. There are uniform spaces whose uniform box product is normal and countably paracompact [1] but not paracompact [7].

In this section we show that the countable uniform box product of a certain class of σ-compact, locally compact spaces is pseudonormal. Recall that a topological space is *pseudonormal* provided that, for any two disjoint closed sets, if one of them is countable, then they can be separated by disjoint open sets.

5.1. Scattered spaces.

A topological space is *scattered* if every subset has an isolated point. Assuming CH, the box product of compact scattered spaces is paracompact [5]. A *Fort space* (the one-point compactification of an uncountable discrete space) is an example of a compact scattered space whose countable uniform box product is normal [1]. However, it is unknown whether the countable uniform box product of a compact scattered space is necessarily normal.

For any space X, the Cantor-Bendixson sequence is formed inductively as follows: $X^0 = X$, $X^{\alpha+1}$ is the set of all non-isolated points of X^{α}, and if α is a limit ordinal, $X^{\alpha} = \bigcap_{\gamma<\alpha} X^{\gamma}$. A space is scattered if and only if there exists an ordinal α such that $X^{\alpha} = \emptyset$; when it exists, the smallest such α is the *rank* of X. If a scattered space is compact, then its rank is a successor ordinal $\beta + 1$ and X^{β} is finite. Compact scattered spaces are zero dimensional.

For an element x of a scattered space X, let $rank(x)$ be the ordinal α such that $x \in X^{\alpha} \setminus X^{\alpha+1}$, and for $A \subseteq X$, we let $rank(A)$ be the least ordinal α such that $A \subseteq X^{\alpha}$.

We will need the following facts, which we show here for completeness.

Fact 5.1. *A locally compact scattered Hausdorff space is zero dimensional.*

Proof. Let X be a locally compact scattered space and fix $x \in X$. Let $U \subseteq X$ be a compact set containing x. Since U is closed and every compact scattered space is zero-dimensional, there is a clopen set $V \subseteq U$ with $x \in V$. $\square$

Fact 5.2. *Suppose X is a compact scattered space. Then X is sequentially compact.*

Proof. Suppose $\{x_1, x_2, \ldots\}$ is a sequence of points in X. Let Y be the set of all limit points of $\{x_1, x_2, \ldots\}$. Then Y contains an isolated point x. So there is a neighborhood U of x such that $U \cap Y = \{x\}$, and U contains infinitely many points of $\{x_1, x_2, \ldots\}$. Thus, there is a convergent subsequence. □

5.2. **Countable-closure spaces.**

We will need a special topological property which we define here.

Definition 5.3. A Hausdorff space X is a *countable-closure space* if for every countable subset $A \subseteq X$, the closure of A is countable.

The following is folklore.

Lemma 5.4. *If X is a locally compact Hausdorff space such that the closure of every countable set has cardinality less than the continuum, then X is scattered.*

Proof. Suppose X is locally compact Hausdorff and not scattered. Since X is not scattered, there is an infinite closed subspace A with no isolated points. Fix $x \in A$ and let U be a compact, regular-closed neighborhood of x. Then $A \cap U$ has no isolated points, and so any infinite subset of $A \cap U$ has closure of cardinality at least continuum. □

Thus, by Lemma 5.4, a locally compact countable-closure space is scattered. Not every countable-closure space is scattered; in a private communication, Peter Nyikos provided an example of an uncountable Lindelöf countable-closure space with no isolated points. Furthermore, not every compact scattered space is a countable-closure space. The following example may be found in [10].

Example 5.5. A compact scattered Hausdorff space which is not a countable-closure space: the Cantor Tree. Take the one point compactification of all sequences of zeros and ones of length $\leq \omega$ topologized as follows: Every finite sequence is isolated and a neighborhood of an infinite sequence α contains α as well as all but finitely many of its initial segments. Then let C be all the isolated points; there are countably many, but the closure of C is the entire space, which has continuum many points.

The next lemma is obvious: The countable-closure property is finitely productive.

Lemma 5.6. *If $X_1, \ldots, X_n$ are countable-closure spaces, then so is $\prod_{i=1}^n X_i$.*

The following partial converse to Lemma 5.4 is due to Gadi Moran [6].

Theorem 5.7 ([6])**.** *Let K be a non-empty compact linearly ordered space. Then K is scattered if and only if for every $A \subseteq K$, $|cl(A)| = |A|$.*

In particular, it follows that every compact ordinal space is a countable-closure space. Since countable-closure is a hereditary property, any ordinal space is a countable-closure space. Finite products of ordinal spaces are also countable-closure spaces. There are compact countable-closure spaces which are not linearly ordered; a Fort space is one such example.

5.3. Convergent sequences.

Let K be a σ-compact, locally compact countable-closure space and let $\prod^{\omega} K$ be the countable uniform box product of K. We will need some conditions which guarantee that a sequence in $\prod^{\omega} K$ converges. First we fix some notation. We may write $K = \bigcup_{n \in \mathbb{N}} X_n$ where each X_n is compact clopen and $X_n \subseteq X_{n+1}$ for all n. We assume this factorization of K is fixed. If K is itself compact, then we consider $X_n = K$ for all $n \in \mathbb{N}$ in all definitions in this section.

Basic clopen sets in K are defined as follows.

Definition 5.8. Suppose K is a σ-compact, locally compact scattered space, where $K = \bigcup_{n \in \mathbb{N}} X_n$ and each X_n is compact clopen and $X_n \subseteq X_{n+1}$. A clopen set $A \subseteq K$ is *basic* provided

(1) if there is a point in A of maximal rank, then
   (a) if $rank\,(A) = \alpha$, then there is a unique point $z \in A$ with $rank\,(z) = \alpha$ and
   (b) if n is least such that $z \in X_n$, then $A \subseteq X_n$;
(2) if A has no points of maximal rank, then $A = K \setminus X_n$ for some integer n.

A finite partition consisting of basic sets will be called a basic partition.

Definition 5.9. A finite partition $P = \{A_1, A_2, \ldots, A_n\}$ of K is *basic* if P is a finite set and each A_i is a basic clopen set.

We use the basic partitions to generate a uniformity base. If $P = \{A_1, A_2, \ldots A_n\}$ is a basic partition of K, denote the corresponding element of $K \times K$ by D_P. That is,

$$D_P = \bigcup_{i \leq n} (A_i \times A_i)\,.$$

Then for $x \in K$, $D_P\,(x) = A_i$ where $x \in A_i$ (i.e, $D_P\,(x)$ chooses which piece of P contains x). $\{D_P : P \text{ is a basic partition}\}$ is a basis for the coarsest totally bounded uniformity on K.

For any set $A \subseteq K$, define the *top* of A to be the set

$$top\,(A) = \{a \in A : rank\,(a) = rank\,(A)\}\,.$$

If A is basic, then $top\,(A)$ is empty or a one point set. Since each X_n is compact scattered, $top\,(X_n)$ is non-empty and finite for each n.

Suppose P is a partition of K. The *top* of P is

$$top\,(P) = \bigcup_{A \in P} top\,(A)\,.$$

A sequence of points $x_1, x_2, \ldots$ in K is *cofinal* provided that, for all n, there is $i \geq n$ such that for all $j > i$ $x_j \notin X_n$. This is equivalent to converging to the extra point in the one-point compactification of K. There are no cofinal sequences when K is compact.

Lemma 5.10. *Let K be a σ-compact, locally compact countable-closure space, suppose $a_1, a_2, \ldots$ is a sequence of points in K, and suppose $P_1, P_2, \ldots$ is a sequence of basic partitions of K such that for all i, $a_{i+1} \in D_{P_i}(a_i)$, P_{i+1} refines P_i, and $cl\,(\{a_1, a_2, \ldots\}) \subseteq \bigcup_{i \in \mathbb{N}} top\,(P_i)$. Then either $a_1, a_2, \ldots$ is cofinal in K or there exists $b \in K$ such that $a_1, a_2, \ldots$ converges to b.*

Proof. Suppose $a_1, a_2, \ldots$ is not cofinal in K. Then there exists n so that $\{i : a_i \in X_n\}$ is infinite. Since X_n is sequentially compact, there is $b \in X_n$ and a subsequence of $a_1, a_2, \ldots$ which converges to b. Since $b \in cl\,(\{a_1, a_2, \ldots\})$ there is m large enough so that $b \in top\,(P_m)$. Choose $j > m$ such that $a_j \in D_{P_m}(b)$. Since $D_{P_j}(a_j) \supseteq \{a_j, a_{j+1}, \ldots\}$, we have $b \in D_{P_j}(a_j)$. Thus, $top\left(D_{P_j}(a_j)\right) = b$. Suppose c is another limit point of $a_1, a_2, \ldots$. Then there is $k > m$ with $c \in top\,(P_k)$. Choose $r > k$ so that $a_r \in D_{P_k}(c)$. Then $D_{P_r}(a_r) \supseteq \{a_r, a_{r+1}, \ldots\}$ and so $c \in D_{p_r}(a_r)$ and $top\,(D_{P_r}(a_r)) = c$. But $b \in D_{P_r}(a_r)$ and so $b = c$. Thus, $a_1, a_2, \ldots$ converges to b. □

Definition 5.11. Suppose K is a σ-compact, locally compact scattered space of rank β, where $K = \bigcup_{n \in \mathbb{N}} X_n$, $X_1 \subseteq X_2 \subseteq \ldots$ and each X_n is compact clopen. A sequence of basic partitions $P_1, P_2, \ldots$ of K is *tidy* with respect to $X_1 \subseteq X_2 \subseteq \ldots$ provided that, for all i,

(1) P_{i+1} refines P_i;
(2) $top\,(X_i) \subseteq top\,(P_i)$;
(3) if K is not compact, then the unique element of P_i with rank β is contained in $K \backslash X_i$.

Note that if P_i and P_{i+1} are elements of a tidy sequence of partitions, then $top\,(P_i) \subseteq top\,(P_{i+1})$. Notice also that by (3) above and Definition 5.8(2), if K is not compact, the unique element of P_i with rank β is some $K \backslash X_j$ where $j \geq i$. Thus, the restriction of P_i to X_j is a partition of X_j.

Lemma 5.12. *Let $K = \bigcup_{n\in\mathbb{N}} X_n$ be a σ-compact, locally compact countable-closure space written as an increasing union of compact open sets, and let $x_1, x_2, \ldots$ be a sequence of points in $\prod^\omega K$. Let $A = \{x_i(n) : i < \omega, n < \omega\}$ and let $f : \mathbb{N} \to cl(A)$ be a bijection. If there exists a sequence of tidy partitions $P_1, P_2, \ldots$ of K such that, for all i,*

(1) $x_{i+1} \in \bar{D}_{P_i}(x_i)$ *and*
(2) $f(i) \in top(P_i)$,

then either $\bigcap_{n\in\mathbb{N}} \bar{D}_{P_n}(x_n) = \emptyset$ or there exists a point $z \in \prod^\omega K$ with $z \in \bigcap_{n\in\mathbb{N}} \bar{D}_{P_n}(x_n)$ for which $x_1, x_2, \ldots$ converges to z.

Proof. Suppose $\bigcap_{n\in\mathbb{N}} \bar{D}_{P_n}(x_n) \neq \emptyset$. First we show that in every coordinate n, $D_{P_i}(x_i(n)) \subseteq X_k$ for some i and some k. This is clear if K is compact. If K is not compact, then if $D_{P_i}(x_i(n)) \nsubseteq X_k$ for any k, then for all i, $D_{P_i}(x_i(n)) \subseteq K \backslash X_i$ as the sequence of partitions is tidy; hence, $\langle x_i(n) : i \in \mathbb{N}\rangle$ is cofinal and so $\bigcap_{i\in\mathbb{N}} D_{P_i}(x_i(n)) = \emptyset$.

So for each n, let i and k be such that $D_{P_i}(x_i(n)) \subseteq X_k$. Then in each coordinate n, $\langle x_i(n) : i \in \mathbb{N}\rangle$ satisfies the hypotheses of Lemma 5.10, and since it is not a cofinal sequence, there is $z(n) \in K$ such that $x_1(n), x_2(n), \ldots$ converges to $z(n)$. Define a point $z \in \prod^\omega K$ as $z = (z(1), z(2), \ldots)$.

By the first condition, for each m and each i we have $D_{P_m}(x_m(i)) \supseteq \{x_{m+1}(i), x_{m+2}(i), \ldots\}$, so that $z(i) \in D_{P_m}(x_m(i))$. We show $x_1, x_2, \ldots$ converges to z.

Suppose not, and let P be a basic partition of K such that there is an infinite set $I \subseteq \mathbb{N}$ so that, for every $n \in I$, there exists $j_n \in \mathbb{N}$ such that $x_{j_n}(n) \notin D_P(z(n))$. By pointwise convergence, for every n $\{m : x_m(n) \notin D_P(z(n))\}$ is finite, so we may assume $j_1 < j_2 < \ldots$.

We show $\langle x_{j_n}(n) : n \in I\rangle$ cannot be a cofinal sequence in K; i.e., we find $s \in \mathbb{N}$ so that X_s contains infinitely many terms from this sequence. This is clear if K is compact. If K is not compact, there are two cases to consider.

Case 1: If $\{z(n) : n \in I\} \nsubseteq X_t$ for any $t \in \mathbb{N}$, then there exists an infinite subset $J \subseteq I$ such that $D_P(z(n)) = K\backslash X_s$ where $K\backslash X_s$ is the unique element of P without maximal rank. Since by assumption $x_{j_n}(n) \notin D_P(z(n))$, for each $i \in J$ we have $x_{j_i}(i) \in X_s$.

Case 2: Suppose $\{z(n) : n \in I\} \subseteq X_t$ for some $t \in \mathbb{N}$. The unique element of P_t without maximal rank is $K\backslash X_s$ for some s where $s \geq t$. We cannot have $D_{P_t}(x_t(n)) = K\backslash X_s$ for any $n \in I$ since if so, then because $z(n) \in D_{P_t}(x_t(n)) = K\backslash X_s$, then $z(n) \notin X_s \supseteq X_t$ whence $z(n) \notin X_t$. Thus, for each $n \in I$, $D_{P_t}(x_t(n)) \subseteq X_s$ and so for every $m \in I$ with $m \geq t$, $x_{j_m}(m) \in X_s$.

Since X_s contains infinitely many terms from $\langle x_{j_n}(n) : n \in I\rangle$, there exists an infinite subset $J \subseteq I$ and a $y \in X_s$ such that $\langle x_{j_n}(n) : n \in J\rangle$ is a sequence in X_s which converges to y. We show $\langle z(n) : n \in J\rangle$ has a subsequence which converges to y.

Choose $m \in \mathbb{N}$ large enough so that $y \in top(P_m)$ and $m \geq s$. Choose l so that $D_{P_m}(y)$ contains all but the first l terms of $\langle x_{j_n}(n) : n \in J\rangle$. Fix $q > \max\{m, l\}$ so that for all $i > q$ with $i \in J$, $j_i > \max\{m, l\}$ as well. Then, for $i > q$ with $i \in J$, $x_{j_i}(i) \in D_{P_m}(x_m(i))$ and also $x_{j_i}(i) \in D_{P_m}(y)$; since P_m is a partition, we have $D_{P_m}(x_m(i)) = D_{P_m}(y)$. Thus, we have $z(i) \in D_{P_m}(x_m(i)) = D_{P_m}(y)$. Thus,

$$D_{P_m}(y) \supseteq \{z(i) : i \in J \text{ and } i > q\},$$

and so $\langle z(n) : n \in J\rangle$ has a limit point w. We must have $w = y$: Choose $m' \geq m$ such that $w \in top(P_{m'})$, which can be done since $w \in cl(A)$. Then either $D_{P_{m'}}(y) \cap D_{P_{m'}}(w) = \emptyset$ or $w = y$. By similar reasoning, $D_{P_{m'}}(y)$ contains all but finitely many terms of $\langle z(n) : n \in J\rangle$, and so also contains w. Thus, $w = y$ by choice of m'.

But then $D_P(z(i)) = D_P(y)$ for infinitely many $i \in J$, and $x_{j_n}(n) \notin D_P(z(n))$ for all $n \in J \subseteq I$ by assumption. So $D_P(y)$ is a neighborhood of y which misses infinitely many terms of $\langle x_{j_n}(n) : n \in J\rangle$, contradicting that $\langle x_{j_n}(n) : n \in J\rangle$ converges to y. $\square$

5.4. Pseudonormality.

Our strategy to prove the main result of this section is to inductively build a tree of clopen sets surrounding a given countable closed set which is contained in a given open set. This tree will be constructed so that if an infinite branch exists, the intersection of the sets in that branch is empty. Then we observe that the union of certain elements of the tree is closed.

Theorem 5.13. *The countable uniform box product of a σ-compact, locally compact countable-closure space is pseudonormal.*

Proof. Let K be a σ-compact, locally compact countable-closure space of rank β. Suppose $C = \{c_1, c_2, \ldots\} \subseteq \prod^\omega K$ is countable and closed with $C \subseteq G$ where G is open in $\prod^\omega K$. We will construct a tree of clopen sets by recursion. Write $K = \bigcup_{n\in\mathbb{N}} X_n$ where each X_n is compact clopen and $X_n \subseteq X_{n+1}$ for all n. If K is compact, then $X_n = K$ for each n.

Let $A = \{c_i(n) : i \in \mathbb{N}, n \in \mathbb{N}\}$ and fix $f : \mathbb{N} \to cl(A)$ a bijection. Let $k_1 = c_1$ and choose a basic clopen partition P_1 such that $\bar{D}_{P_1}(k_1) \subseteq G$, $top(P_1) \supseteq \{f(1)\} \cup top(X_1)$, and, if K is not compact, the unique element of P_1 with rank β is contained in $K \backslash X_1$. Let $T_1 = Q_1 \cup R_1$ where

$$Q_1 = \left\{\bar{D}_{P_1}(c) : c \in C \text{ and } \bar{D}_{P_1}(c) \subseteq G\right\}$$

$$R_1 = \left\{\bar{D}_{P_1}(c) : c \in C \text{ and } \bar{D}_{P_1}(c) \nsubseteq G\right\}.$$

At stage $n+1$, let k_{n+1} be c_j, where $j = \min\{i : c_i \in C \cap (\bigcup R_n)\}$, that is, the first member of C which is not in any Q_i yet. Choose a basic clopen partition P_{n+1} so that P_{n+1} refines P_n, $\bar{D}_{P_{n+1}}(k_{n+1}) \subseteq G$, $top(P_{n+1}) \supseteq \{f(n+1)\} \cup top(X_{n+1})$ and if K is not compact, the unique element of P_{n+1} with rank β is contained in $K \backslash X_{n+1}$. Let $T_{n+1} = Q_{n+1} \cup R_{n+1}$ where

$$Q_{n+1} = \left\{\bar{D}_{P_{n+1}}(c) : c \in C \cap \left(\bigcup R_n\right) \text{ and } \bar{D}_{P_{n+1}}(c) \subseteq G\right\}$$

$$R_{n+1} = \left\{\bar{D}_{P_{n+1}}(c) : c \in C \cap \left(\bigcup R_n\right) \text{ and } \bar{D}_{P_{n+1}}(c) \nsubseteq G\right\}.$$

So for every $c \in C$, there is n such that $c \in A \in Q_n$. We claim $H = \bigcup_{n\in\mathbb{N}} \bigcup Q_n$ is closed. Let $y \in cl\left(\bigcup_{n\in\mathbb{N}} \bigcup Q_n\right)$ and suppose $y \notin \bigcup Q_n$ for any n. Then $\bar{D}_{P_1}(y) \cap R_1 \neq \emptyset$ and so there exists a unique (since R_1 is pairwise disjoint) $A_1 \in R_1$ with $A_1 = \bar{D}_{P_1}(y)$; A_1 cannot be in Q_1 since $y \notin \bigcup Q_1$. So, for every n, there is a unique $A_n \in R_n$ with $\bar{D}_{P_n}(y) = A_n$. By construction, we have $A_1 \supseteq A_2 \supseteq \ldots$ and so $y \in \bigcap_{n\in\mathbb{N}} A_n$. But $\{k_1, k_2, k_3, \ldots\} \subseteq C$ and $P_1, P_2, \ldots$ satisfy the hypotheses of Lemma 5.12, and hence $\{k_1, k_2, k_3, \ldots\}$ converges to a point in C since $\bigcap_{n\in\mathbb{N}} D_{P_n}(k_n) \neq \emptyset$; however, this point must be in $\bigcup Q_r$ for some r, a contradiction. Therefore, $y \in \bigcup_{n\in\mathbb{N}} \bigcup Q_n$ and so H is closed. Clearly, H is also open and $C \subseteq H \subseteq G$. □

In particular, the countable uniform box product of a compact countable-closure space is pseudonormal.

6. Ordinal Spaces

Ordinal spaces are of particular interest, since it is not known if the uniform box product of the simplest non-trivial compact ordinal space, $[0, \omega_1]$, is normal. Nyikos [8] has shown that the uniform box product of $[0, \omega_1)$ is collectionwise normal.

Theorem 6.1. *The countable uniform box product of any ordinal space is pseudonormal.*

Proof. Since successor ordinals are compact countable-closure spaces, and limit ordinals with countable cofinality are σ-compact, locally compact countable-closure spaces, as a corollary to Theorem 5.13, we have that the countable uniform box product of an ordinal space α is pseudonormal if α is a successor ordinal or is of countable cofinality.

Let α be an ordinal of uncountable cofinality. Suppose $C \subseteq \prod^{\omega}[0, \alpha)$ is countable and closed, with $C \subseteq G$ where G is open. Then

$$Y = \{c_i(n) : i \in \mathbb{N}, n \in \mathbb{N}\}$$

is a countable collection of ordinals, and hence has a supremum β. Since α has uncountable cofinality, $\beta + 1 < \alpha$ and $C \subseteq \prod^{\omega}[0, \beta+1]$. Since $\prod^{\omega}[0, \beta+1]$ is an open subspace of $\prod^{\omega}[0, \alpha)$, $G \cap \prod^{\omega}[0, \beta+1]$ is open; since $\prod^{\omega}[0, \beta+1]$ is pseudonormal, there is an open set $H \subseteq \prod^{\omega}[0, \beta+1]$ with $C \subseteq H \subseteq G$. $\square$

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