http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 48, 2016

Pages 49–63

http://topology.nipissingu.ca/tp/

Characterizations of the Pseudo-Arc

by

Wayne Lewis

Electronically published on April 14, 2015

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 48 (2016)
Pages 49-63

E-Published on April 14, 2015

CHARACTERIZATIONS OF THE PSEUDO-ARC

WAYNE LEWIS

Abstract. The pseudo-arc has many interesting properties and is still the subject of many significant questions. It has several characterizations. Some of these are well known while others are less well known. We review many of these characterizations. We also consider several possible additional characterizations as well as conditions which are known not to characterize the pseudo-arc.

1. Introduction

A *continuum* is a compact, connected metric space. A continuum X is *chainable* if, for every $\epsilon > 0$, there is an open cover $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of X such that $\text{diam}(C_i) < \epsilon$ for each $1 \leq i \leq n$ and $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Nondegenerate chainable continua are also referred to as *arc-like* since they are homeomorphic to inverse limits of arcs and, for each $\epsilon > 0$, they admit a continuous surjection $f : X \to [0, 1]$ with $\text{diam}(f^{-1}(t)) < \epsilon$ for each $0 \leq t \leq 1$.

A continuum is *indecomposable* if it is not the union of two proper subcontinua and *hereditarily indecomposable* if every subcontinuum of it is indecomposable.

A continuum X is *homogeneous* if, for each $x_1, x_2 \in X$, there is a homeomorphism $h : X \to X$ with $h(x_1) = x_2$.

2. First Characterizations

In 1922, B. Knaster [34] gave the first example of a nondegenerate hereditarily indecomposable continuum. He constructed it as the intersection of a nested sequence of strips, using what he termed the *method*

2010 *Mathematics Subject Classification.* Primary 54F65, 54F15; Secondary 54F50, 54G20.

Key words and phrases. chainable, hereditarily indecomposable, pseudo-arc.

des bandes. It was what, by later terminology, would be called a chainable continuum.

In 1948, Edwin E. Moise [55] constructed a family of nondegenerate indecomposable chainable continua in the plane, each having the property that it was homeomorphic to each of its nondegenerate subcontinua, thus answering in the negative a question originally posed by M. Mazurkiewicz [51] in 1921. This property has subsequently been termed *hereditary equivalence*. Thus, each of these continua was hereditarily indecomposable. Since each was chainable and shared the property with the arc of being hereditarily equivalent, Moise called each of his continua a *pseudo-arc*.

In 1951, R H Bing [4] proved the following result.

Characterization 1. *Any two nondegenerate hereditarily indecomposable chainable continua are homeomorphic.*

Thus, all of the continua described by Moise are homeomorphic to each other as well as being homeomorphic to the continuum described by Knaster. This was the first characterization of the pseudo-arc and is now used as the definition of the pseudo-arc. Thus, a *pseudo-arc* is any nondegenerate hereditarily indecomposable chainable continuum. Such a continuum is topologically unique. This has also been the most important and most widely used characterization of the pseudo-arc.

As defined here, continua are normally considered to be metric spaces. However, compact connected nonmetrizable Hausdorff spaces are sometimes considered as generalized continua. Michel Smith [65] has given examples of topologically distinct generalized continua which are hereditarily indecomposable and the inverse limits of pseudo-arcs with ω_1 factors. Thus, Characterization 1 cannot be extended to generalized continua.

However, Moise introduced the term pseudo-arc because the continua which he constructed were hereditarily equivalent. George W. Henderson [25] has shown that the arc is the only decomposable continuum which is hereditarily equivalent. H. Cook [15] has shown that every hereditarily equivalent continuum is tree-like. Any hereditarily equivalent indecomposable continuum is hereditarily indecomposable. If it is also chainable and nondegenerate, then it is a pseudo-arc, by Bing's characterization.

Possible Characterization 1. *Is the pseudo-arc the only nondegenerate indecomposable hereditarily equivalent continuum?*

Here also, the extension to generalized continua gives a noncharacterization. Lee Mohler and Lex G. Oversteegen [54] have constructed an example of a generalized (nonmetrizable) continuum which is decomposable and hereditarily equivalent. Smith [65] has constructed an example of a generalized (nonmetrizable) continuum which is indecomposable and hereditarily equivalent.

The characterization by Bing cannot be weakened either by replacing hereditarily indecomposable simply with indecomposable or by replacing chainable with tree-like or nonseparating planar. William Thomas Watkins [68] has shown that there are uncountably many topologically distinct continua obtainable as inverse limits of the arc with open bonding maps, generally referred to as Knaster-type continua. W. T. Ingram [29] has constructed uncountably many topologically distinct hereditarily indecomposable tree-like continua in the plane.

Noncharacterization 1. *There exist* $\mathfrak{c} = 2^{\aleph_0}$ *topologically distinct chainable indecomposable continua* [68]. *There also exist* $\mathfrak{c} = 2^{\aleph_0}$ *topologically distinct hereditarily indecomposable tree-like (or nonseparating planar) continua* [29].

Since chainability has turned out to be important in many areas, but is not easy to characterize or to determine if a continuum is not specifically constructed either using a sequence of chains or as an inverse limit of arcs, in 1964, A. Lelek [39] introduced the concept of span of a continuum. For any continuum X, the *span of* X is given by $\sigma(X) = \sup\{s(A) \mid A \text{ is a subcontinuum of } X \times X \text{ with } \pi_1(A) = \pi_2(A)\}$ where $s(A) = \inf\{\mathrm{dist}(x, y) \mid (x, y) \in A\}$. Every chainable continuum has span 0 and Lelek asked if span 0 characterized chainable continua. L. C. Hoehn [26] has recently constructed an example of a nonchainable continuum with span 0. However, recently Hoehn and Oversteegen [27] have shown that every nondegenerate hereditarily indecomposable continuum with span 0 is chainable and hence a pseudo-arc. Thus, one has a generalization of Bing's initial characterization.

Characterization 2. *The pseudo-arc is the only nondegenerate hereditarily indecomposable continuum with span 0.*

There are two other ways to weaken chainability in the characterization by Bing, one gives a possible characterization and the other one does not.

A *weak chain* is a collection of open sets $\mathcal{W} = \{W_1, W_2, ..., W_n\}$ such that $W_i \cap W_{i+1} \neq \emptyset$ for each $1 \leq i < n$. In a weak chain, nonconsecutively indexed open sets may also intersect. A continuum X is *weakly chainable* if there is a sequence $\{\mathcal{W}_i\}_{i=1}^{\infty}$ of weak chains such that each $\mathcal{W}_i$ is an open cover of X, each element of $\mathcal{W}_i$ has diameter less than $\frac{1}{i}$, $\mathcal{W}_{i+1}$ refines $\mathcal{W}_i$ for each i, and there is a sequence of functions $\{f_i\}_{i=1}^{\infty}$ such that f_i is a function from the index set of $\mathcal{W}_{i+1}$ to the index set of $\mathcal{W}_i$ such that $W_{i+1}^j \subset W_i^{f_i(j)}$ for each j in the index set of $\mathcal{W}_{i+1}$ and $|f_i(j+1)-f_i(j)| \leq 1$ if both j and $j+1$ are in the index set of $\mathcal{W}_{i+1}$.

Lelek [38] and Lawrence Fearnley [17] have shown independently that a continuum is weakly chainable if and only if it is the continuous image of the pseudo-arc.

While it is known [2] that the pseudo-arc is the continuous image of every nondegenerate hereditarily indecomposable continuum, there is no known nondegenerate nonchainable hereditarily indecomposable continuum which is the continuous image of the pseudo-arc.

Possible Characterization 2. *Is the pseudo-arc the only nondegenerate hereditarily indecomposable weakly chainable continuum?*

Some consider the above one of the most significant open questions about the pseudo-arc. A positive answer to it would give an alternate proof that the classification of homogeneous plane continua is complete.

A continuous surjection $f : X \to Y$ of continuum X onto continuum Y is *confluent* if, for every subcontinuum H of Y and every component K of $f^{-1}(H)$, $f(K) = H$. It is known that the class of confluent surjections includes the classes of open surjections and of monotone surjections. T. Maćkowiak [50] has shown that a continuum is hereditarily indecomposable if and only if every continuous surjection from a continuum onto it is confluent. T. Bruce McLean [52] has proven that the confluent image of a tree-like continuum is tree-like. Thus, every weakly chainable hereditarily indecomposable continuum is tree-like, being the confluent image of a chainable continuum. It is unknown if the confluent image of a chainable continuum is chainable. The major difficulty in determining such a result is that we do not have a good way in general of determining when a continuum is chainable.

However, Oversteegen and E. D. Tymchatyn [57] have shown that every continuum of span 0 is weakly chainable. As indicated above, Hoehn [26] has given an example of a continuum with span 0 which is nonchainable. However, Hoehn and Oversteegen [27] have recently shown that every hereditarily indecomposable continuum of span 0 is not only weakly chainable but is actually chainable.

A continuum X is *almost chainable* if, for every $\epsilon > 0$, there exists an open cover $\mathcal{C} = \{C_1, C_2, ..., C_n\}$ of X such that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$, $\text{diam}(C_i) < \epsilon$ for each $1 \leq i < n$ ($\text{diam}(C_n)$ is not constrained) and $\text{dist}(x, C_1 \cup C_2 \cup \ldots C_{n-1}) < \epsilon$ for each $x \in X$. Thus, $\mathcal{C}$ is a chain covering X such that every link except possibly the last one is small and every point is close to one of the small links.

Bing [4] has constructed the nonchainable circle-like hereditarily indecomposable pseudo-circle; Fearnley [20] and James T. Rogers, Jr. [59] have studied the analogous nonchainable circle-like hereditarily indecomposable pseudo-solenoids. Each of these continua has the property that every nondegenerate proper subcontinuum is a pseudo-arc. There are also simple-triod-like nonchainable continua with every nondegenerate proper subcontinuum a pseudo-arc constructed by Ingram [28] and, for each n, continua separating the plane into n complementary domains with every nondegenerate proper subcontinuum a pseudo-arc constructed by Cook [14]. It is an easy observation that any continuum with every nondegenerate proper subcontinuum a pseudo-arc is almost chainable.

Noncharacterization 2. *There exist* $\mathfrak{c} = 2^{\aleph_0}$ *topologically distinct hereditarily indecomposable almost chainable continua. There are* $\mathfrak{c} = 2^{\aleph_0}$ *such which are planar* [14] [28], *as well as* $\mathfrak{c} = 2^{\aleph_0}$ *such which are nonplanar and circle-like* [20] [59].

A continuum X is *2-equivalent* if there are exactly two nonhomeomorphic types of nondegenerate subcontinua of X.

Each of the examples described just above is 2-equivalent, since each nondegenerate proper subcontinuum is a pseudo-arc, while the entire continuum is not a pseudo-arc.

Noncharacterization 3. *There exist* $\mathfrak{c} = 2^{\aleph_0}$ *topologically distinct hereditarily indecomposable 2-equivalent continua.*

3. Homogeneity

In 1948, Bing [3] and in 1949, Moise [56] independently proved that the pseudo-arc is homogeneous, thus providing a negative answer to a question of Knaster and C. Kuratowski [35] in 1920. Actually, since this was before Bing's first characterization of the pseudo-arc, Bing constructed a nondegenerate planar hereditarily indecomposable chainable continuum which was homogeneous and Moise showed that each of the continua which he had recently constructed was homogeneous. In each of the constructions by Bing and Moise there are points which are clearly endpoints and there are points which are clearly midpoints in terms of the defining chains in the construction of the continuum. At first glance these may seem to be distinct sets of points.

For a point x of a nondegenerate continuum X, consider the following three conditions.

(a) For each $\epsilon > 0$, there exists an ϵ-chain $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ covering X with $x \in C_1$.

(b) If H and K are subcontinua of X containing x, then either $H \subseteq K$ or $K \subseteq H$.

(c) If H is any nondegenerate subcontinuum of X with $x \in H$, then H is irreducible between x and another point y of H.

Condition (a) is what is normally understood by claiming that x is an *endpoint* of chainable continuum X. In 1951, Bing [5] showed that, for any point x of any nondegenerate continuum X, conditions (b) and (c) are equivalent and if X is chainable, then all three conditions are equivalent. Condition (b) at every point of X is equivalent to X being hereditarily indecomposable.

This thus provides another characterization of the pseudo-arc.

Characterization 3. *Every nondegenerate chainable continuum with the property that every point is an endpoint of the continuum is a pseudo-arc.*

In the above result it is not sufficient that the chainable continuum have a dense set of endpoints. For example, the union of two pseudo-arcs with a single point in common has a dense set of endpoints but is decomposable. The compactification of a pseudo-arc minus a point with remainder an interval is a nondegenerate indecomposable chainable continuum with a dense set of endpoints.

Noncharacterization 4. *The pseudo-arc is not the only nondegenerate chainable (indecomposable) continuum with a dense set of endpoints.*

In 1959, Bing [6] proved another characterization of the pseudo-arc.

Characterization 4. *The only nondegenerate homogeneous chainable continuum is the pseudo-arc.*

By the previous characterization, it suffices to show that some point (and hence, by homogeneity, every point) of the continuum is an endpoint. However, there are chainable continua which have no endpoints. Bing observed that every chainable continuum has a pseudo-endpoint. (A point x of chainable continuum X is a *pseudo-endpoint* of X if, for every $\epsilon > 0$, there exists an ϵ-chain $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ covering X such that C_1 is contained in the ϵ-neighborhood of x. Thus, x need not be in an end-link of an ϵ-chain covering X, but there is an ϵ-chain covering X with an end-link close to x.) Then, using the homogeneity of the continuum, one can produce an endpoint of the continuum.

C. E. Burgess [8] asked if every continuum with the property that each of its nondegenerate proper subcontinua was homogeneous was itself homogeneous. A negative answer was provided when Fearnley [19] and Rogers [58] independently showed that the pseudo-circle and pseudo-solenoids were not homogeneous. Each of these continua is almost homogeneous. A continuum X is *almost homogeneous* if, for each $x_1, x_2 \in X$ and each $\epsilon > 0$, there exists a homeomorphism $h : X \to X$ with $\text{dist}(h(x_1), x_2) < \epsilon$; i.e., each point has a dense orbit under the action of its homeomorphism group.

This author [42] in 1980 extended Bing's characterization [6] by using a consequence of a theorem of Edward G. Effros [16] to show that any nondegenerate homogeneous almost chainable continuum was chainable.

Characterization 5. *The pseudo-arc is the only nondegenerate homogeneous almost chainable continuum.*

The pseudo-arc has for some time been the only known nondegenerate homogeneous nonseparating plane continuum. F. Burton Jones [33] has shown that any homogeneous nonseparating plane continuum must be indecomposable and Charles L. Hagopian [22] has shown that any such continuum must be hereditarily indecomposable. Several further conditions which any nondegenerate homogeneous nonseparating plane continuum must satisfy are also known. Any one-dimensional nonseparating plane continuum must be tree-like. Oversteegen and Tymchatyn [57] have shown that every nonseparating homogeneous plane continuum must have span 0 and be weakly chainable. Recently Hoehn and Oversteegen [27] have shown that every hereditarily indecomposable continuum with span 0 is chainable, yielding the following result.

Characterization 6. *The pseudo-arc is the only nondegenerate homogeneous nonseparating plane continuum.*

This completes the classification of homogeneous plane continua with the four examples of the point, the pseudo-arc, the simple closed curve, and the circle of pseudo-arcs.

While the pseudo-arc is chainable (or arc-like), it is also circle-like. Circle-like homogeneous continua are completely classified [3], [24], [43], [56], [60], [67] and are known to consist of the pseudo-arc, the simple closed curve, the circle of pseudo-arcs, the solenoids, and the solenoids of pseudo-arcs, with a unique solenoid of pseudo-arcs corresponding to each solenoid. Every nondegenerate homogeneous plane continuum is circle-like.

Thus, we have the following classification.

Characterization 7. *The pseudo-arc is the only acyclic homogeneous circle-like continuum.*

Because of the similarity in names, persons occasionally confuse the circle of pseudo-arcs and the pseudo-circle. Both are circle-like continua which separate the plane. As its name suggests, the circle of pseudo-arcs admits a continuous decomposition into pseudo-arcs with decomposition space a circle. It is decomposable and homogeneous. The pseudo-circle is hereditarily indecomposable and is not homogeneous. The pseudo-circle was described by Bing [4] and the circle of pseudo-arcs was described by Bing and Jones [7].

The pseudo-arc is both hereditarily indecomposable and homogeneous. Either of these two conditions in combination with chainability and nondegenerate characterizes the pseudo-arc.

It is not known if the two conditions in combination with each other characterize the pseudo-arc.

Possible Characterization 3. *Is the pseudo-arc the only nondegenerate homogeneous hereditarily indecomposable continuum?*

Rogers [62] has shown that every homogeneous hereditarily indecomposable continuum is tree-like.

Paweł Krupski and Janusz R. Prajs [36] have shown that every homogeneous tree-like continuum is hereditarily indecomposable.

Since it is now known [26] that there are continua with span 0 which are not chainable, the following result is of interest.

Characterization 8. *The pseudo-arc is the only nondegenerate homogeneous continuum with span 0.*

Every continuum of span 0 is tree-like. If homogeneous, it is hereditarily indecomposable. Being of span 0 and hereditarily indecomposable, it is by the recent result of Hoehn and Oversteegen [27] chainable and hence a pseudo-arc

Possible Characterization 4. *Is the pseudo-arc the only nondegenerate homogeneous tree-like continuum?*

As indicated above, if it can be shown that every homogeneous hereditarily indecomposable continuum has span 0, then the recent result of Hoehn and Oversteegen [27] shows that each of the two previous questions has a positive answer. By the results of Rogers [62] and of Krupski and Prajs [36] these two questions are equivalent.

Rogers [61] has shown that every indecomposable planar homogeneous continuum fails to separate the plane and hence is tree-like. By a result of Hagopian [22], it is hereditarily indecomposable. The recent result of Hoehn and Oversteegen [27] shows the following characterization. By the Jones aposyndetic decomposition theorem [32], every decomposable homogeneous plane continuum admits a continuous decomposition into homogeneous nonseparating plane continua with decomposition space a simple closed curve. Since the simple closed curve and the circle of pseudo-arcs are the unique continua with such decompositions into points or pseudo-arcs, they are the only decomposable homogeneous plane continua.

Characterization 9. *The pseudo-arc is the only nondegenerate indecomposable homogeneous plane continuum.*

Solenoids are indecomposable nonplanar homogeneous continua. They have every nondegenerate proper subcontinuum an arc and hence are not hereditarily indecomposable.

Noncharacterization 5. *There are $\mathfrak{c} = 2^{\aleph_0}$ topologically distinct indecomposable homogeneous nonplanar continua.*

Possible Characterization 5. *Is the pseudo-arc the only nondegenerate chainable continuum which is almost homogeneous?*

It follows from a result of the author [41] that if X is a nondegenerate continuum with the property that every nondegenerate proper subcontinuum of X is a pseudo-arc and $x_1, x_2 \in X$ are points in the same composant of X, then there is a homeomorphism $h : X \to X$ with $h(x_1) = x_2$. Thus, such continua as the pseudo-circle, pseudo-solenoids, examples constructed by Ingram [29], examples constructed by Cook [14], and others are almost homogeneous, though being almost chainable but not chainable they are not homogeneous.

Noncharacterization 6. *There exist $\mathfrak{c} = 2^{\aleph_0}$ topologically distinct continua which are almost chainable and almost homogeneous.*

Any locally connected continuum is weakly chainable. Thus, the simple closed curve, the Menger universal curve [1], any manifold, or any product of such continua, are examples of weakly chainable homogeneous continua.

Noncharacterization 7. *There are $\mathfrak{c} = 2^{\aleph_0}$ many topologically distinct weakly chainable homogeneous continua.*

Oversteegen and Tymchatyn [57] have proven that every nonseparating homogeneous plane continuum is weakly chainable. With the recent result of Hoehn and Oversteegen [27], we now know that every homogeneous plane continuum is weakly chainable, since the simple closed curve and the circle of pseudo-arcs are. However, since nontrivial solenoids are not weakly chainable, there are one-dimensional nonplanar homogeneous continua which are not weakly chainable.

Noncharacterization 8. *Every homogeneous plane continuum is weakly chainable. There exist one-dimensional nonplanar homogeneous continua (e.g., the solenoids and solenoids of pseudo-arcs) which are not weakly chainable.*

A continuum X is *homogeneous with respect to the class* $\mathcal{M}$ if, for each $x_1, x_2 \in X$, there is a continuous surjection $f : X \to X$ such that $f \in \mathcal{M}$ and $f(x_1) = x_2$. Thus, a continuum is homogeneous if it is homogeneous with respect to homeomorphisms.

Janusz J. Charatonik [11] has proven the following generalizations of a characterization by Bing.

Characterization 10. *The pseudo-arc is the only nondegenerate chainable continuum which is homogeneous with respect to open maps.*

Characterization 11. *The pseudo-arc is the only nondegenerate chainable continuum which is homogeneous with respect to open monotone maps.*

A continuous function $f : X \to Y$ from continuum X to continuum Y is *weakly confluent* if, for each subcontinuum H of Y, there exists a component K of $f^{-1}(H)$ such that $f(K) = H$. It is known that every continuous surjection $f : X \to Y$ of a continuum X onto a chainable continuum Y is weakly confluent.

Charatonik and Maćkowiak [12] have obtained the following generalization of the above two characterizations.

Characterization 12. *The pseudo-arc is the only nondegenerate chainable continuum which is homogeneous with respect to confluent maps.*

Possible Characterization 6. *Is the pseudo-arc the only nondegenerate hereditarily indecomposable continuum which is homogeneous with respect to the class of confluent maps?*

Since every map of a continuum onto a hereditarily indecomposable continuum is confluent, any hereditarily indecomposable continuum which is homogeneous with respect to continuous surjections is also homogeneous with respect to confluent maps.

Frank Sturm [66] has recently announced that the pseudo-circle and pseudo-solenoids are not homogeneous with respect to the class of continuous surjections.

Noncharacterization 9. *The arc is a nondegenerate chainable continuum which is homogeneous with respect to weakly confluent maps. The Knaster simplest nondegenerate indecomposable continuum is a nondegenerate indecomposable chainable continuum which is homogeneous with respect to weakly confluent maps* [12].

4. Other Characterizations

There are a few other characterizations of the pseudo-arc which are of a different nature from those above.

Bing [4] proved the following characterization of the pseudo-arc.

Characterization 13. *In any Euclidean space of dimension $n \geq 2$ and in the Hilbert cube, the collection of all subcontinua which are homeomorphic to the pseudo-arc is a dense G_δ in the hyperspace of all subcontinua.*

This is frequently shortened to the statement that "most continua are pseudo-arcs." Since any two dense G_δ's in a complete metric space must intersect, the pseudo-arc is the only continuum with this property and hence, this is a characterization.

A compactum X has *property HN* (for " homeomorphically near") if, for every embedding of X in a metric space Z and every $\epsilon > 0$, there exists $\delta > 0$ such that if Y is a copy of X also embedded in Z such that the Hausdorff distance $d_H(X, Y) < \delta$, then there is a homeomorphism $h : X \to Y$ such that $\text{dist}(x, h(x)) < \epsilon$ for every $x \in X$; i.e., any two copies of X which are setwise close are homeomorphically close.

This author [48] has shown that this property characterizes the pseudo-arc. Jan van Mill [53] has recently given an alternate proof of this result.

Characterization 14. *Every compactum which has property HN is either a finite set, a Cantor set, or a finite set of pseudo-arcs. Thus, the pseudo-arc is the only nondegenerate continuum with property HN.*

This author [44] has shown that the pseudo-arc has the property indicated in the following question. It is not known if it is the only nondegenerate continuum with this property. Any continuum with this property will have to be both arc-like and circle-like and hence either indecomposable or 2-indecomposable [9]. If it must be hereditarily indecomposable, then it is a pseudo-arc.

Possible Characterization 7. *Is the pseudo-arc the only nondegenerate continuum X with the property that if $f : X \to Y$ is a continuous surjection of X onto nondegenerate continuum Y and $\epsilon > 0$, then there exists a continuous surjection $g : X \to Y$ such that diam$(g^{-1}(y)) < \epsilon$ for every $y \in Y$?*

For the pseudo-arc, the continuous surjection g described above can always be of the form $g = f \circ h$ where h is a self-homeomorphism of the pseudo-arc.

This author [46] has shown that every homeomorphism of the pseudo-arc is conjugate to an ϵ-homeomorphism of the pseudo-arc, for every $\epsilon > 0$. Thus, all of the behavior exhibited by homeomorphisms of the pseudo-arc occurs in arbitrarily small neighborhoods of the identity.

Trevor Irwin and Sławomir Solecki [30] have a characterization of the pseudo-arc with a property similar to that in the above question as well as similar to the above property.

Characterization 15. *The pseudo-arc P is the only nondegenerate chainable continuum with the property that, for any chainable continuum X, any continuous surjections f_1 and f_2 from P onto X and any $\epsilon > 0$, there exists a homeomorphism $h : P \to P$ such that dist$(f_1(p), f_2 \circ h(p)) < \epsilon$ for each $p \in P$.*

This author [45] and Smith [64] independently have shown that every continuous surjection of the pseudo-arc onto itself is a near homeomorphism; i.e., if $f : P \to P$ is a continuous surjection of the pseudo-arc and $\epsilon > 0$, then there exists a homeomorphism $h : P \to P$ such that $\mathrm{dist}(f(p), h(p)) < \epsilon$ for each $p \in P$. There are other nondegenerate continua which have this property almost trivially since they have few self-homeomorphisms and every continuous self-surjection is a homeomorphism, e.g., examples by Cook [13]. However, this property is unusual for continua which have an abundance of self-homeomorphisms.

Possible Characterization 8. *Is the pseudo-arc the only nondegenerate homogeneous continuum with the property that every continuous self-surjection is a near homeomorphism; i.e., the family of self-homeomorphisms is a dense G_δ in the family of all continuous self-surjections?*

We have presented a wide variety of characterizations, possible characterizations, and known noncharacterizations of the pseudo-arc. Some others can be produced by combining some of the ones presented here or by replacing a property in one characterization by a stronger property which the pseudo-arc is known to have. For example, the pseudo-arc is the only nondegenerate bihomogeneous chainable continuum. It would not be surprising if the pseudo-arc has some other characterizations quite distinct from those presented here. There is still much to discover about it, especially about its mapping properties. It may well be a "meta-characterization" that the pseudo-arc is the only nondegenerate continuum with so many conceptually distinct characterizations.

References

[1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. (2) **67** (1958), 313–324.

[2] David P. Bellamy, *Mapping hereditarily indecomposable continua onto a pseudo-arc* in Topology Conference (Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973). Ed. Raymond F. Dickman, Jr. and Peter Fletcher. Lecture Notes in Math., Vol. 375. Berlin: Springer, 1974. 6–14.

[3] R H Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. **15** (1948), no. 3, 729–742.

[4] ______, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), no. 1, 43–51.

[5] ______, *Snake-like continua*, Duke Math. J. **18** (1951), no. 3, 653–663.

[6] ______, *Each homogeneous nondegenerate chainable continuum is a pseudo-arc*, Proc. Amer. Math. Soc. **10** (1959), no. 3, 345–346.

[7] R H Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc. **90** (1959), no. 1, 171–192.

[8] C. E. Burgess, *Homogeneous continua* in Summer Institute on Set Theoretic Topology: Summary of Lectures and Seminars. Madison, Wisconsin: University of Wisconsin, 1955. (revised 1958). 75–78.

[9] ______, *Chainable continua and indecomposability*, Pacific J. Math. **9** (1959), 653–659.

[10] ______, *Homogeneous continua which are almost chainable*, Canad. J. Math. **13** (1961), 519–528.

[11] Janusz J. Charatonik, *A characterization of the pseudo-arc*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 11, 901–903 (1979).

[12] J. J. Charatonik and T. Maćkowiak, *Confluent and related mappings on arc-like continua—an application to homogeneity*, Topology Appl. **23** (1986), no. 1, 29–39.

[13] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. **60** (1967), 241–249.

[14] ______, *Concerning three questions of Burgess about homogeneous continua*, Colloq. Math. **19** (1968), 241–244.

[15] ______, *Tree-likeness of hereditarily equivalent continua*, Fund. Math. **68** (1970), 203–205.

[16] Edward G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55.

[17] Lawrence Fearnley, *Characterizations of the continuous images of the pseudo-arc*, Trans. Amer. Math. Soc. **111** (1964), 380–399.

[18] ______, *Topological operations on the class of continuous images of all snake-like continua*, Proc. London Math. Soc. (3) **15** (1965), 289–300.

[19] ______, *The pseudo-circle is not homogeneous*, Bull. Amer. Math. Soc. **75** (1969), 554–558.

[20] ______, *Classification of all hereditarily indecomposable circularly chainable continua*, Trans. Amer. Math. Soc. **168** (1972), 387–401.

[21] Charles L. Hagopian, *Homogeneous plane continua*, Houston J. Math. **1** (1975), 35–41.

[22] ______, *Indecomposable homogeneous plane continua are hereditarily indecomposable*, Trans. Amer. Math. Soc. **224** (1976), no. 2, 339–350 (1977).

[23] ______, *Homogeneous continua in 2-manifolds*, Topology Appl. **19** (1985), no. 2, 157–163.

[24] Charles L. Hagopian and James T. Rogers, Jr. *A classification of homogeneous, circle-like continua*, Houston J. Math. **3** (1977), no. 4, 471–474.

[25] George W. Henderson, *Proof that every compact decomposable continuum which is topologically equivalent to each of its nondegenerate subcontinua is an arc*, Ann. of Math. (2) **72** (1960), 421–428.

[26] L. C. Hoehn, *A non-chainable plane continuum with span zero*, Fund. Math. **211** (2011), no. 2, 149–174.

[27] L. C. Hoehn and L. G. Oversteegen, *A complete classification of homogeneous plane continua*. Preprint. Available at arXiv:1409.6324v1 [math.GN].

[28] W. T. Ingram, *Hereditarily indecomposable tree-like continua*, Fund. Math. **103** (1979), no. 1, 61–64.

[29] ———, *Hereditarily indecomposable tree-like continua. II*, Fund. Math. **111** (1981), no. 2, 95–106.

[30] Trevor Irwin and Sławomir Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. **358** (2006), no. 7, 3077–3096.

[31] F. Burton Jones, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. **2** (1951), 855–859.

[32] ———, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. **6** (1955), 735–740.

[33] ———, *On homogeneity* in Summer Institute on Set Theoretic Topology: Summary of Lectures and Seminars. Madison, Wisconsin: University of Wisconsin, 1955. (revised 1958). 68–70.

[34] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. **3** (1922), no. 1, 247–286.

[35] B. Knaster and C. Kuratowski, *Problèmes. 2)*, Fund. Math. **1** (1920), no. 1, 223.

[36] Paweł Krupski and Janusz R. Prajs, *Outlet points and homogeneous continua*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 123–141.

[37] Krystyna Kuperberg, *On the bihomogeneity problem of Knaster*, Trans. Amer. Math. Soc. **321** (1990), no. 1, 129–143.

[38] A. Lelek, *On weakly chainable continua*, Fund. Math. **51** 1962/1963, 271–282.

[39] ———, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214.

[40] ———, *Some problems concerning curves*, Colloq. Math. **23** (1971), 93–98, 176.

[41] Wayne Lewis, *Stable homeomorphisms of the pseudo-arc*, Canad. J. Math. **31** (1979), no. 2, 363–374.

[42] ———, *Almost chainable homogeneous continua are chainable*, Houston J. Math. **7** (1981), no. 3, 373–377.

[43] ———, *Homogeneous circlelike continua*, Proc. Amer. Math. Soc. **89** (1983), no. 1, 163–168.

[44] ———, *Observations on the pseudo-arc*, Topology Proc. **9** (1984), no. 2, 329–337.

[45] ———, *Most maps of the pseudo-arc are homeomorphisms*, Proc. Amer. Math. Soc. **91** (1984), no. 1, 147–154.

[46] ———, *Homeomorphisms of the pseudo-arc are essentially small*, Topology Appl. **34** (1990), no. 2, 203–206.

[47] ———, *The classification of homogeneous continua*, Soochow J. Math. **18** (1992), no. 1, 85–121.

[48] ———, *Another characterization of the pseudo-arc*, Topology Proc. **23** (1998), Spring, 235–244.

[49] ———, *The pseudo-arc*, Bol. Soc. Mat. Méxicana (3) **5** (1999), no. 1, 25–77.

[50] T. Maćkowiak, *The hereditary classes of mappings*, Fund. Math. **97** (1977), no. 2, 123–150.

[51] M. Mazurkiewicz, *Problèmes. 14)*, Fund. Math. **2** (1921), no. 1, 286.

[52] T. Bruce McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J. **39** (1972), 465–473.

[53] Jan van Mill, *A note on an unusual characterization of the pseudo-arc*, Topology Proc. **45** (2015), 1–4.

[54] Lee Mohler and Lex G. Oversteegen, *On hereditarily decomposable hereditarily equivalent nonmetric continua*, Fund. Math. **136** (1990), no. 1, 1–12.

[55] Edwin E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. **63** (1948), 581–594.

[56] ______, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. **67** (1949), 57–58.

[57] Lex G. Oversteegen and E. D. Tymchatyn, *On span and weakly chainable continua*, Fund. Math. **122** (1984), no. 2, 159–174.

[58] James T. Rogers, Jr., *The pseudo-circle is not homogeneous*, Trans. Amer. Math. Soc. **148** (1970), 417–428.

[59] ______, *Pseudo-circles and universal circularly chainable continua*, Illinois J. Math. **14** (1970), 222–237.

[60] ______, *Solenoids of pseudo-arcs*, Houston J. Math. **3** (1977), no. 4, 531–537.

[61] ______, *Homogeneous, separating plane continua are decomposable*, Michigan Math. J. **28** (1981), no. 3, 317–322.

[62] ______, *Homogeneous hereditarily indecomposable continua are tree-like*, Houston J. Math. **8** (1982), no. 3, 421–428.

[63] Ira Rosenholtz, *Open maps of chainable continua*, Proc. Amer. Math. Soc. **42** (1974), 258–264.

[64] Michel Smith, *Every mapping of the pseudo-arc onto itself is a near homeomorphism*, Proc. Amer. Math. Soc. **91** (1984), no. 1, 163–166.

[65] ______, *On nonmetric pseudo-arcs*, Topology Proc. **10** (1985), no. 2, 385–397.

[66] Frank Sturm, *Pseudo-solenoids are not continuously homogeneous*, Topology Appl. **171** (2014), 71–86.

[67] D. van Dantzig, *Ueber topologisch homogene Kontinua*, Fund. Math. **15**, (1930), no. 1, 102–125.

[68] William Thomas Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. **103** (1982), no. 2, 589–601.

Department of Mathematics & Statistics; Texas Tech University; Lubbock, Texas 79409-1042

E-mail address: `wayne.lewis@ttu.edu`