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by

Yuli Rudyak

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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ON TOPOLOGICAL COMPLEXITY OF EILENBERG–MACLANE SPACES

YULI RUDYAK

Abstract. We note that, for any natural k and every natural l between k and $2k$, there exists a group π with $\operatorname{cat} K(\pi,1)=k$ and $\mathrm{TC}(K(\pi,1))=l$. Because of this, we can set up a problem for searching for a purely group-theoretical description of $\mathrm{TC}(K(\pi,1))$ as an invariant of π.

Below $\operatorname{cat} X$ denotes the Lusternik–Schnirelmann category (normalized, i.e., $\operatorname{cat} S^n=1$, see [2]). Furthermore, we denote by $\mathrm{TC}(X)$ the topological complexity of X defined by Michael Farber [5], but we use the normalized version in [7], [8].

Because of results of Alexander Dranishnikov [3, Lemma 2.7 and Theorem 3.6], we get the following inequalities:

$$\operatorname{cat}(G\times H)\le \mathrm{TC}(G\vee H)\le \operatorname{cat} G+\operatorname{cat} H. \tag{1}$$

Farber asked about calculation of $\mathrm{TC}(K(\pi,1))$'s. It is known that $\operatorname{cat} X\le \mathrm{TC}(X)\le \operatorname{cat}(X\times X)$ for all X [5]. The following observation tells us that, in the class of $K(\pi,1)$-spaces, the above mentioned inequality gets no new bounds.

Theorem 1. *For every natural k and every natural l with $k\le l\le 2k$, there exists a discrete group π such that π with $\operatorname{cat} K(\pi,1)=k$ and $\mathrm{TC}(K(\pi,1))=l$. In fact, we can put $\pi=\mathbb{Z}^k*\mathbb{Z}^{l-k}$.*

Proof. Let T^m be the m-torus. Then $\operatorname{cat} T^m=m$. Put $r=l-k$ and consider the free product $\pi:=\mathbb{Z}^k*\mathbb{Z}^r$. Then $K(\pi,1)=T^k\vee T^r$, because

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$\operatorname{cat}(X \vee Y) = \max(\operatorname{cat} X, \operatorname{cat} Y)$ (for good enough spaces X and Y, like CW spaces). So $\operatorname{cat}(K(\pi, 1)) = k$. On the other hand, because of (1), we have

$$\begin{aligned} l = \operatorname{cat}(T^l) = \operatorname{cat}(T^k \times T^r) &\leq \mathrm{TC}(T^l \vee T^r) \\ = \mathrm{TC}(K(\pi, 1)) &\leq \operatorname{cat} T^k + \operatorname{cat} T^r = k + r = l. \end{aligned}$$

Thus, $\mathrm{TC}(K(\pi, 1)) = l$. □

The TC of groups $\mathbb{Z}^k * \mathbb{Z}^r$ also appear (implicitly) in [1].

Note that the invariant $\operatorname{cat}(K(\pi, 1))$ has a known purely group-theoretical description. In fact, $\operatorname{cat} K(\pi, 1)$ is equal to the cohomological dimension $\operatorname{cd}(\pi)$ of π. Indeed, Samuel Eilenberg and Tudor Ganea [4] proved that $\operatorname{cat} K(\pi, 1) = \operatorname{cd}(\pi)$ except, possibly, in the following case: $\operatorname{cat}(K(\pi, 1)) = 2$ while $\operatorname{cd}(\pi) = 1$. However, John Stallings [9] and Richard G. Swan [10] proved that if $\operatorname{cd}(\pi) = 1$, then π is free. So, in this case, $K(\pi, 1)$ is homotopy equivalent to a wedge of circles, and thus $\operatorname{cat}(K(\pi, 1)) = 1$.

Now, in view of Theorem 1, the problem of describing of $\mathrm{TC}(K(\pi, 1))$ in purely group-theoretical terms turns out to be essential.

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Department of Mathematics; 1400 Stadium Rd.; University of Florida; Gainesville, FL 32611

E-mail address: rudyak@ufl.edu