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ENDPOINTS OF INVERSE LIMITS WITH SET-VALUED FUNCTIONS

JAMES P. KELLY

Abstract. Suppose that $\{\mathbf{X},\mathbf{F}\}$ is an inverse sequence where, for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous, and suppose that $\mathbf{p}$ is a point of the inverse limit of this inverse sequence. We show that $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$ provided that for infinitely many $n \in \mathbb{N}$, $(p_1, \ldots, p_n)$ is an endpoint of $\Gamma_n = \{\mathbf{x} \in \prod_{i=1}^{n} X_i : x_i \in F_i(x_{i+1}) \text{ for } 1 \leq i < n\}$.

Additionally, in the special case that each bonding function has its inverse equal to the union of mappings, we show that $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$ if and only if $(p_1, \ldots, p_n)$ is an endpoint of Γ_n for all $n \in \mathbb{N}$. We show some examples of how this characterization of endpoints of an inverse limit may be used to show that two inverse limits are not homeomorphic.

We also demonstrate how these results may be applied to inverse limits with continuous, single-valued bonding functions.

Introduction

We begin with some definitions for the terms used in this paper.

A *compactum* is a non-empty, compact metric space. A *continuum* is a connected compactum. A continuum which is a subset of a compactum X is called a *subcontinuum* of X.

A point p in a compactum X is called an *endpoint* of X if, for any two subcontinua H and K of X which both contain p, either $H \subseteq K$ or $K \subseteq H$.

Given a compactum X, we define 2^X to be the space consisting of all non-empty, compact subsets of X. Given a function $F : X \to 2^Y$, we

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define its *graph* to be the set

$$\Gamma(F) = \{(x, y) : y \in F(x)\}.$$

It was shown in [12] that F is upper semi-continuous if and only if $\Gamma(F)$ is closed in $X \times Y$.

Let $\mathbf{X}$ be a sequence of compacta, and let $\mathbf{F}$ be a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$. Then the pair $\{\mathbf{X}, \mathbf{F}\}$ is called an *inverse sequence*. The *inverse limit* of the inverse sequence is the set

$$\varprojlim \mathbf{F} = \{\mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$

(In this paper, sequences—both finite and infinite—are written in bold and their terms are written in italics.) The terms of the sequence $\mathbf{X}$ are called the *factor spaces*, and the terms of the sequence $\mathbf{F}$ are called the *bonding functions*.

Given an inverse sequence $\{\mathbf{X}, \mathbf{F}\}$ and $n \in \mathbb{N}$, we define

$$\Gamma_n = \{\mathbf{x} \in \prod_{i=1}^{n} X_i : x_i \in F_i(x_{i+1}) \text{ for } 1 \leq i < n\}.$$

Also, for each $n \in \mathbb{N}$, we define projection mappings

$$\pi_n : \varprojlim \mathbf{F} \to X_n \text{ and } \pi_{[1,n]} \varprojlim \mathbf{F} \to \Gamma_n$$

by $\pi_n(\mathbf{x}) = x_n$, and $\pi_{[1,n]}(\mathbf{x}) = (x_1, \ldots, x_n)$.

Finally, given a compactum X and an upper semi-continuous function $F : X \to 2^X$, there is a naturally induced inverse sequence $\{\mathbf{X}, \mathbf{F}\}$ where, for each $i \in \mathbb{N}$, $X_i = X$ and $F_i = F$.

Endpoints of inverse limits have been studied a great deal in the past. In the classical setting, inverse sequences were defined to be a pair $\{\mathbf{X}, \mathbf{f}\}$ where for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to X_i$ is continuous. Much has been written concerning endpoints of classical inverse limits. In [5], Marcy Barge and Joe Martin gave a characterization of endpoints of inverse limits with a single continuous bonding function on $[0, 1]$. They also showed that the study of endpoints of the inverse limit can be related to the study of the dynamics of the function. Since then, there have been many more results concerning endpoints and other characterizations (see [1]– [3], [7]– [9], and [13]). These have all been in the case of a single bonding function on $[0, 1]$, and most of them focus on unimodal functions.

One of the main reasons endpoints of inverse limit spaces are studied is that endpoints are a topological invariant, so they can be used to show that two inverse limit spaces are not homeomorphic. William

Thomas Watkins used this in his classification of the inverse limits of certain piecewise linear open functions in [22], and the study of endpoints played a large role in the work leading to the proof of the Ingram Conjecture which was ultimately proven in [4].

In 2004 William S. Mahavier and in 2006 W. T. Ingram and Mahavier began the study of inverse limits with upper semi-continuous set-valued functions (see [19], [12]). In this paper, we will discuss endpoints of these more general inverse limits. Specifically, we will show in Theorem 1.2, that if $\{\mathbf{X}, \mathbf{F}\}$ is an inverse sequence and $\mathbf{p} \in \varprojlim \mathbf{F}$, then if $(p_1, \ldots, p_n)$ is an endpoint of Γ_n for infinitely many $n \in \mathbb{N}$, it follows that $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$. This will bring us to our main theorem which we state here. (It is proven in Theorem 1.3.)

Theorem. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. Suppose that for each $i \in \mathbb{N}$ there exists a collection, $\{f_\alpha^{(i)} : X_i \to X_{i+1}\}_{\alpha \in A_i}$, of continuous functions such that*

$$\Gamma\left(F_i^{-1}\right) = \bigcup_{\alpha \in A_i} \Gamma\left(f_\alpha^{(i)}\right).$$

Then for every $\mathbf{p} \in \varprojlim \mathbf{F}$, the following are equivalent.

(1) $\mathbf{p}$ *is an endpoint of* $\varprojlim \mathbf{F}$.
(2) $(p_1, \ldots, p_n)$ *is an endpoint of* Γ_n *for infinitely many* $n \in \mathbb{N}$.
(3) $(p_1, \ldots, p_n)$ *is an endpoint of* Γ_n *for all* $n \in \mathbb{N}$.

Thus, we have a characterization of endpoints for the inverse limit of a certain type of upper semi-continuous set-valued functions. In particular, irreducible functions (as defined in [16]) are the inverse of a union of mappings, so this characterization applies to their inverse limits. We use this in Example 1.4 to show that four particular irreducible functions have topologically distinct inverse limits by showing that their respective sets of endpoints have different cardinalities. More generally, this characterization of endpoints is used in [14] to help classify the inverse limits of multiple families of irreducible functions.

In section 2, we discuss some applications of this characterization to classical inverse limits and dynamical systems. Finally, in section 3, alternate definitions for endpoint are discussed, and the author leaves as an open question whether or not the main theorem would still hold under these alternate definitions.

1. Endpoints of Inverse Limits

In this section, we give the proofs of our main results. We begin with the following lemma.

Lemma 1.1. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. Let H and K be closed sets in $\varprojlim \mathbf{F}$. If for all $n \in \mathbb{N}$, $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$, then $H \subseteq K$.*

Proof. Let $\mathbf{x} \in H$. Then, for each $n \in \mathbb{N}$, $\pi_{[1,n]}(\mathbf{x}) \in \pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$. Therefore, for each $n \in \mathbb{N}$, there exists a point $\mathbf{y}(n) \in K$ such that $\pi_{[1,n]}(\mathbf{y}(n)) = \pi_{[1,n]}(\mathbf{x})$. It follows that $\mathbf{y}(n) \to \mathbf{x}$ as $n \to \infty$, so since K is closed, and each $\mathbf{y}(n) \in K$, we have that $\mathbf{x} \in K$. $\square$

This brings us to the following result which gives a sufficient condition for a point of the inverse limit space to be an endpoint.

Theorem 1.2. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. For any point $\mathbf{p} \in \varprojlim \mathbf{F}$, if $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n for infinitely many $n \in \mathbb{N}$, then $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$.*

Proof. Let $H, K \subseteq \varprojlim \mathbf{F}$ be two continua, each containing $\mathbf{p}$. We will show that either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ will hold for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n. Note that each of $\pi_{[1,n]}(H)$ and $\pi_{[1,n]}(K)$ is a subcontinuum of Γ_n containing $\pi_{[1,n]}(\mathbf{p})$, so either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$.

Hence, for all $n \in \mathbb{N}$ for which $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n, we have that the continua $\pi_{[1,n]}(H)$ and $\pi_{[1,n]}(K)$ are nested. Since there are infinitely many such $n \in \mathbb{N}$, it follows that either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ for infinitely many $n \in \mathbb{N}$, or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ for infinitely many $n \in \mathbb{N}$.

Now, note that if, for some $N \in \mathbb{N}$, $\pi_{[1,N]}(H) \subseteq \pi_{[1,N]}(K)$, then $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ for all $n \leq N$. Therefore, if $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ holds for infinitely many $n \in \mathbb{N}$, then it holds for all $n \in \mathbb{N}$. Likewise, if $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ holds for infinitely many $n \in \mathbb{N}$, then it holds for all $n \in \mathbb{N}$.

It follows then from Lemma 1.1 that either $H \subseteq K$ or $K \subseteq H$. Therefore, $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$. $\square$

The main result of this paper deals with the special case where each bonding function is the inverse of a union of maps. In this case, we have a characterization of the endpoints of the inverse limit.

Theorem 1.3. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. Suppose that, for each $i \in \mathbb{N}$, there exists a collection $\{f_\alpha^{(i)} : X_i \to X_{i+1}\}_{\alpha \in A_i}$ of continuous functions such that*

$$\Gamma\left(F_i^{-1}\right) = \bigcup_{\alpha \in A_i} \Gamma\left(f_\alpha^{(i)}\right).$$

Then for every $\mathbf{p} \in \varprojlim \mathbf{F}$, the following are equivalent.

(1) $\mathbf{p}$ *is an endpoint of* $\varprojlim \mathbf{F}$.

(2) $\pi_{[1,n]}(\mathbf{p})$ *is an endpoint of* Γ_n *for infinitely many* $n \in \mathbb{N}$.
(3) $\pi_{[1,n]}(\mathbf{p})$ *is an endpoint of* Γ_n *for all* $n \in \mathbb{N}$.

Proof. Clearly, (3) implies (2), and by Theorem 1.2, (2) implies (1). Thus, we must only show that (1) implies (3).

We will show that the negation of (3) implies the negation of (1). Suppose that $\mathbf{p} \in \varprojlim \mathbf{F}$ and there exists an $n \in \mathbb{N}$ such that $\pi_{[1,n]}(\mathbf{p})$ is not an endpoint of Γ_n. Then there exist two continua $H, K \subseteq \Gamma_n$ such that $\pi_{[1,n]}(\mathbf{p}) \in H \cap K$, and neither H nor K is contained in the other.

By assumption, for each $i \in \mathbb{N}$ and each $x \in X_i$,

$$F_i^{-1}(x) = \bigcup_{\alpha \in A_i} f_\alpha^{(i)}(x).$$

Thus, since for each $i \in \mathbb{N}$, $p_{i+1} \in F_i^{-1}(p_i)$, there exists a sequence $(\alpha_i)_{i=1}^\infty$ with $\alpha_i \in A_i$ such that $p_{i+1} = f_{\alpha_i}^{(i)}(p_i)$ for all $i \in \mathbb{N}$. Define two sets $\widetilde{H}$ and $\widetilde{K}$ by

$$\begin{aligned} \widetilde{H} &= \{\mathbf{x} : (x_i)_{i=1}^n \in H, \text{ and } x_{i+1} = f_{\alpha_i}^{(i)}(x_i) \text{ for } i \geq n\} \text{ and} \\ \widetilde{K} &= \{\mathbf{x} : (x_i)_{i=1}^n \in K, \text{ and } x_{i+1} = f_{\alpha_i}^{(i)}(x_i) \text{ for } i \geq n\}. \end{aligned}$$

Then each of $\widetilde{H}$ and $\widetilde{K}$ is a subcontinuum of $\varprojlim \mathbf{F}$, each contains $\mathbf{p}$, and neither is contained in the other. Therefore, $\mathbf{p}$ is not an endpoint of $\varprojlim \mathbf{F}$. □

We conclude this section with an example in which we use Theorem 1.3 to show that different inverse limits are not homeomorphic. The inverse limits of F, G, Φ, and Ψ (pictured in figures 1, 2, 3, and 4, respectively) have many properties in common with each other. In particular, it follows from a result of Scott Varagona, [21, Theorem 3.2], that the inverse limit of each of these functions is an indecomposable continuum. Ingram shows in [11, Example 5.4] that the inverse limit of Φ is chainable, and it follows from [15, Theorem 4.5] that each one of these functions has a chainable inverse limit.

However, we show in the following example that each inverse limit has a different number of endpoints, so no two of $\varprojlim \mathbf{F}$, $\varprojlim \mathbf{G}$, $\varprojlim \mathbf{\Phi}$, and $\varprojlim \mathbf{\Psi}$ are homeomorphic.

Example 1.4. Let F, G, Φ, and Ψ be the upper semi-continuous set-valued functions pictured in figures 1, 2, 3, and 4, respectively. Then $\varprojlim \mathbf{F}$ has exactly one endpoint, $\varprojlim \mathbf{G}$ has exactly two endpoints, $\varprojlim \mathbf{\Phi}$ has countably many endpoints, and $\varprojlim \mathbf{\Psi}$ has uncountably many endpoints.

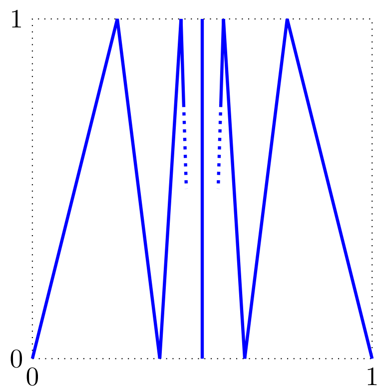

FIGURE 1. F

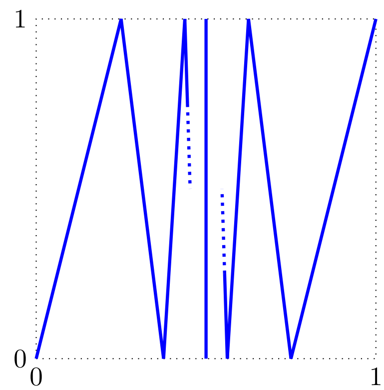

FIGURE 2. G

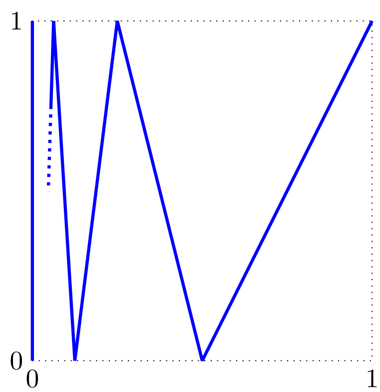

FIGURE 3. Φ

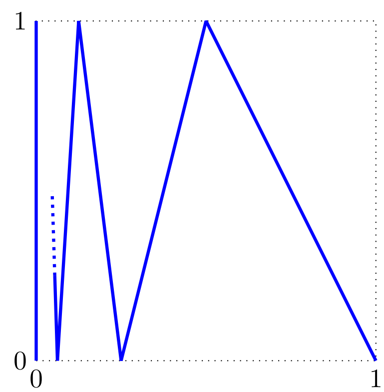

FIGURE 4. Ψ

Proof. First, we claim that for any $n \in \mathbb{N}$ and any of the four functions discussed in Example 1.4, the endpoints of Γ_n are precisely those points which are also in $\{0,1\}^n$.

We prove the claim for Φ (Figure 3). The proofs for the other three functions are similar.

Define a sequence $(\varphi_i)_{i=1}^{\infty}$ of continuous functions as follows: For all $i \in \mathbb{N}$ $\varphi_i : [0,1] \to [0,1]$ such that if i is odd, then the graph of φ_i is the straight line joining $(1, 1/2^{i-1})$ and $(0, 1/2^i)$, and if i is even, then the graph of φ_i is the straight line joining $(0, 1/2^{i-1})$ and $(1, 1/2^i)$. Also, let $\varphi_0 : [0,1] \to [0,1]$ be the function given by $\varphi_0(x) = 0$ for all $x \in [0,1]$.

Then we have that

$$\Gamma(\Phi) = \bigcup_{i=0}^{\infty} \Gamma\left(\varphi_i^{-1}\right).$$

For each $n \in \mathbb{N}$, let E_n be the set of endpoints of Γ_n. Note that $\Gamma_1 = [0,1]$ and $E_1 = \{0,1\}$. Proceeding by induction, suppose that for some $n \in \mathbb{N}$, $E_n = \Gamma_n \cap \{0,1\}^n$.

We show that $E_{n+1} \subseteq \{0,1\}^{n+1}$ by showing that $\Gamma_{n+1} \setminus \{0,1\}^{n+1} \subseteq \Gamma_{n+1} \setminus E_{n+1}$. Let $\mathbf{p} \in \Gamma_{n+1} \setminus \{0,1\}^{n+1}$. If $(p_1, \ldots, p_n)$ is not in $\{0,1\}^n$, then by the induction hypothesis, it is not an endpoint of Γ_n, and thus, $\mathbf{p}$ is not an endpoint of Γ_{n+1}.

If $(p_1, \ldots, p_n)$ is in $\{0,1\}^n$, then we have that $p_{n+1} \notin \{0,1\}$. Note that since

$$\Gamma(\Phi) = \bigcup_{i=0}^{\infty} \Gamma\left(\varphi_i^{-1}\right),$$

we have that $p_{n+1} = \varphi_i(p_n)$ for some $i \in \mathbb{N}$. In fact, since p_n is in $\{0,1\}$ while p_{n+1} is not, there exists $i \in \mathbb{N}$ such that $p_{n+1} = \varphi_i(p_n) = \varphi_{i+1}(p_n)$. Let

$$\begin{aligned} A_i &= \{\mathbf{x} \in \Gamma_{n+1} : x_{n+1} = \varphi_i(x_n)\} \text{ and} \\ A_{i+1} &= \{\mathbf{x} \in \Gamma_{n+1} : x_{n+1} = \varphi_{i+1}(x_n)\}. \end{aligned}$$

Each of these is a continuum containing $\mathbf{p}$, but neither is a subset of the other. Thus, $\mathbf{p}$ is not an endpoint of Γ_{n+1}.

Next, we show that $\{0,1\}^{n+1} \cap \Gamma_{n+1} \subseteq E_{n+1}$. Let $\mathbf{p} \in \{0,1\}^{n+1} \cap \Gamma_{n+1}$. Then, in particular, $(p_1, \ldots, p_n) \in \{0,1\}^n$, so it is an endpoint of Γ_n.

Ingram showed in [11, Example 5.4] that the point $(1, \ldots, 1)$ is an endpoint of Γ_{n+1}. Since this is the only point in Γ_{n+1} for which $x_{n+1} = 1$, we need only to consider the case where $p_{n+1} = 0$. Let

$$A_0 = \{\mathbf{x} \in \Gamma_{n+1} : x_{n+1} = 0\}.$$

Note that A_0 is homeomorphic to Γ_n by way of the projection map $\pi_{[1,n]}$ which implies that $\mathbf{p}$ is an endpoint of A_0. In [11, Example 5.4], Ingram shows that A_0 has the following properties:

(1) If K is a subcontinuum of Γ_{n+1} which intersects A_0 and its complement, then K contains A_0.
(2) If K and L are subcontinua of Γ_{n+1}, each of which intersects A_0, then either $K \subseteq L \cup A_0$ or $L \subseteq K \cup A_0$.

It follows from these properties and the fact that $\mathbf{p}$ is an endpoint of A_0 that $\mathbf{p}$ is an endpoint of Γ_{n+1}. This proves the claim for Φ.

It then follows from Theorem 1.3 that for each of the inverse limits under consideration, its set of endpoints is equal to its intersection with $\{0,1\}^{\mathbb{N}}$. Therefore, to count the endpoints of the respective inverse limits, we need only determine which sequences in $\{0,1\}^{\mathbb{N}}$ are elements of each inverse limit.

Let us first consider F, pictured in Figure 1. Note that $F^{-1}(1)$ contains neither 0 nor 1, so for any $\mathbf{x} \in \varprojlim \mathbf{F}$, if $x_i = 1$ for some $i \in \mathbb{N}$, then $x_{i+1} \notin \{0,1\}$. Hence, if $\mathbf{x} \in \varprojlim \mathbf{F}$ and $x_i = 1$ for some $i \in \mathbb{N}$, then $\mathbf{x} \notin \{0,1\}^{\mathbb{N}}$. Therefore, the only point in $\{0,1\}^{\mathbb{N}} \cap \varprojlim \mathbf{F}$, and hence the only endpoint of $\varprojlim \mathbf{F}$, is $(0,0,0,\ldots)$.

Next, consider G, pictured in Figure 2. Since $G(0) = \{0\}$ and $G(1) = \{1\}$, the only points in $\{0,1\}^{\mathbb{N}} \cap \varprojlim \mathbf{G}$ are $(0,0,0,\ldots)$ and $(1,1,1,\ldots)$.

Now consider Φ, pictured in Figure 3. In this case, $\Phi(0)$ contains both 0 and 1, but $\Phi(1)$ contains only 1. Thus, if $\mathbf{p} \in \varprojlim \mathbf{\Phi} \cap \{0,1\}^{\mathbb{N}}$ and, for some $n \in \mathbb{N}$, $p_n = 0$, then $p_i = 0$ for all $i \geq n$. However, if for some $n \in \mathbb{N}$, $p_n = 1$, then p_{n+1} could be either 0 or 1. This implies that $\varprojlim \mathbf{\Phi} \cap \{0,1\}^{\mathbb{N}}$ contains $(0,0,0,\ldots)$, $(1,1,1,\ldots)$, and any point $\mathbf{p}$ such that there exists $n \in \mathbb{N}$ with $p_i = 1$ for $i \leq n$ and $p_i = 0$ for $i > n$. Hence, $\varprojlim \mathbf{\Phi}$ has countably many endpoints.

Finally, we consider Ψ, pictured in Figure 4. At a glance, it might seem as though $\varprojlim \mathbf{\Psi}$ would have the same number of endpoints as $\varprojlim \mathbf{\Phi}$, but this is not the case. If $\mathbf{p} \in \varprojlim \mathbf{\Psi} \cap \{0,1\}^{\mathbb{N}}$ and $p_i = 1$, then $p_{i+1} = 0$. However, if $p_i = 0$, then p_{i+1} could be either 0 or 1. Thus, $\varprojlim \mathbf{\Psi} \cap \{0,1\}^{\mathbb{N}}$ is equal to the set

$$\left\{\mathbf{x} \in \{0,1\}^{\mathbb{N}} : \text{ if for some } i \in \mathbb{N},\ x_i = 1,\ \text{then } x_{i+1} = 0\right\}.$$

This set contains uncountably many elements. To show this, we will define an injection from $\mathbb{N}^{\mathbb{N}}$ into $\varprojlim \mathbf{\Psi} \cap \{0,1\}^{\mathbb{N}}$. Let $h : \mathbb{N}^{\mathbb{N}} \to \varprojlim \mathbf{\Psi} \cap \{0,1\}^{\mathbb{N}}$ be defined by setting $h(n_1, n_2, n_3, \ldots)$ equal to the sequence which begins with a one followed by n_1 many zeros, which are followed by a one followed by n_2 many zeros, which are followed by a one and so on. It follows that $\varprojlim \mathbf{\Psi}$ has at least as many endpoints as the cardinality of the set $\mathbb{N}^{\mathbb{N}}$ which is uncountable. □

In addition to satisfying the requirements Varagona outlined in [21, Theorem 3.2], these functions also satisfy the conditions of [17, Theorem 28] and, more generally, the definition of an irreducible function given in [16, definitions 3.1 & 3.7].

Not every inverse limit of an irreducible function has endpoints, and for those that do, the sets of endpoints are not necessarily as simple to determine as they are for the four functions in Example 1.4. However, the properties these four functions possess which make determining the endpoints of the inverse limits so attainable can be found in a broad class of irreducible functions. This, in part, is what makes possible a classification of the inverse limits of certain families of irreducible functions in [14].

2. Applications to Classical Inverse Limits and Dynamical Systems

We now consider how Theorem 1.3 may be applied to the special case where $\{\mathbf{X}, \mathbf{f}\}$ is an inverse sequence such that for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to X_i$ is continuous. In this case, for each $n \in \mathbb{N}$, Γ_n is homeomorphic to X_n with the projection mapping as the homeomorphism. Hence, a point $(p_1, \ldots, p_n)$ in Γ_n is an endpoint if and only if p_n is an endpoint of X_n. This makes applying Theorem 1.3 much simpler.

The following theorem is a restatement of Theorem 1.3 in the setting of classical inverse limits. Note that the additional assumption this theorem places on each f_i (that its inverse is a union of continuous functions) is satisfied, in particular, if each f_i is an open mapping on the unit interval.

Theorem 2.1. *Let $\{\mathbf{X}, \mathbf{f}\}$ be an inverse sequence where for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to X_i$ is continuous. Suppose that for each $i \in \mathbb{N}$ there exists a collection $\{g_\alpha^{(i)} : X_i \to X_{i+1}\}_{\alpha \in A_i}$ of continuous functions such that*

$$\Gamma\left(f^{-1}\right) = \bigcup_{\alpha \in A_i} \Gamma\left(g_\alpha^{(i)}\right).$$

Then for every $\mathbf{p} \in \varprojlim \mathbf{f}$, the following are equivalent.

1. *$\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{f}$.*
2. *p_i is an endpoint of X_i for infinitely many $i \in \mathbb{N}$.*
3. *p_i is an endpoint of X_i for all $i \in \mathbb{N}$.*

Of particular note is the equivalence of (2) and (3) above. The following corollary expresses two particular implications of this equivalence.

Corollary 2.2. *Let X be a compactum and let $f : X \to X$ be continuous. Suppose that there exists a collection $\{f_\alpha : X \to X\}_{\alpha \in A}$ of continuous functions such that*

$$\Gamma\left(f^{-1}\right) = \bigcup_{\alpha \in A} \Gamma\left(f_\alpha\right).$$

Let E be the set of endpoints of X and suppose that E is non-empty. Then the following hold.

1. *If for some $x \in E$ there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, then $f^i(x) \in E$ for all $i \in \mathbb{N}$.*
2. *If $f^n(E) = E$ for some $n \in \mathbb{N}$, then $f(E) = E$.*

Proof. To see that (1) holds, suppose that $x \in E$, $n \in \mathbb{N}$, and $f^n(x) = x$. Then the point $(x, f^{n-1}(x), \ldots, f(x), x, f^{n-1}(x), \ldots)$ is an element of $\varprojlim \mathbf{f}$. Moreover, infinitely many of this point's coordinates are equal to x which is an endpoint of X, so by Theorem 2.1, all of its coordinates must be endpoints of X. This implies that $f^i(x) \in E$ for $i = 1, \ldots, n-1$, and hence, $f^i(x) \in E$ for all $i \in \mathbb{N}$.

To prove (2), suppose that $f^n(E) = E$ for some $n \in \mathbb{N}$ and fix $x_0 \in E$. Since $f^n(E) = E$, there exists $x_1 \in E$ such that $f^n(x_1) = x_0$. Likewise, if for some $j \in \mathbb{N}$, $x_j \in E$ has been chosen, then we may choose $x_{j+1} \in E$ with $f^n(x_{j+1}) = x_j$.

Then consider the point

$$\big(f(x_0), x_0, f^{n-1}(x_1), \ldots f(x_1), x_1, f^{n-1}(x_2), \ldots\big)$$

in $\varprojlim \mathbf{f}$. Infinitely many of its coordinates are endpoints of X, so, by Theorem 2.1, all of its coordinates must be endpoints of X. In particular, $f^{n-1}(x_1)$ is in E and $f(f^{n-1}(x_1)) = x_0$, so $x_0 \in f(E)$. Also, $f(x_0) \in E$.

Therefore, we have that $f(E) \subseteq E \subseteq f(E)$, so $f(E) = E$. $\square$

3. Other Notions of “Endpoint”

There are many ways to define what it means for a point (or more generally a subcontinuum) to lie at the “end” of a continuum. An overview of some of the ways this has been defined in the past can be found in [6].We will focus on two particular notions of “endpoint”: A. Lelek’s definition of an endpoint of an arcwise connected continuum in [18] and Harlan C. Miller’s definition of a terminal point of a continuum in [20].

Lelek’s definition states that a point p is an endpoint of an arcwise connected continuum X if p is an endpoint of any arc in X which contains p.

Miller’s definition states that a point p is a terminal point in a continuum X if every irreducible continuum in X containing p is irreducible between p and some other point.

Question 3.1. (A) Does Theorem 1.3 hold if we define endpoint as Lelek does?
(B) Does Theorem 1.3 hold if the word “endpoint” is replaced with “terminal point” (as defined by Miller)?

We will say that a point which satisfies Lelek’s definition of an endpoint satisfies *Property L* and that a point which satisfies Miller’s definition of a terminal point satisfies *Property M*.

In the special case discussed in Theorem 1.3 (where each bonding function has its inverse equal to a union of continuous functions), an argument similar to the one used in the proof of Theorem 1.3 could be used to show that if $\mathbf{p} \in \varprojlim \mathbf{F}$ and $\pi_{[1,n]}(\mathbf{p})$ failed to satisfy Property L (Property M) in Γ_n for some $n \in \mathbb{N}$, then $\mathbf{p}$ fails to satisfy Property L (Property M) in $\varprojlim \mathbf{F}$.

It is less clear whether or not $\pi_{[1,n]}(\mathbf{p})$ satisfying Property L (Property M) in Γ_n for infinitely many $n \in \mathbb{N}$ (or even for all $n \in \mathbb{N}$) would imply that $\mathbf{p}$ satisfies Property L (Property M) in $\varprojlim \mathbf{F}$.

We conclude this paper with two examples of functions for which a statement such as Theorem 1.3 would hold (with either of the alternate notions of endpoint).

Example 3.2. The function pictured in Figure 5 was discussed in [10, Example 2.14]. Its inverse limit is a harmonic fan, and for each $n \geq 3$, Γ_n is a simple n-od. Moreover, given a point $\mathbf{p}$ in its inverse limit, $\mathbf{p}$ satisfies Property L (Property M) in the inverse limit if and only if $\pi_{[1,n]}(\mathbf{p})$ satisfies Property L (Property M) in Γ_n for all $n \in \mathbb{N}$.

Example 3.3. The function pictured in Figure 6 was discussed in [10, Example 2.7]. Its inverse limit is a cone over a Cantor set, and for each $n \geq 2$, Γ_n is a simple 2^n-od. Just as in Example 3.2, in this case too, the points which satisfy Property L (Property M) in the inverse limit are precisely those points whose projections satisfy Property L (Property M) in Γ_n.

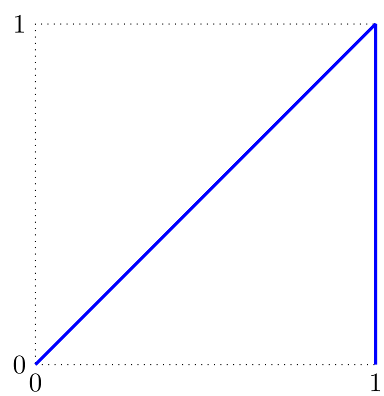

FIGURE 5

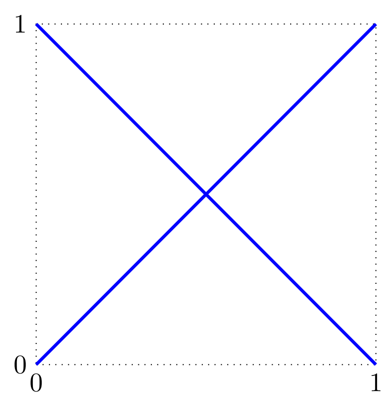

FIGURE 6

REFERENCES

[1] Lori Alvin, *Hofbauer towers and inverse limit spaces*, Proc. Amer. Math. Soc. **141** (2013), no. 11, 4039–4048.

[2] Lori Alvin and Karen Brucks, *Adding machines, endpoints, and inverse limit spaces*, Fund. Math. **209** (2010), no. 1, 81–93.

[3] ———, *Adding machines, kneading maps, and endpoints*, Topology Appl. **158** (2011), no. 3, 542–550.

[4] Marcy Barge, Henk Bruin, and Sonja Štimac, *The Ingram conjecture*, Geom. Topol. **16** (2012), no. 4, 2481–2516.

[5] Marcy Barge and Joe Martin, *Endpoints of inverse limit spaces and dynamics* in Continua (Cincinnati, OH, 1994). Ed. Howard Cook, et al. Lecture Notes in Pure and Appl. Math., 170. New York: Dekker, 1995. 165–182.

[6] D. E. Bennett and J. B. Fugate, *Continua and Their Non-Separating Subcontinua.* Dissertationes Math. (Rozprawy Mat.) Vol. 149, 1977.

[7] Louis Block, James Keesling, Brian Raines, and Sonja Štimac, *Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point*, Topology Appl. **156** (2009), no. 15, 2417–2425.

[8] Henk Bruin, *Planar embeddings of inverse limit spaces of unimodal maps*, Topology Appl. **96** (1999), no. 3, 191–208.

[9] W. T. Ingram, *Invariant sets and inverse limits*, Topology Appl. **126** (2002), no. 3, 393–408.

[10] ———, *An Introduction to Inverse Limits with Set-Valued Functions.* Springer Briefs in Mathematics. New York: Springer, 2012.

[11] ———, *Concerning chainability of inverse limits on* $[0,1]$ *with set-valued functions*, Topology Proc. **42** (2013), 327–340.

[12] W. T. Ingram and William S. Mahavier, *Inverse limits of upper semi-continuous set valued functions*, Houston J. Math. **32** (2006), no. 1, 119–130.

[13] Leslie Braziel Jones, *Adding machines and endpoints*, Topology Appl. **156** (2009), no. 17, 2899–2905.

[14] James P. Kelly, *A partial classification of inverse limits with irreducible functions.* Preprint.

[15] ———, *Chainability of inverse limits with a single irreducible function on* $[0,1]$, Topology Appl. **176** (2014), 57–75.

[16] ———, *Inverse limits with irreducible set-valued functions*, Topology Appl. **166** (2014), 15–31.

[17] James P. Kelly and Jonathan Meddaugh, *Indecomposability in inverse limits with set-valued functions*, Topology Appl. **160** (2013), no. 13, 1720–1731.

[18] A. Lelek, *On plane dendroids and their end points in the classical sense*, Fund. Math. **49** (1960/1961), 301–319.

[19] William S. Mahavier, *Inverse limits with subsets of* $[0,1]\times[0,1]$, Topology Appl. **141** (2004), no. 1-3, 225–231.

[20] Harlan C. Miller, *On unicoherent continua*, Trans. Amer. Math. Soc. **69** (1950), 179–194.

[21] Scott Varagona, *Inverse limits with upper semi-continuous bonding functions and indecomposability*, Houston J. Math. **37** (2011), no. 3, 1017–1034.

[22] William Thomas Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps*, Pacific J. Math. **103** (1982), no. 2, 589–601.

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