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SELECTIBILITY IS NOT PRESERVED UNDER OPEN LIGHT MAPPINGS BETWEEN FANS

FELIX CAPULÍN, LEONARDO JUÁREZ-VILLA,
AND FERNANDO OROZCO-ZITLI

Abstract. In this paper, we give an example of an open-light mapping between fans that does not preserve selectibility, which is an answer to the following question posed by Tadeusz Maćkowiak in *Continuous selections for* $C(X)$ [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 6]: Does it follow that an open image of a selectible fan is selectible? Further, it is an answer to the following question posed by J. J. Charatonik, W. J. Charatonik, and S. Miklos in *Confluent mappings of fans* [Dissertationes Math. (Rozprawy Mat.) **301**, 1990]: Is selectibility invariant under mappings of fans that are (1) light and open, (2) open, (3) light and confluent?

1. Introduction

A *continuum* means a nonempty compact and connected metric space. A mapping is a continuous function. A continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. An *arc* is understood as a homeomorphic image of a closed unit interval of the real line. If any two points of a space Z can be joined by an arc lying in Z, then Z is said to be *arcwise connected*.

A *dendroid* is defined as an arcwise connected and hereditarily unicoherent continuum. A point p of a dendroid X is called a *ramification point* of X (in the classical sense) if there exist three arcs emanating from

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p in X, with the intersection of each pair of them being just the singleton $\{p\}$. A *fan* means a dendroid having exactly one ramification point and this point is called its *top*.

Let X be a continuum. The hyperspace of all nonempty closed subsets of X is denoted by 2^X, the hyperspace of all subcontinua of X is denoted by $C(X)$ and the hyperspace of singletons is denoted by $F_1(X) = \{\{x\} : x \in X\}$, all equipped with the Hausdorff metric. Since $F_1(X)$ is homeomorphic to X, we may assume that $X \subset C(X)$.

By a *selection* for $C(X)$ we mean a mapping $\sigma : C(X) \to X$ such that $\sigma(A) \in A$ for each $A \in C(X)$. Then, X is said to be *selectible* provided that there is a selection for $C(X)$.

A mapping $g : X \longrightarrow Y$ between continua is said to be

- *monotone*, provided for each subcontinuum Q of Y, $g^{-1}(Q)$ is a continuum in X;
- *confluent*, provided for each subcontinuum Q of Y and each component C of $g^{-1}(Q)$, we have $g(C) = Q$;
- *light*, if the preimage $g^{-1}(y)$ is totally disconnected, for every $y \in Y$;
- *open*, if for any open set U in X, $g(U)$ is open in $f(X)$.

Let $p, q \in X$. We say that X is of *type N between p and q* if there exist in X an arc $A = \widehat{pq}$, two sequences of arcs $\{A_i\}_{i=1}^{\infty} = \{\widehat{p_i p'_i}\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty} = \{\widehat{q_i q'_i}\}_{i=1}^{\infty}$, and points $p''_i \in B_i \setminus \{q_i, q'_i\}$ and $q''_i \in A_i \setminus \{p_i, p'_i\}$ for each $i \in \mathbb{N}$ (the symbol $\mathbb{N}$ stands for the set of all positive integers) such that

(1) $A = \mathrm{Lim} A_i = \mathrm{Lim} B_i$;
(2) $p = \lim p_i = \lim p'_i = \lim p''_i$;
(3) $q = \lim q_i = \lim q'_i = \lim q''_i$;
(4) each arc in X joining p_i and p'_i contains q''_i;
(5) each arc in X joining q_i and q'_i contains p''_i.

We say that a continuum X is of *type N* if X is of type N between two points in X.

The following definition was introduced by Tadeusz Maćkowiak in [5]. Let A be a subcontinuum of a continuum X and let $B \subset A$. We say that B is a *bend set* of A if there exist two sequences of subcontinua $\{A_n\}_{n=1}^{\infty}$ and $\{A'_n\}_{n=1}^{\infty}$ of X satisfying the following conditions:

(1) $A_n \cap A'_n \neq \emptyset$ for each $n \in \mathbb{N}$;
(2) $A = \mathrm{Lim} A_n = \mathrm{Lim} A'_n$;
(3) $B = \mathrm{Lim}(A_n \cap A'_n)$.

A continuum X is said to have the *bend intersection property* provided that, for each subcontinuum A of X, the intersection of all its bend sets is nonempty.

It is known that every selectible dendroid has the bend intersection property; see [5, Corollary, p. 548].

Is not difficult to prove that every dendroid of type N is not selectible.

The following question was formulated for the first time by Maćkowiak in [5, Problem, p. 550].

Question 1.1. Is selectibility between fans preserved under open mappings?

In this direction, J. J. Charatonik, W. J. Charatonik, and S. Miklos asked in [4] the following question.

Question 1.2. Is selectibility invariant under mappings of fans that are (1) open and light, (2) open, (3) confluent and light?

This question is a particular case of a more general one; see [4, Question 14.14].

Question 1.3. What kind of confluent mappings preserve selectibility (nonselectibility) of fans?

It is well known that the image of a selectible (nonselectible) fan under a monotone mapping need not be selectible (nonselectible); see [5, p. 549] and [4, Corollary 14.13].

In [2, examples 2.1 and 2.2] the authors gave examples to show that the image of a selectible fan (nonselectible) under a light confluent mapping is not selectible (nonselectible).

In the same paper the authors gave an example to show that nonselectibility is not preserved under light open mappings.

In this paper we are going to give a selectible fan and a light open mapping such that the image is a nonselectible fan. So selectibility between fans is not preserved under light open mappings, answering Question 1.1 and questions 1.2 and 1.3 for the light open mappings.

2. Example

Example 2.1. There are a selectible fan X and an light open mapping g such that $g(X)$ is a nonselectible fan.

For each x and y in the 3-dimensional Euclidean space $\mathbb{R}^3$, denote by xy the convex arc joining x and y.

Now consider the following points in cylindrical coordinates in $\mathbb{R}^3$.

$p = (0,0,0)$, $a_0 = (\frac{1}{2}, \frac{3\pi}{4}, 0)$, $a_0' = (\frac{1}{2}, \frac{5\pi}{4}, 0)$,
$a_n = (\frac{1}{2^{n-1}}, \frac{\pi}{2^n}, 0)$
$a_{0,m} = (\frac{1}{2}, \frac{3\pi}{4}, \frac{1}{m})$, $a_{0,m}' = (\frac{1}{2}, \frac{5\pi}{4}, \frac{1}{m})$,
$p_{0,m} = (\frac{1}{2^m}, \frac{5\pi}{8}, \frac{1}{m})$, $p_{0,m}' = (\frac{1}{2^m}, \pi, \frac{1}{m})$,

$$a_{n,m} = (\tfrac{1}{2^{n-1}}(1+\tfrac{1}{m}), \tfrac{\pi}{2^n}, 0),\ p_{n,m} = (\tfrac{1}{m2^n}, \tfrac{3\pi}{2^{n+2}}, 0),$$

for every $n, m \in \mathbb{N}$.

Consider $T = \bigcup\{pa_n : n \in \mathbb{N}\} \cup pa_0 \cup pa'_0$. For each $m \in \mathbb{N}$, put $\widehat{pa'_m} = a'_{0,m}p'_{0,m} \cup p'_{0,m}a_{0,m} \cup a_{0,m}p_{0,m} \cup p_{0,m}a_{1,m} \cup \bigcup\{a_{n,m}p_{n,m} \cup p_{n,m}a_{n+1,m} : n \in \mathbb{N}\} \cup \{p\}$.

Note that the sequence $\{\widehat{pa'_m}\}$ converges to T. So

$$Y = T \cup (\bigcup\{\widehat{pa'_m} : m \in \mathbb{N}\})$$

is a countable fan with top p.

Note that the fan Y is homeomorphic to the plane fan Y' in Figure 1. Moreover, Y' is a rotation about 90° with respect to the fan Y.

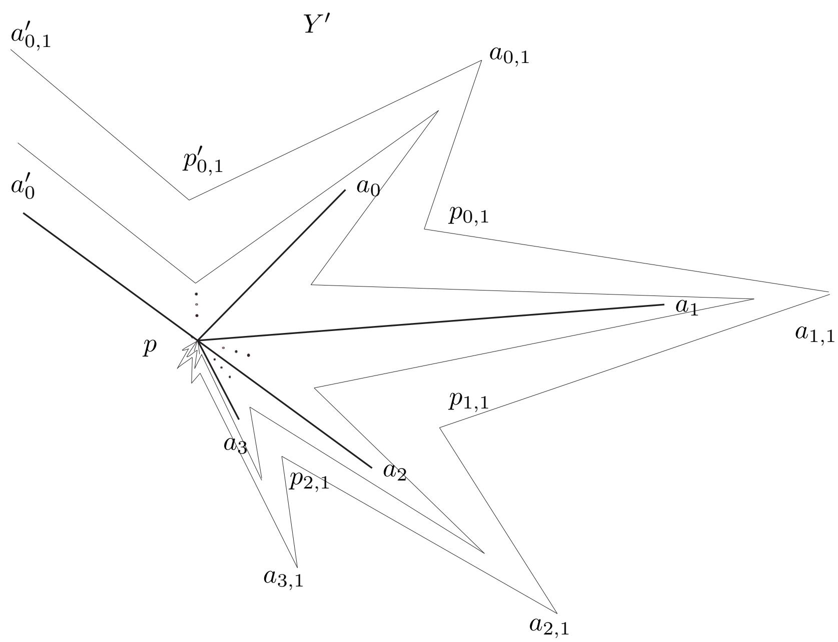

Figure 1: Fan Y'

Let $f : Y - (pa_0 \cup pa'_0) \to \mathbb{R}^3$ be defined by $f((r, \theta, z)) = (r, -\theta, -z)$. Put

$$X = Y \cup f(Y - (pa_0 \cup pa'_0)).$$

Then X is a countable plane fan with top p, which is homeomorphic to the plane fan X' in Figure 2.

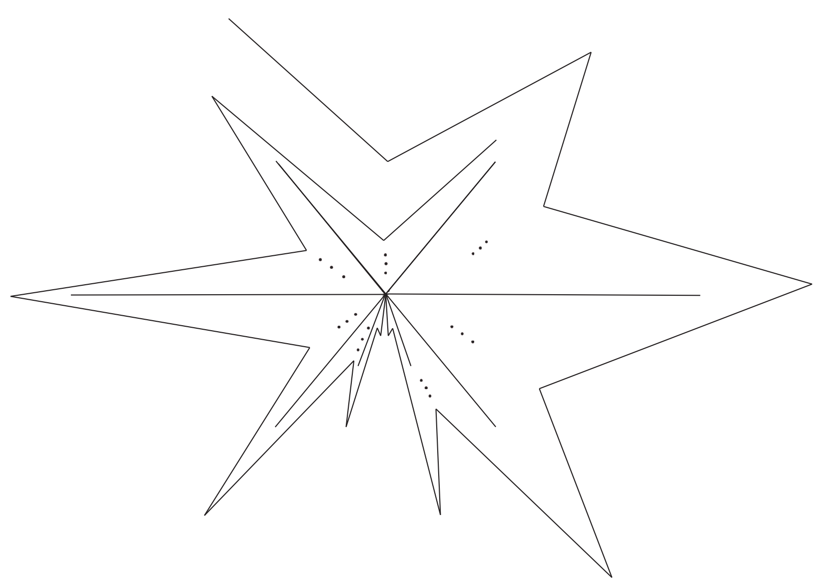

Figure 2: Fan X'

CLAIM: X is selectible.

In order to describe the selection of $C(X)$, let us consider the following set

$$F = \bigcup\{pa_n : n \in \mathbb{N} \cup \{0\}\} \cup \bigcup\{pa'_n : n \in \mathbb{N} \cup \{0\}\}$$
$$= T \cup f(T - (pa_0 \cup pa'_0)),$$

where $a'_n = f(a_n)$ for each $n \in \mathbb{N}$.

For each $x \neq y \in F$ such that $p \in xy$, let $s(x, y)$ be the point in F such that $||s(x, y)|| = \frac{\big| ||x|| - ||y|| \big|}{2}$ in such a way that either $s(x, y) \in px$ if $||x|| > ||y||$ or $s(x, y) \in py$ if $||y|\,| > ||x||$; in other words, $s(x, y)$ is the middle point into the arc $xp \cup py$ ($|| * ||$ denotes the Euclidean norm of a point in $\mathbb{R}^3$ and $\big| * \big|$ denotes the absolute value of a real number).

Let $C(F,p) = \{K \in C(F) : p \in K\}$ and we define

$$\alpha : C(F,p) \times \mathbb{R}^2 \longrightarrow F$$

a mapping such that $\alpha(K,z)$ is the point of $pz \cap K$ having the greatest norm.

We define a selection $\sigma' : C(F) \longrightarrow F$ in the following way: Let $K \in C(F)$. If $p \notin K$, K is an arc $xy \subset F$. Without loss of generality, we can suppose that $||x|| \geq ||y||$, so it is natural to put $\sigma'(K) = x$.

If $p \in K$, put

$t_n = \alpha(K, a_n)$ and $t'_n = \alpha(K, a'_n)$ for each $n \in \mathbb{N} \cup \{0\}$,
$x_1 = \alpha(K, 4s(s(t_0, 2t'_1), s(t'_0, 2t_1)))$,
$x_n = 2s(t_n, t'_n)$ for each $n > 1$,
$y_0 = p$,
$y_n = \alpha(K, 2s(y_{n-1}, 3x_n))$ for each $n \in \mathbb{N}$.

So define

$$\sigma'(K) = \alpha(K, \lim y_n).$$

To show that σ' is a mapping. We will consider all possible cases for $\sigma'(K)$ for every $K \in C(F)$.

(1) If $K = pt_n$ or $K = pt'_n$ for any $n \in \mathbb{N} \cup \{0\}$, then $\sigma'(K) = t_n$ or $\sigma'(K) = t'_n$.
   (a) $K = pt_0$.
   Then $s(t_0, 2t'_1) = \{\frac{t_0}{2}\}$ and $s(t'_0, 2t_1) = p$, so $x_1 = t_0$ and $x_n = p$ for each $n \in \mathbb{N}$; therefore, the sequence $\{y_n\}$ converges to t_0 and $\sigma'(K) = t_0$. Analogously, if $K = pt'_0$.
   (b) $K = pt_1$.
   We have that $s(t_0, 2t'_1) = p$ and $s(t'_0, 2t_1) = t_1$, so $x_1 = t_1$ and $x_n = p$ for each $n > 1$. Thus, the sequence $\{y_n\}$ converges to t_1 and $\sigma'(K) = t_1$. Similarly, if $K = pt'_1$.
   (c) $K = pt_n$ for any $n > 1$.
   Then $x_j = p$ for every $j \neq n$ and $x_j = t_j$ if $j = n$, so $y_n = t_n$ for each $j > n$. Hence, the sequence $\{y_n\}$ converges to t_n and $\sigma'(K) = t_n$. Analogously, if $K = pt'_n$.
(2) If $K = t_n t'_n$ for any $n \in \mathbb{N} \cup \{0\}$, then $\sigma'(K) \in pt_n$ if $||t_n|| > ||t'_n||$ or $\sigma'(K) \in pt'_n$ if $||t'_n|| > ||t_n||$.
   (a) Suppose that $K = t_0 t'_0$ and $||t_0|| > ||t'_0||$.
   We have that $s(t_0, 2t'_1) = \{\frac{t_0}{2}\}$ and $s(t'_0, 2t_1) = \frac{t'_0}{2}$. So $s(s(t_0, 2t'_1), s(t'_0, 2t_1)) \in p\frac{t_0}{4}$, then $x_1 \in pt_0$ and $x_n = p$ for

each $n \in \mathbb{N}$. Thus, the sequence $\{y_n\}$ converges to x_1 and $\sigma'(K) \in pt_0$.

(b) Suppose that $K = t_1 t_1'$ and $||t_1|| > ||t_1'||$.
So $s(t_0, 2t_1') = t_1'$, $s(t_0', 2t_1) = t_1$, and $s(s(t_0, 2t_1'), s(t_0', 2t_1)) \in p\frac{t_1}{2}$. Thus, $x_1 \in pt_1$ and $x_n = p$ for each $n \in \mathbb{N}$ and the sequence $\{y_n\}$ converges to x_1 and $\sigma'(K) \in pt_1$.

(c) $K = t_n t_n'$ and $||t_n|| > ||t_n'||$ for $n > 1$.
We have that $x_j = p$ for each $j \neq n$ and $x_j \in pt_j$ to $j = n$, so the sequence $\{y_m\}$ converges to x_n and $\sigma'(K) \in pt_n$.

(3) If $a_0', a_0 \in K \subset pa_0' \cup pa_0 \cup pa_1$, then $\sigma'(K) \in pa_0 \cup pa_1$.

(a) $||t_1|| \leq 1/4$.
We have that $s(t_0, 2t_1') = \{\frac{a_0}{2}\}$ and $s(t_0', 2t_1) \in p\frac{a_0'}{2}$. So $s(s(t_0, 2t_1'), s(t_0', 2t_1)) \in p\frac{a_0}{4}$; in the other words, $x_1 \in pa_0$ and $x_n = p$ for each $n > 1$. Thus, the sequence $\{y_n\}$ converges to x_1. Hence, $\sigma'(K) \in pa_0$. Notice that $\sigma'(K)$ runs through the segment pa_0 in the sense from p to a_0, when $||t_1||$ goes from 0 to 1/4.

(b) $1/4 < ||t_1|| < 1/2$.
Then $s(t_0, 2t_1') = \{\frac{a_0}{2}\}$ and $s(t_0', 2t_1) \in p\frac{a_0'}{4}$. Since $||s(t_0, 2t_1')|| \geq ||s(t_0', 2t_1)||$, $s(s(t_0, 2t_1'), s(t_0', 2t_1)) \in p\frac{a_0}{4}$; in other words, $x_1 \in pa_0$ and $x_n = p$ for each $n > 1$. So the sequence $\{y_n\}$ converges to x_1 and $\sigma'(K) \in pa_0$. In this case, $\sigma'(K)$ runs through the segment pa_0 in the sense from a_0 to p, when $||t_1||$ goes from 1/4 to 1/2.

(c) $1/2 < ||t_1|| < 1$.
So $s(t_0, 2t_1') = \{\frac{a_0}{2}\}$ and $s(t_0', 2t_1) \in \frac{a_1}{4}\frac{3a_1}{4}$. Note that $||s(t_0, 2t_1')|| \geq ||s(t_0', 2t_1)||$. Thus, $s(s(t_0, 2t_1'), s(t_0', 2t_1)) \in p\frac{a_1}{4}$; in other words, $x_1 \in pa_1$ and $x_n = p$ for each $n > 1$. Hence, the sequence $\{y_n\}$ converges to x_1 and $\sigma'(K) \in pa_1$. Here, $\sigma'(K)$ runs through the segment pa_1 in the sense from p to a_1, when $||t_1||$ goes from 1/2 to 1.

(4) If $a_1 \in K \subset pa_0' \cup pa_0 \cup pa_1$, then $\sigma'(K) = a_1$.

In this case, we have that $s(t_0, 2t_1') \in p\{\frac{a_0}{2}\}$ and $s(t_0', 2t_1) \in \frac{3a_1}{4}a_1$. So $s(s(t_0, 2t_1'), s(t_0', 2t_1)) \in \frac{a_1}{4}\frac{a_1}{2}$; in other words, $x_1 = a_1$ and $x_n = p$ for each $n > 1$. Thus, the sequence $\{y_n\}$ converges to x_1 and $\sigma'(K) = a_1$.

(5) If $K = F$, then $\sigma'(F) = p$.

Since $K = F$, $t_n = a_n$, $t_n' = a_n'$, and $||t_n|| = ||t_n'||$ for all $n \in \mathbb{N} \cup \{0\}$. So $x_n = p$ for $n \in \mathbb{N}$, the sequence $\{y_n\}$ converges to p, and $\sigma'(F) = p$.

Using conditions (1)–(5), we can see that σ' is a continuous selection for $C(F)$.

Now let $\beta : X \longrightarrow F$ be a retraction from X onto F such that for each $(n,m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$, $\beta(a_{n,m}) = a_n$, $\beta(f(a_{n,m})) = a'_n$, and $\beta(p_{n,m}) = p = \beta(f(p_{n,m}))$.

We define a partial order $\leq_p$ on X with respect to the point p as follows: Let $x, y \in X$, $x \leq_p y$ if and only if $px \subset py$.

Notice that if $K \in C(X)$ such that $K \cap F = \emptyset$, then K is an arc such that $K \subset \widehat{pa'_m}$ or $K \subset f(\widehat{pa'_m})$ for some $m \in \mathbb{N}$, and the set

$$\beta^{-1}(\sigma'(\beta(K))) \cap K = (\beta|_K)^{-1}(\sigma'(\beta(K)))$$

is finite.

We define a function $\sigma : C(X) \longrightarrow X$ by

$$\sigma(K) = \begin{cases} \sigma'(K \cap F), & \text{if } K \cap F \neq \emptyset, \\ \min_{\leq_p} (\beta|_K)^{-1}(\sigma'(\beta(K))), & \text{if } K \cap F = \emptyset. \end{cases}$$

One can verify that σ is a selection for $C(X)$, and so X is a selectible fan.

Now we define an equivalence relation in X: Let $x, y \in X$ and let $x \sim y$ if and only if either $y = f(x)$ or $x \in pa_0$, $y \in pa'_0$, and $||x|| = ||y||$.

Consider $Z = X/\sim$. Clearly, Z is homeomorphic to the fan in Figure 3.

Let g be the quotient mapping from X to Z. Notice that if $\widehat{z} \in Z$, $g^{-1}(\widehat{z}) = \{z, f(z)\}$, when $\widehat{z} \neq \widehat{p}$, and $g^{-1}(\widehat{z}) = \{p\}$, if $\widehat{z} = \widehat{p}$. So g is a light mapping. In order to see that g is open we will use the following theorem ([1, Theorem. 2.4]).

Theorem 2.2. *Let $f : X \longrightarrow Y$ be a map between continua. Then f is open if and only if for each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $\lim_{n\to\infty} y_n = y$, for some point $y \in Y$, and for any $x \in f^{-1}(y)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} x_n = x$ and $x_n \in f^{-1}(y_n)$, for each $n \in \mathbb{N}$.*

Let $\{\widehat{z}_n\}_{n\in\mathbb{N}}$ be a sequence of Z such that $\lim_{n\to\infty} \widehat{z}_n = \widehat{z}$ for some $\widehat{z} \in Z$. Since $g^{-1}(\widehat{z}) = \{z, f(z)\}$ and $g^{-1}(\widehat{z}_n) = \{z_n, f(z_n)\}$ for each $n \in \mathbb{N}$, by the construction of Z, the sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to z and by continuity of f, $\{f(z_n)\}_{n\in\mathbb{N}}$ converges to $f(z)$. Then, by Theorem 2.2, g is open.

Notice that Z is of type N between the point $\widehat{p}$ and the point $\widehat{a}_0 = \{a_0, a'_0\}$. So Z does not have the bend intersection property. Thus, by [5, Corollary, p. 548], Z is a non-selectible fan.

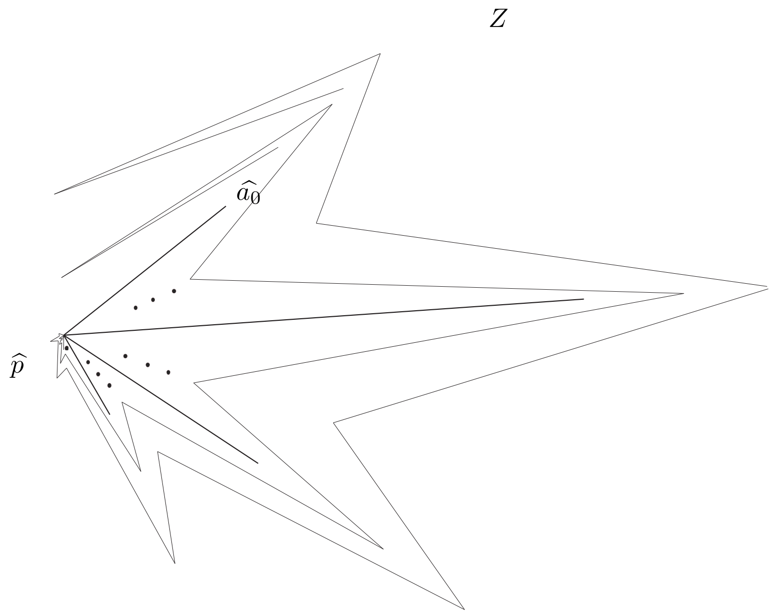

Figure 3: Fan Z

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