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by

Naoki Kato

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LIE FLOWS ON CONTACT MANIFOLDS

NAOKI KATO

Abstract. M. Llabrés and A. Reventós gave a necessary and sufficient condition for a Lie $\mathfrak{g}$-flow on a closed 3-manifold to be the characteristic foliation of a contact form. The aim of this paper is to generalize the Llabrés and Reventós's result for an arbitrary odd-dimensional closed manifold. Moreover, for a Lie flow $\mathcal{F}$ satisfying some cohomological condition, we will construct a good contact form such that the characteristic foliation coincides with $\mathcal{F}$.

1. Introduction

Throughout this paper, we suppose all manifolds to be closed, smooth, and orientable and all foliations to be smooth and transversely orientable.

Let M be a closed $(2n+1)$-dimensional manifold and let α be a contact form on M. The Reeb vector field X on the contact manifold (M, α) is the vector field on M defined by the equations $\alpha(X) = 1$ and $i_X\alpha = 0$. The one-dimensional foliation $\mathcal{F}$ defined by X is called the *characteristic foliation of* (M, α). The topology of the Reeb vector field X or the characteristic foliation $\mathcal{F}$ is, in general, quite complicated. Therefore, for a given contact manifold, to decide the topology of the characteristic foliation is an important problem.

Related to this problem, for a given one-dimensional foliation $\mathcal{F}$, it is also an important problem to decide whether the foliation can be realized as the characteristic foliation of a contact form. In this paper, we study on this problem in the case where $\mathcal{F}$ is a Lie foliation.

M. Nicolau and A. Reventós [7] gave a necessary and sufficient condition for a Seifert fibration $\mathcal{F}$ on a closed 3-manifold to be the characteristic

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foliation of a contact form. M. Saralegui [12] gave a necessary and sufficient condition in the case where $\mathcal{F}$ is an isometric flow on a closed 3-manifold. In [5], M. Llabrés and Reventós gave a necessary and sufficient condition in the case where $\mathcal{F}$ is a Lie $\mathfrak{g}$-flow on a closed 3-manifold, which is a special case of Saralegui's result.

In this paper, we generalize the result of Llabrés and Reventós for an arbitrary odd-dimensional closed manifold. We obtained the following theorem.

Theorem 3.1. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Then $\mathcal{F}$ is a contact Lie $\mathfrak{g}$-foliation if and only if $\mathcal{F}$ satisfies the following conditions.*

(a) *Any orbit of $\mathcal{F}$ is closed.*
(b) *There exists a symplectic form ω on the leaf space $N = M/\mathcal{F}$ such that $[\omega] \in H^2(N;\mathbb{Z})$ and $e(\mathcal{F}) = -[\omega]$.*

If a Lie $\mathfrak{g}$-flow $\mathcal{F}$ satisfies some cohomological condition, then $\mathcal{F}$ is homogeneous. In this case, we can construct a homogeneous contact form such that $\mathcal{F}$ coincides with the characteristic foliation of the homogeneous contact manifold.

Theorem 4.1. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Suppose that $\mathcal{F}$ satisfies the conditions* (a) and

(b′) There exists an algebraic symplectic form $\omega \in A^2_b(M,\mathcal{F}) = A^2(N)$ such that $[\omega] \in H^2(N;\mathbb{Z})$ and $e(\mathcal{F}) = -[\omega]$.

Then

(i) the Lie $\mathfrak{g}$-flow $\mathcal{F}$ is homogeneous and
(ii) there exists a contact form α on M such that α is homogeneous and the characteristic foliation of (M,α) coincides with $\mathcal{F}$.

If the Lie algebra $\mathfrak{g}$ is nilpotent, more generally of type (R), then the condition (b′) is equivalent to the condition (b). In this case, we can show that any contact form whose characteristic foliation is a Lie $\mathfrak{g}$-flow is homotopic to a homogeneous contact form. More precisely, we obtained the following corollary.

Corollary 4.3. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Suppose that $\mathfrak{g}$ is of type* (R)*. Then any contact form on M with the characteristic foliation $\mathcal{F}$ is homotopic to a homogeneous contact form as a non-singular one-form.*

2. Preliminaries

In this section we recall some basic definitions of Lie foliations and contact manifolds. We also describe Llabrés and Reventós's theorem.

2.1. Lie foliations.

Let M be an n-dimensional closed manifold and let $\mathcal{F}$ be a codimension q foliation of M. Let $\mathfrak{X}(M)$ be the set of vector fields on M, let $\mathfrak{X}(\mathcal{F})$ be the set of vector fields on M which are tangent to the leaves of $\mathcal{F}$, and let

$$L(M,\mathcal{F}) = \{\, X \in \mathfrak{X}(M) \mid [X, \mathfrak{X}(\mathcal{F})] \subset \mathfrak{X}(\mathcal{F}) \,\}$$

be the set of projectable vector fields. By the bracket of vector fields, the set $\mathfrak{X}(M)$ is a Lie algebra. Then $\mathfrak{X}(\mathcal{F})$ and $L(M,\mathcal{F})$ are Lie subalgebras of $\mathfrak{X}(M)$. By the definition, $\mathfrak{X}(\mathcal{F})$ is an ideal of $L(M,\mathcal{F})$ and, hence, the quotient

$$l(M,\mathcal{F}) = L(M,\mathcal{F}) / \, \mathfrak{X}(\mathcal{F})$$

is a Lie algebra. We call it the *Lie algebra of transverse vector fields*.

A family $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of transverse vector fields which is linearly independent everywhere is called a *transverse parallelism of* $\mathcal{F}$. If there exists a transverse parallelism of $\mathcal{F}$, then the foliation $\mathcal{F}$ is called *transversely parallelizable*.

For a transverse parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of $\mathcal{F}$, the $A^0_b(M,\mathcal{F})$-submodule spanned by $\{\bar{X}_1, \ldots \bar{X}_q\}$ is a Lie subalgebra of $l(M,\mathcal{F})$, where

$$A^0_b(M,\mathcal{F}) = \{\, f \in C^\infty(M) \mid {}^\forall X \in \mathfrak{X}(\mathcal{F}), X(f) = 0 \,\}$$

is the set of basic functions on $(M,\mathcal{F})$. Note that, in general, the vector subspace $\langle \bar{X}_1, \ldots, \bar{X}_q \rangle_{\mathbb{R}}$ spanned by $\{\bar{X}_1, \ldots, \bar{X}_q\}$ over $\mathbb{R}$ may not be a Lie subalgebra; that is, it may not be closed under the Lie bracket.

Definition 2.1. Let $\mathfrak{g}$ be a q-dimensional Lie algebra. A codimension q foliation $\mathcal{F}$ of M is a *Lie* $\mathfrak{g}$*-foliation* if there exists a transverse parallelism $\{\bar{X}_1, \ldots \bar{X}_q\}$ of $\mathcal{F}$ such that the vector subspace $\langle \bar{X}_1, \ldots \bar{X}_q \rangle_{\mathbb{R}}$ is a Lie subalgebra of $l(M,\mathcal{F})$ and is isomorphic to $\mathfrak{g}$.

We call such transverse parallelisms *transverse Lie* $\mathfrak{g}$*-parallelisms*.

Edmond Fedida [2] proved that Lie $\mathfrak{g}$-foliations have the following special property.

Theorem 2.2 ([2])**.** *Let $\mathcal{F}$ be a codimension q Lie $\mathfrak{g}$-foliation of a closed manifold M and let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let $p\colon \widetilde{M} \to M$ be the universal covering of M. Then there exists a locally trivial fibration $D\colon \widetilde{M} \to G$ and a homomorphism $h\colon \pi_1(M) \to G$ such that*

(1) *$D(\alpha \cdot \widetilde{x}) = h(\alpha) \cdot D(\widetilde{x})$ for any $\alpha \in \pi_1(M)$ and any $\widetilde{x} \in \widetilde{M}$ and*
(2) *the lifted foliation $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$ coincides with the foliation defined by the fibers of the fibration D.*

The fibration $D\colon \widetilde{M} \to G$ is called the developing map, the homomorphism $h\colon \pi_1(M) \to G$ is called the holonomy homomorphism, and the image of h is called the holonomy group of $(M, \mathcal{F})$.

Conversely, if there exist such D and h satisfying condition (1) above, then the foliation of $\widetilde{M}$ defined by the fibers of the fibration D induces a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of M such that the developing map is D and the holonomy homomorphism is h.

Example 2.3. Let

$$1 \to K \to H \xrightarrow{D} G \to 1$$

be a exact sequence of simply connected Lie groups. Assume that H has a uniform lattice Δ. Then the surjective homomorphism $D\colon H \to G$ defines a Lie $\mathfrak{g}$-foliation $\mathcal{F}_0$ of the homogeneous space $\Delta\backslash H$. We call this Lie $\mathfrak{g}$-foliation a homogeneous Lie $\mathfrak{g}$-foliation.

Definition 2.4. A Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M is said to be *homogeneous* if there exists a homogeneous Lie $\mathfrak{g}$-foliation $\mathcal{F}_0$ of a homogeneous space $\Delta\backslash H$ such that $(M, \mathcal{F})$ is diffeomorphic to $(\Delta\backslash H, \mathcal{F}_0)$; that is, there exists a diffeomorphism $f\colon M \to \Delta\backslash H$ such that $f(L) \in \mathcal{F}_0$ for any $L \in \mathcal{F}$.

2.2. Basic cohomology.

Let $\mathcal{F}$ be a codimension q foliation of a closed manifold M. Let $A^k(M)$ be the set of differential k-forms on M. A k-form ω is said to be basic if ω satisfies $i_X\omega = 0$ and $i_X d\omega = 0$ for any $X \in \mathfrak{X}(\mathcal{F})$. We denote the set of basic k-forms by $A_b^k(M, \mathcal{F})$.

For a basic form ω, the exterior derivative $d\omega$ of ω is also a basic form. Hence, the set of basic k-forms $A_b^*(M, \mathcal{F})$ is a subcomplex of the de Rham complex $A^*(M)$. The cohomology defined by the subcomplex $A_b^*(M, \mathcal{F})$ is called the *basic cohomology* of the foliated manifold $(M, \mathcal{F})$ and denoted by $H_b^*(M, \mathcal{F})$.

If M is a fiber bundle over a closed manifold N and $\mathcal{F}$ is a foliation of M defined by the set of fibers, then the set of basic k-forms $A_b^k(M, \mathcal{F})$ is identified with the set of k-forms $A^k(N)$ on N via the fibration $\pi\colon M \to N$. In this paper, if M is a fiber bundle over N and $\mathcal{F}$ is a foliation defined by the set of fibers, we always identify the set of basic k-forms of $(M, \mathcal{F})$ with the set of k-forms on N.

2.3. Contact manifolds.

Let M be a $(2n + 1)$-dimensional closed manifold.

Definition 2.5. A non-singular one-form α on M is called a *contact form* on M if α satisfies

$$\alpha \wedge (d\alpha)^n \neq 0 \tag{2.1}$$

everywhere on M.

The pair (M, α) is called a contact manifold.

Definition 2.6. Let (M, α) be a contact manifold. The non-singular vector field X on M defined by the following equations

$$\begin{aligned} i_X \alpha = 1 \quad &\text{and} \\ i_X d\alpha = 0. & \end{aligned} \tag{2.2}$$

is called the *Reeb vector field* or the *characteristic vector field* of the contact manifold (M, α).

The one-dimensional foliation $\mathcal{F}$ defined by the integral curves of X is called the *characteristic foliation* of the contact manifold (M, α).

By equations (2.1) and (2.2), the exterior derivative $d\alpha$ of the contact form α is a basic 2-form and a transverse symplectic form of the characteristic foliation $\mathcal{F}$.

Example 2.7. Let H be a simply connected Lie group with a uniform lattice Δ. Suppose that H has a left-invariant contact form $\widetilde{\alpha}_0$. Then $\widetilde{\alpha}_0$ induces a contact form α_0 on the homogeneous space $\Delta \backslash H$.

We call the contact form α_0 on the homogeneous space $\Delta \backslash H$ a homogeneous contact form and call the contact manifold $(\Delta \backslash H, \alpha_0)$ a homogeneous contact manifold.

Definition 2.8. A contact form α on a closed manifold M is said to be *homogeneous* if there exists a homogeneous contact form α_0 on a homogeneous space $\Delta \backslash H$ and a diffeomorphism $f \colon M \to \Delta \backslash H$ such that $f^* \alpha_0 = \alpha$. A contact manifold (M, α) is said to be homogeneous if the contact form α is homogeneous.

2.4. Contactization.

Let ω be a symplectic form on a $2n$-dimensional closed manifold N. We assume that the cohomology class $[\omega]$ of ω is in the second cohomology of integer coefficients $H^2(N; \mathbb{Z})$.

Let M be a principal S^1-bundle over N whose Euler class is $-[\omega]$ and $\mathcal{F}$ be the flow on M defined by the fundamental vector field X on M. Then we can take a connection form α on M such that the curvature form $d\alpha$ coincides with ω as follows.

First, we take an arbitrary connection form α' on M. Then the curvature form is given by $d\alpha' \in \mathrm{A}_b^2(M, \mathcal{F}) = A^2(N)$ and the Euler class of the

S^1-bundle is given by $-[d\alpha']$. On the other hand, since the Euler class of M is $-[\omega]$, we have $-[\omega] = -[d\alpha'] \in H^2_{dR}(N) = H^2_b(M, \mathcal{F})$. Hence, there exists a basic one-form $\beta \in A^1_b(M, \mathcal{F})$ such that $-\omega = -d\alpha' + d\beta$. Take $\alpha = \alpha' - \beta$. Then α is a connection form and the curvature form is

$$d\alpha = d(\alpha' - \beta) = \omega.$$

Let α be a connection form on M such that $d\alpha = \omega$. Then α satisfies

$$i_X(\alpha \wedge (d\alpha)^n) = (d\alpha)^n = \omega^n. \tag{2.3}$$

Since ω is a symplectic form, equation (2.3) shows that the connection form α is a contact form on M.

This contact manifold (M, α) is called the *contactization* of the symplectic manifold (N, ω).

By the construction, the characteristic foliation of the contact manifold (M, α) coincides with the foliation $\mathcal{F}$ defined by the fibers.

2.5. Llabrés and Reventós's theorem.

A Lie $\mathfrak{g}$-flow $\mathcal{F}$ on a closed manifold M is said to be a contact Lie $\mathfrak{g}$-foliation if there exists a contact form α on M such that the one-dimensional foliation $\mathcal{F}$ coincides with the characteristic foliation of the contact manifold (M, α).

Llabrés and Reventós gave a necessary and sufficient condition for a Lie $\mathfrak{g}$-flow on a closed 3-manifold to be a contact Lie $\mathfrak{g}$-foliation.

Theorem 2.9 ([5], Theorem 2)**.** *Let $\mathcal{F}$ be a unimodular Lie $\mathfrak{g}$-flow on a closed 3-manifold M. Then the following conditions are equivalent.*

(1) *The Euler class $e(\mathcal{F})$ of $\mathcal{F}$ is non-zero.*
(2) *$\mathcal{F}$ is a contact Lie $\mathfrak{g}$-foliation.*

A codimension q foliation $\mathcal{F}$ is said to be unimodular if $H^q_b(M, \mathcal{F}) \neq \{0\}$. If $\mathcal{F}$ is a unimodular Lie $\mathfrak{g}$-flow, then $\mathcal{F}$ is an isometric flow with respect to some Riemannian metric g. The Euler class $e(\mathcal{F})$ means the Euler class of the isometric flow $(\mathcal{F}, g)$. This class does not depend on the choice of Riemannian metric g up to a non-zero constant multiple. In particular, the vanishing of the Euler class of $\mathcal{F}$ does not depend on the choice of the Riemannian metric g.

If $\mathcal{F}$ is a unimodular Lie $\mathfrak{g}$-flow on a closed 3-manifold, then the dimension of $\mathfrak{g}$ is two. Since $\mathcal{F}$ is unimodular, the Lie algebra $\mathfrak{g}$ is unimodular and, hence, is isomorphic to $\mathbb{R}^2$. If the one-dimensional Lie $\mathbb{R}^2$-foliation $\mathcal{F}$ has no closed orbits, the M is diffeomorphic to the 3-dimensional torus T^3 and $\mathcal{F}$ is diffeomorphic to a linear flow on T^3 (see [1]). Hence, if $\mathcal{F}$ has no closed orbits, then $e(\mathcal{F}) = 0$. Thus, in this case, the condition that $e(\mathcal{F}) \neq 0$ is equivalent to the condition that any orbit of $\mathcal{F}$ is closed and the Euler class of the oriented S^1-bundle $\pi\colon M \to M/\mathcal{F}$ is non-trivial.

3. A Generalization of Llabrés and Reventós's Theorem

In this section, we prove the following theorem, which is a generalization of Theorem 2.9.

Theorem 3.1. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Then $\mathcal{F}$ is a contact Lie $\mathfrak{g}$-foliation if and only if $\mathcal{F}$ satisfies the following conditions.*

(a) *Any orbit of $\mathcal{F}$ is closed.*
(b) *There exists a symplectic form ω on the leaf space $N = M/\mathcal{F}$ such that $[\omega] \in H^2(N;\mathbb{Z})$ and $e(\mathcal{F}) = -[\omega]$.*

Proof. Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold. Let $\{\bar{X}_1, \ldots, \bar{X}_{2n}\}$ be a transverse Lie $\mathfrak{g}$-parallelism of $\mathcal{F}$.

Suppose that there exists a contact form α on M such that $\mathcal{F}$ is a characteristic foliation of the contact manifold (M,α). Let X be the Reeb vector field of the contact manifold (M,α).

Since Lie foliations are Riemannian foliations, by [10, Proposition 1] and [11, Corollary 1], there exists a closed orbit of $\mathcal{F}$. Hence, any orbit of $\mathcal{F}$ is closed.

Since any orbit of the Lie $\mathfrak{g}$-flow $\mathcal{F}$ is closed, the leaf space $M/\mathcal{F}$ is a closed manifold and M is a principal S^1-bundle over $M/\mathcal{F}$. The transverse symplectic form $d\alpha$ defines a symplectic form on $M/\mathcal{F}$. The cohomology class of this 2-form is opposite of the Euler class $e(\mathcal{F})$, so is an integral class.

Conversely, we assume that the Lie $\mathfrak{g}$-flow $\mathcal{F}$ satisfies conditions (a) and (b). Then the leaf space $N = M/\mathcal{F}$ is a closed manifold and M is a principal S^1-bundle over N with the Euler class $-[\omega]$.

By using the contactization of the symplectic manifold (N,ω), we obtain a contact manifold (M',α'), where the manifold M' is a principal S^1-bundle M' over N whose Euler class is $-[\omega]$. Since $e(\mathcal{F}) = -[\omega]$, the principal S^1-bundle M is diffeomorphic to the principal S^1-bundle M'. Hence, M has a contact form α such that the characteristic foliation coincides with $\mathcal{F}$. □

Remark 3.2. In the case where $n = 1$, conditions (a) and (b) are equivalent to the conditions that $\mathcal{F}$ is unimodular and $e(\mathcal{F}) \neq 0$. Hence, Theorem 3.1 is a generalization of Theorem 2.9.

4. Contact Lie Foliations with the Algebraic Euler Class

In the case where the Euler class of $\mathcal{F}$ is algebraic, then we can construct a homogeneous contact form on M.

Fix a developing map $D\colon \widetilde{M} \to G$ and a holonomy homomorphism $h\colon \pi_1(M) \to G$. Let Γ be the holonomy group. Then the set of basic k-forms $A_b^k(M,\mathcal{F})$ is identified with the set of Γ-invariant k-forms on G

$$A_\Gamma^k(G) = \{\omega \in A^k(G) \mid {}^\forall\gamma \in \Gamma, \gamma^*\omega = \omega\}$$

via the developing map D. On the other hand, by identifying $\mathfrak{g}$ with the set of left-invariant vector fields on G, we identify $\bigwedge^k \mathfrak{g}^*$ with the set of left-invariant k-forms on G. Hence, we obtain the natural inclusion map

$$\iota\colon \bigwedge^k \mathfrak{g}^* \to A_\Gamma^k(G).$$

A basic k-form $\omega \in A_b^k(M,\mathcal{F})$ is said to be algebraic if ω is in $\bigwedge^k \mathfrak{g}^*$ via the above identification. A cohomology class $[\omega]$ is said to be algebraic if $[\omega]$ is represented by an algebraic form.

Theorem 4.1. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Suppose that $\mathcal{F}$ satisfies conditions* (a) *and*

(b′) *there exists an algebraic symplectic form $\omega \in A_b^2(M,\mathcal{F}) = A^2(N)$ such that $[\omega] \in H^2(N;\mathbb{Z})$ and $e(\mathcal{F}) = -[\omega]$.*

Then

(i) *the Lie $\mathfrak{g}$-flow $\mathcal{F}$ is homogeneous and*

(ii) *there exists a contact form α on M such that α is homogeneous and the characteristic foliation of (M,α) coincides with $\mathcal{F}$.*

To prove Theorem 4.1, we use the following theorem (see [4, Theorem 5.1] and its proof.

Theorem 4.2 ([4]). *Let $\mathcal{F}$ be a unimodular Lie $\mathfrak{g}$-flow on a closed manifold M. If the Euler class of $\mathcal{F}$ is algebraic, then $\mathcal{F}$ is homogeneous.*

Sketch of Proof: Let $\mathcal{F}$ be a unimodular Lie $\mathfrak{g}$-flow on a closed manifold M. Suppose that there exists a 2-form $\beta \in \bigwedge^2 \mathfrak{g}^*$ such that the Euler class $e(\mathcal{F})$ of $\mathcal{F}$ is represented by an algebraic 2-form $-\iota(\beta) \in A_b^2(M,\mathcal{F})$. Then there exists a transverse Lie $\mathfrak{g}$-parallelism $\{\bar{X}_1,\dots,\bar{X}_{2n}\}$ of $\mathcal{F}$ and a non-singular vector field $X \in \mathfrak{X}(\mathcal{F})$ which satisfy the following equations

$$\begin{aligned}[X_i,X_j] &= \sum_{i=1}^{2n} a_{ij}^k X_k + b_{ij}X \quad \text{and} \\ [X,X_j] &= 0,\end{aligned} \tag{4.1}$$

where a_{ij}^k are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\bar{X}_1,\dots,\bar{X}_{2n}\}$ and $b_{ij} = -2\beta(\bar{X}_i,\bar{X}_j)$ are constants.

Hence, $\langle X, X_1,\dots,X_{2n}\rangle_{\mathbb{R}}$ is a Lie algebra, which is a central extension

$$0 \to \mathbb{R} \to \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

of $\mathfrak{g}$ by $\mathbb{R}$ with the Euler class $[-2\beta] \in H^2(\mathfrak{g})$. Then the simply connected Lie group H of $\mathfrak{h}$ acts transitively on M.

This action induces a diffeomorphism $f\colon (\Delta\backslash H, \mathcal{F}_0) \to (M, \mathcal{F})$, where Δ is an isotropy subgroup of H, which is a uniform lattice of H, and $\mathcal{F}_0$ is the homogeneous flow on $\Delta\backslash H$ defined by $p\colon \mathfrak{h} \to \mathfrak{g}$. □

Now, we prove Theorem 4.1.

Proof of Theorem 4.1: Let $\omega = \iota(\beta) \in A_b^2(M, \mathcal{F})$ be an algebraic symplectic form such that $e(\mathcal{F}) = -[\omega]$.

By assumption (a), $\mathcal{F}$ is unimodular. Hence, by Theorem 4.2, the Lie $\mathfrak{g}$-flow $\mathcal{F}$ is homogeneous.

Let $(\Delta\backslash H, \mathcal{F}_0)$ be the homogeneous Lie $\mathfrak{g}$-flow and $f\colon \Delta\backslash H \to M$ be the diffeomorphism constructed in the proof of Theorem 4.2. Let $\{\eta, \eta^1, \ldots, \eta^{2n}\}$ be the dual basis of $\mathfrak{h}^*$ for the basis $\{X, X_1, \ldots, X_{2n}\}$ of $\mathfrak{h}$. By equations (4.1), we have

$$d\eta = -\frac{1}{2}\sum_{i<j} b_{ij}\eta^i \wedge \eta^j.$$

On the other hand, we have

$$f^*\omega(X_i, X_j) = \omega(X_i, X_j) = \beta(\bar{X}_i, \bar{X}_j) = -\frac{1}{2}b_{ij} \quad \text{and}$$
$$f^*\omega(X, X_i) = 0$$

for each i and j. Hence, we have

$$f^*\omega = -\frac{1}{2}\sum_{i<j} b_{ij}\eta^i \wedge \eta^j.$$

Therefore, the one-form η satisfies

$$d\eta = f^*\omega.$$

By assumption (b′), the 2-form $\omega \in \bigwedge^2 \mathfrak{g}^*$ is a symplectic form. Hence, the one-form $\eta \in \mathfrak{h}^*$ is a left-invariant contact form on H. Therefore, $(M, f^*\eta)$ is a contact manifold with the characteristic foliation $\mathcal{F}$ which is diffeomorphic to the homogeneous contact manifold $(\Delta\backslash H, \eta)$. □

Let $\mathfrak{g}$ be a Lie algebra, let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$, and let Γ be a uniform lattice of G. If the cohomology group $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is isomorphic to the de Rham cohomology group $H^*_{dR}(\Gamma\backslash G)$ of the homogeneous space $\Gamma\backslash G$ via the map

$$\iota\colon H^*(\mathfrak{g}) \to H^*_{dR}(\Gamma\backslash G)$$

which is induced by the natural inclusion map $\iota\colon \bigwedge^* \mathfrak{g}^* \to A^*_\Gamma(G)$, then condition (b) is equivalent to condition (b′).

The condition that $H^*(\mathfrak{g})$ is isomorphic to $H^*_{dR}(\Gamma\backslash G)$ always holds if $\mathfrak{g}$ is nilpotent (see [8]). More generally, if $\mathfrak{g}$ is of type (R), then $H^*(\mathfrak{g})$ is isomorphic to $H^*_{dR}(\Gamma\backslash G)$ (see [3]), where a Lie algebra $\mathfrak{g}$ is of type (R) if all the eigenvalues of the adjoint representation $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ are real for any $X \in \mathfrak{g}$. Thus, we have the following corollary.

Corollary 4.3. *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(2n+1)$-dimensional closed manifold M. Suppose that $\mathfrak{g}$ is of type* (R)*. Then any contact form on M with the characteristic foliation $\mathcal{F}$ is homotopic to a homogeneous contact form as a non-singular one-form.*

Proof. Let α_0 be a contact form on M with the characteristic foliation $\mathcal{F}$. Let X be the Reeb vector field of (M, α_0). By Theorem 3.1, Theorem 4.1, and the above remark, $(M, \mathcal{F})$ is diffeomorphic to a homogeneous Lie $\mathfrak{g}$-flow and there exists an homogeneous contact form α_1 on M such that the characteristic foliation of (M, α_1) is $\mathcal{F}$. Then two contact forms α_0 and α_1 have the same characteristic foliation. Since $\mathcal{F}$ is the characteristic foliation of (M, α_1), we may assume that $\alpha_1(X) > 0$.

Then $\alpha_t = t\alpha_0 + (1-t)\alpha_1$ is a homotopy between non-singular one-forms α_0 and α_1. Since $\alpha_0(X) = 1$ and $\alpha_1(X) > 0$, α_t is non-singular for any $t \in [0, 1]$. □

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Graduate School of Mathematical Sciences; University of Tokyo; 3-8-1 Komaba Meguro-ku; Tokyo 153-9814, Japan

E-mail address: knaoki@ms.u-tokyo.ac.jp