Lie Flows on Contact Manifolds

by

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foliation of a contact form. Their work was extended by M. Llabrés and A. Reventós [6] for a Lie $\mathfrak{g}$-flow on a closed 3-manifold. More precisely, they gave a necessary and sufficient condition for a Lie $\mathfrak{g}$-flow $\mathcal{F}$ on a closed 3-manifold M to be the characteristic foliation of a contact form α. The aim of this paper is to generalize their result for an arbitrary odd-dimensional closed manifold.

Let M be a closed $(2n+1)$-dimensional manifold and let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on M. Then there exists a 1-form ω on M which defines $\mathcal{F}$. We denote by X the vector field on M with $\omega(X) = 1$. Let g be the dimension of the Lie algebra $\mathfrak{g}$.

The following theorem is our main result.

Theorem 1.1. Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a closed $(2n+1)$-dimensional manifold M defined by a 1-form ω. Then $\mathcal{F}$ is the characteristic foliation of a contact form if and only if there exists a closed 2-form β on M such that $\beta^n \wedge \omega$ is nowhere zero.

We will prove this theorem in Section 3.

In Section 4, for a Lie flow $\mathcal{F}$ satisfying some cohomological condition, we will construct a good contact form such that the characteristic foliation coincides with $\mathcal{F}$.

2. Preliminaries

In this section, we recall some definitions and basic facts on Lie foliations. See [8] for more details.

Definition 2.1. Let M be a closed manifold and let $\mathfrak{g}$ be a Lie algebra of dimension q. A codimension q foliation $\mathcal{F}$ on M is a Lie $\mathfrak{g}$-foliation if there exist a $\mathfrak{g}$-valued 1-form ω on M and a Lie algebra $\mathfrak{g}$ such that

(1) $\omega_x : T_x M \to \mathfrak{g}$ is surjective for each $x \in M$,

(2) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

Then the foliation $\mathcal{F}$ is defined by the equation $\omega = 0$.

For a Lie $\mathfrak{g}$-foliation, the following fundamental theorem is known.

Theorem 2.2 (Fedida [2], [3]). Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation on a closed manifold M. Let $\widetilde{M}$ be the universal covering of M. Then there exist a homomorphism $D : \pi_1(M) \to G$, a fibration