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THE SHARKOVSKY PROGRAM OF CLASSIFICATION OF TRIANGULAR MAPS – A SURVEY

MARTA ŠTEFÁNKOVÁ

Abstract. For continuous interval maps there are more than 50 mutually equivalent conditions characterizing maps with zero topological entropy. At the end of the 1980s, A. N. Sharkovsky proposed to verify which of the implications among these conditions are valid in the class of triangular maps of the unit square. Since some conditions are not applicable to maps of the square, whereas some new conditions have been added thereafter, the contemporary list usually contains 32 conditions which means nearly 1,000 possible implications. This huge program has been recently completed and the aim of this paper is to give a survey of the results.

1. Introduction

Recently a paper concluding the so-called Sharkovsky classification program of triangular maps was published (see [10]). The program was announced in late 1980s and the work on it lasted for 25 years. During this time 16 authors created 24 papers devoted directly to this problem (these papers are in the references marked by *) and many other authors contributed to this subject, for example, by providing alternative and more straightforward solutions of problems formulated in this program. (Almost 70 papers related to this program are known to the author.)

Let (X, ρ) be a compact metric space with a metric ρ, let $\mathcal{C}(X)$ be the class of continuous maps from X to itself, and let $I = [0, 1]$ be the

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compact unit interval. A *triangular map* is a map $F \in \mathcal{C}(I \times I)$ of the form $F(x, y) = (f(x), g_x(y))$. The map $f \in \mathcal{C}(I)$ is the *base map* of F, and $g_x : I \to I,\ x \in I$, is a family of *fiber maps* from $\mathcal{C}(I)$ depending continuously on x.

In 1979 Peter E. Kloeden [17] proved that the Sharkovsky theorem on coexistence of periods of interval maps is valid also for triangular maps. On the other hand, in 1989 S. F. Kolyada proved that there are triangular maps of type 2^∞ with positive topological entropy ([21], see also [22]), which is impossible for interval maps. Inspired by these results, A. N. Sharkovsky in his talk (based on his joint work with Kolyada [23],) at the European Conference on Iteration Theory: ECIT 89, formulated his program of classification of triangular maps: For continuous interval maps there are more than 50 conditions characterizing maps with zero topological entropy. These properties are, for example, zero topological entropy on sets of recurrence, non-chaoticity of a map restricted to a set of recurrence (by "chaos" it is meant either the chaos in the sense of Li and Yorke or the distributional chaos), types of ω-limit sets.

He proposed to check which of the implications among these conditions are valid also for the triangular maps. Some of Sharkovsky's conditions are not applicable for triangular maps (e.g., the non-existence of a horseshoe). On the other hand, in the course of time new conditions appeared which are equivalent to positive topological entropy for the interval maps and are applicable for triangular maps (e.g., the conditions related to distributional chaos). The current list consists of 32 conditions; that is, there are 992 possible implications which must be verified. The most recent paper where all these conditions are listed is [26]; let us also note that the systematic approach was commenced by Zdeněk Kočan in 1999 (see [18]).

Since 2006, 23 equivalence classes have been known (see [7], [18], and [20]), which gives "only" 506 implications or non-implications. The last question which remained open was whether zero topological entropy on the set of almost periodic points is equivalent to zero entropy on the set of uniformly recurrent points. Note that this problem was solved negatively in [10].

The aim of the present paper is to provide a survey of the results with some comments concerning, for example, methods of proofs, and to give an extensive list of references. The paper is organized as follows. In the next section we give the complete list of properties of the classification program together with all definitions. Then we describe a useful method of construction of a parametric family of triangular maps. In section 4 we give a diagram and a table summarizing the results. The last two sections are devoted to some comments concerning, for example, methods of proofs or results for the triangular maps monotone on the fibers.

2. Properties and Definitions

The complete list of 32 properties in 23 equivalence classes is given below. Note that letters are used to denote the equivalent conditions. The explanation of symbols and notions follow the list.

(1) (a) $h(f) = 0$
(b) $h(f|\mathrm{CR}(f)) = 0$
(c) $h(f|\Omega(f)) = 0$
(d) $h(f|\omega(f)) = 0$
(e) $h(f|\mathrm{C}(f)) = 0$
(f) $h(f|\mathrm{Rec}(f)) = 0$
(2) (a) $h(f|\mathrm{UR}(f)) = 0$
(b) there is no minimal set with positive topological entropy
(3) $h(f|\mathrm{AP}(f)) = 0$
(4) (a) $h(f|\mathrm{Per}(f)) = 0$
(b) the period of every cycle of f is a power of 2
(c) every cycle is simple
(5) f has no homoclinic trajectory
(6) $f|\mathrm{CR}(f)$ is not LYC
(7) $f|\Omega(f)$ is not LYC
(8) $f|\omega(f)$ is not LYC
(9) $f|\mathrm{C}(f)$ is not LYC
(10) $f|\mathrm{Rec}(f)$ is not LYC
(11) $f|\mathrm{UR}(f)$ is not LYC
(12) $\mathrm{UR}(f) = \mathrm{Rec}(f)$
(13) (a) no infinite ω-limit set contains a cycle
(b) every ω-limit set either is a cycle or contains no cycle
(14) every ω-limit set contains a unique minimal set
(15) f is not DC1
(16) f is not DC2
(17) f is not DC3
(18) the trajectory of every point is strongly approximable
(19) the trajectory of every point is weakly approximable
(20) if $\omega_f(x) = \omega_{f^2}(x)$, then $\omega_f(x)$ is a fixed point
(21) there is no infinite countable ω-limit set
(22) the trajectories of every two points are correlated
(23) for every closed invariant set A and every $m \in \mathbb{N}$, the map $f^m|A$ cannot be topologically almost conjugate to the shift

Now, let us give the definitions of the notions used in the above mentioned properties. For $f \in C(X)$ and $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers), we denote by f^n the nth *iterate* of f. The *trajectory* of a point x under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$.

The set $\omega_f(x)$ of accumulation points of the trajectory of x under f is the *ω-limit set* of x. A subset M of X is a *minimal set* if it is closed, nonempty, and invariant (i.e., $f(M) \subset M$) and if it contains no proper subset with these three properties. A nonempty closed set $M \subset X$ is minimal if and only if the orbit of every point of M is dense in M or, equivalently, $\omega_f(x) = M$ for every $x \in M$.

Let us now recall notions of recurrence. A point $x \in X$ is called

- *fixed*, if $f(x) = x$;
- *periodic*, if $f^n(x) = x$ for some $n \in \mathbb{N}$;
- *chain recurrent*, if, for any $\varepsilon > 0$, there is a sequence of points $\{x_i\}_{i=0}^{n}$ with $x_0 = x = x_n$ and $\rho(x_{i+1}, f(x_i)) < \varepsilon$, for $i \in \{0, \dots, n-1\}$;
- *non-wandering*, if, for any neighborhood U of x, there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$;
- *recurrent*, if $x \in \omega_f(x)$;
- *uniformly recurrent*, if, for any neighborhood U of x, there exists an $n \in \mathbb{N}$ such that if $f^m(x) \in U$ where $m \geq 0$, then $f^{m+i}(x) \in U$ for some i with $0 < i \leq n$;
- *almost periodic*, if, for any neighborhood U of x, there is an $n \in \mathbb{N}$ such that $f^{in}(x) \in U$, for any $i \in \mathbb{N}$.
- The *centre* of f is the closure of the set of recurrent points.

By $\mathrm{CR}(f)$, $\Omega(f)$, $\omega(f)$, $\mathrm{Rec}(f)$, $\mathrm{UR}(f)$, $\mathrm{AP}(f)$, $\mathrm{Per}(f)$ and $\mathrm{Fix}(f)$ we denote the set of *chain recurrent, non-wandering, ω-limit, recurrent, uniformly recurrent, almost periodic, periodic*, and *fixed points* of f, respectively. Recall that

$$\mathrm{Fix}(f) \subset \mathrm{Per}(f) \subset \mathrm{AP}(f) \subset \mathrm{UR}(f) \subset \mathrm{Rec}(f) \begin{array}{c} \subset \mathrm{C}(f) \\ \subset \omega(f) \end{array} \subset \Omega(f) \subset \mathrm{CR}(f).$$

Now proceed with the definition of topological entropy. A set $A \subset X$ is *(n,ε)-separated* if, for any $x, y \in A$, $x \neq y$, there is an $i \in \mathbb{N}$ with $0 \leq i < n$ and $\rho(f^i(x), f^i(y)) > \varepsilon$. For $Y \subset X$, denote by $s_n(\varepsilon, Y, f)$ the maximal possible number of points in an (n,ε)-separated subset of Y. The *topological entropy of f with respect to Y* is

$$h(f|Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, Y, f),$$

and the *topological entropy of f* is

$$h(f) = h(f|X).$$

We continue with definitions of Li-Yorke and distributional chaos.

A map f is *chaotic in the sense of Li and Yorke* (briefly, *LYC*) if there is an *LY-scrambled pair*, i.e., if there exist $x, y \in X$ and $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > \varepsilon.$$

For any pair (x, y) of points in X and any $n \in \mathbb{N}$, define a *distribution function* $\Phi_{xy}^{(n)} : (0, \infty) \to I$ by

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n}\#\{0 \le i < n;\ \rho(f^i(x), f^i(y)) < t\}.$$

The *lower* and *upper distribution function* generated by f and the pair (x, y) is

$$\Phi_{xy}(t) = \liminf_{n\to\infty}\ \Phi_{xy}^{(n)}(t), \quad \text{and} \quad \Phi_{xy}^*(t) = \limsup_{n\to\infty}\ \Phi_{xy}^{(n)}(t),$$

respectively. Obviously $\Phi_{xy}(t) \le \Phi_{xy}^*(t)$, for any $t \in (0, \infty)$.

If there is a pair (x, y) of points in X such that

(DC1) $$\Phi_{xy}^* \equiv 1 \text{ and } \Phi_{xy}(t) = 0, \text{ for some } t > 0, \text{ or}$$

(DC2) $$\Phi_{xy}^* \equiv 1 \text{ and } \Phi_{xy} < \Phi_{xy}^*, \text{ or}$$

(DC3) $$\Phi_{xy} < \Phi_{xy}^*,$$

where $\Phi_{xy}^* \equiv 1$ means that $\Phi_{xy}^*(t) = 1$ for any $t \in (0, \infty)$, and $\Phi_{xy} < \Phi_{xy}^*$ means that $\Phi_{xy}(t) < \Phi_{xy}^*(t)$ for all t in some nondegenerate interval, then this pair is called the *distributionally scrambled pair* and we say that f exhibits *distributional chaos of type 1-3*, (briefly *DC1, DC2*, and *DC3*, respectively). Directly from the definitions it follows that DC1 is the strongest and DC3 the weakest version of distributional chaos. Note that for $f \in \mathcal{C}(I)$ these three notions are equivalent (see [31]).

We remark that originally in definitions of both Li-Yorke and distributional chaos the existence of an *uncountable* scrambled set $S \subset X$ (i.e., such that any pair of distinct points from S form a scrambled pair) was required. However, in the case of interval maps, the existence of two-points and an uncountable perfect scrambled set are equivalent (for Li-Yorke chaos it was proved in [27] and for distributional chaos in [31]).

Let $\varepsilon > 0$. The trajectory of a point $x \in X$ is *strongly ε-approximable* by the trajectory of a set A if there exists $i \in \mathbb{N}$ such that $\operatorname{diam}(f^i(A)) < \varepsilon$ and

$$\lim_{n\to\infty} \rho(f^n(x), f^n(A)) = 0,$$

and it is *weakly ε-approximable* by the trajectory of a set A if there exist $i, n_0 \in \mathbb{N}$ such that $\operatorname{diam}(f^i(A)) < \varepsilon$ and, for any $n \ge n_0$,

$$\rho(f^n(x), f^n(A)) < \varepsilon.$$

The trajectory of a point $x \in X$ is *strongly* (*weakly*, respectively) *approximable* if it is strongly (weakly, respectively) ε-approximable for any $\varepsilon > 0$.

Let $x \in \mathrm{Fix}(f)$ and let x_n, $n = 1, 2, \ldots$, be distinct points in X such that $f(x_{n+1}) = x_n$, for any n, $f(x_1) = x$, and $\lim_{n\to\infty} x_n = x$. Then $\{x_n\}_{n=1}^{\infty}$ is a *homoclinic trajectory related to the point* x. A homoclinic trajectory related to a periodic orbit is defined similarly, see, e.g., [7].

Trajectories of points $x, y \in X$ are *correlated* if either $\omega_f(x)$ or $\omega_f(y)$ is a fixed point or

$$\omega_{f\times f}(x, y) \neq \omega_f(x) \times \omega_f(y),$$

where the map $f \times f : X \times X \to X \times X$ is given by $(x, y) \mapsto (f(x), f(y))$.

The *shift* (Σ, σ) is the space $\Sigma = \{0, 1\}^{\mathbb{N}}$ of sequences $x_1x_2\ldots$ of two symbols (namely, 0's and 1's) with the map $\sigma : x_1x_2\ldots \mapsto x_2x_3\ldots$. A map $f \in C(X)$ is *topologically almost conjugate* to the shift if there exists a continuous surjective map $\psi : X \to \Sigma$, such that $\psi \circ f = \sigma \circ \psi$ and any point from Σ has at most two preimages in X.

3. A Parametric Family of Maps

During the work on the classification program some useful tools for working with triangular maps have been developed. In my opinion, the most important tool is a method of defining triangular maps using a special parametric family of fiber maps. The idea of this construction appeared for the first time in [14]. As a parametric family it was introduced in [3] and then improved in [9] and [35].

Let us outline first the main ideas and then some properties of this construction.

Let $Q = \{0, 1\}^{\mathbb{N}}$ be the middle-thirds Cantor set represented as the set of sequences of two symbols, and let τ be the (binary) *adding machine* or *odometer* on Q defined by $\tau(x_1x_2x_3\ldots) = x_1x_2x_3\cdots + 1000\ldots$, where the adding is mod 2 with carry; thus, e.g., $\tau(11010\ldots) = 00110\ldots$.

Let the base map $f \in \mathcal{C}$ be an extension of τ such that f is affine on every interval complementary to Q. Then f is of type 2^∞. For every $m = 2^n \in \mathbb{N}$, there is a system $J_0, J_1, \cdots, J_{m-1}$ of m mutually disjoint compact f-periodic intervals with endpoints in Q, forming a periodic orbit of f of period m such that every non-periodic $x \in I$ is by f eventually mapped into J_0. Moreover, for every $i \in \mathbb{N} \cup \{0\}$, $f(J_i) = J_{i+1}$ if i is taken mod(m) and every $x \in J_i \cap Q$ begins with the same block β_i of the first n digits. In particular, $\beta_0 = 0^n$ is the block of n zeros, $\beta_{m-1} = 1^n$, and, for $i < m - 1$, β_i consists of the first n digits of $\tau^i(0^\infty)$.

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers of the form $n_k = 2^{c_k}$, $k, c_k \in \mathbb{N}$, with $c_k \geq 2$. Write any $x = x_1x_2x_3 \cdots \in Q$ in blocks as

$$x = x^1x^2x^3 \ldots, \text{ where } x^j \text{ is the block of } c_j \text{ digits of } x. \tag{3.1}$$

Finally, for any finite block $\alpha = x_sx_{s+1} \cdots x_{s+k}$ of zeros and ones, $e(\alpha) = x_s + 2x_{s+1} + 2^2x_{s+2} + \cdots + 2^kx_{s+k}$ is the *evaluation* of α. For any family

$$\Psi := \{\psi_k^{(j)}; 0 \leq j \leq n_k - 2\}_{k\in\mathbb{N}} \tag{3.2}$$

of maps from $\mathcal{C}(I)$, define a triangular map $F_\Psi : Q \times I \to Q \times I$ by $F_\Psi(x, y) := (\tau(x), y)$ if $x = 1^\infty$ (i.e., if x contains no zero digit). Otherwise, let x^k be the first block in (3.1) containing a zero digit, and let

$$F_\Psi(x, y) := (\tau(x), \psi_k^{(p)}(y)), \text{ where } p = e(x^k).$$

Now, we have to define the fiber maps g_x for $x \in I \setminus Q$. If x belongs to an interval (a, b) complementary to Q, then there is a unique $t_x \in (0, 1)$ such that $x = t_xa + (1 - t_x)b$. Put $g_x = t_xg_a + (1 - t_x)g_b$ and let $F(x, y) = (f(x), g_x(y))$, $x, y \in I$. Obviously, $F : I \times I \to I \times I$ defined in this way is continuous.

If the family Ψ in (3.2) is taken such that

$$\lim_{k\to\infty} \max_j ||\psi_k^{(j)} - \mathrm{Id}|| = 0,$$

where Id denotes the identity map on I and $||\cdot||$ the uniform norm, then F_Ψ (and hence, F) is continuous, and if

$$\psi_k^{(n_k-2)} \circ \psi_k^{(n_k-3)} \circ \cdots \circ \psi_k^{(1)} \circ \psi_k^{(0)} = \psi_k^{(0)} = \mathrm{Id},\ k \in \mathbb{N}, \tag{3.3}$$

then some recurrence formulae are valid.

Lemma 3.1. (See [9].) *For $x \in Q$, $y \in I$, and $i \in \mathbb{N}$, let the symbol $y_x(i)$ denotes the second coordinate of $F_\Psi^i(x, y)$. Then, for any $j, k \in \mathbb{N}$ such that $1 \leq j < n_{k+1}$,* (3.3) *implies*

$$y_0(j \cdot m_k) = \psi_{k+1}^{(j-1)} \circ \psi_{k+1}^{(j-2)} \circ \cdots \circ \psi_{k+1}^{(1)} \circ \psi_{k+1}^{(0)}(y),$$

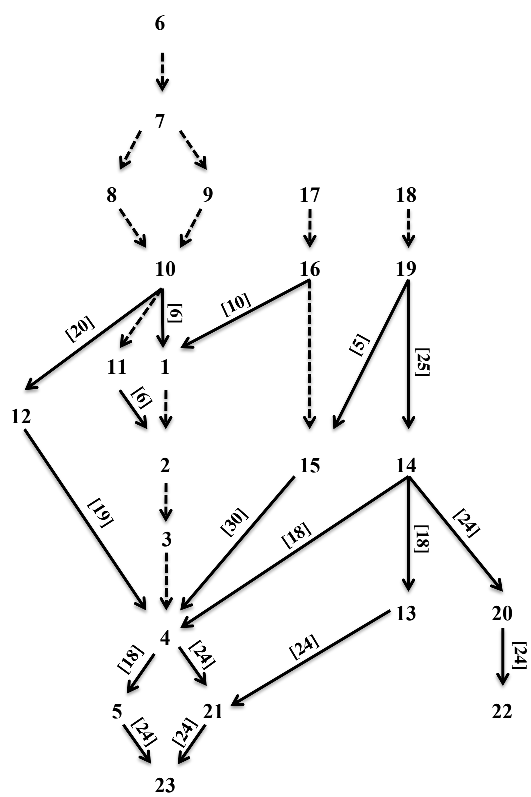
where $m_k := n_1n_2n_3 \cdots n_k$. In particular, $y_0(m_k) = y_0(0)$ $(= y)$.

So we are able to calculate in an easy way second coordinates of some points in the trajectories of points from the fiber I_0. Note that in [35] an even stronger tool can be found; it is related to the calculation of second coordinates of points in the trajectories starting from fibers I_x, where $x \in Q$ begins with a block of zero digits.

4. Diagram and Table

In this section we survey the results first in a diagram displaying vividly the relations between the properties. In this diagram the arrows mean implications, the dashed arrows are the implications which follow directly by the definitions. The implications stated in this diagram (together with implications generated by transitivity) are all existing implications between these properties.

Then a table containing all implications and non-implications is given. The symbol of implication (non-implication, respectively) on the ith row and jth column means that property i implies (does not imply, respectively) property j. The implications following directly by definitions are marked by "def" above the symbol of implication. The remaining implications and non-implications have been either proved or disproved by giving a counterexample (we call them *essential* and they are given with the proper reference), or they follow from transitivity of implications.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	•	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	[19] $\not\Rightarrow$	[19] $\not\Rightarrow$	[19] $\not\Rightarrow$	$\not\Rightarrow$	[19] $\not\Rightarrow$	[15],[19] $\not\Rightarrow$	[14] $\not\Rightarrow$	[18] $\not\Rightarrow$	[18],[22] $\not\Rightarrow$	[15] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\not\Rightarrow$	$\Rightarrow$	[24] $\not\Rightarrow$	$\Rightarrow$
2	[22] $\not\Rightarrow$	•	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[22] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
3	[22] $\not\Rightarrow$	[10] $\not\Rightarrow$	•	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[22] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
4	[22],[4] $\not\Rightarrow$	$\not\Rightarrow$	[34] $\not\Rightarrow$	•	[18] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[16] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[15] $\not\Rightarrow$	$\not\Rightarrow$	[33] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
5	[18] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[18] $\not\Rightarrow$	•	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\not\Rightarrow$	$\not\Rightarrow$	[24] $\Rightarrow$
6	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	•	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	[19] $\not\Rightarrow$	[19] $\not\Rightarrow$	[20] $\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	[26] $\not\Rightarrow$	$\Rightarrow$
7	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	[19] $\not\Rightarrow$	•	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
8	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	[19] $\not\Rightarrow$	•	[19] $\not\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
9	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[19] $\not\Rightarrow$	•	def. $\Rightarrow$	def. $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
10	[6] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	•	def. $\Rightarrow$	[20] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
11	[22] $\not\Rightarrow$	[6] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[15],[19] $\not\Rightarrow$	•	[22] $\not\Rightarrow$	$\not\Rightarrow$	[15],[19] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[3] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
12	[34] $\not\Rightarrow$	$\not\Rightarrow$	[34] $\not\Rightarrow$	[19] $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[19] $\not\Rightarrow$	[16] $\not\Rightarrow$	$\not\Rightarrow$	•	[19] $\not\Rightarrow$	[19] $\not\Rightarrow$	[20] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
13	[34] $\not\Rightarrow$	$\not\Rightarrow$	[34] $\not\Rightarrow$	[16] $\not\Rightarrow$	[1] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[15],[19] $\not\Rightarrow$	•	[22],[18] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\Rightarrow$	[24] $\not\Rightarrow$	$\Rightarrow$
14	[18] $\not\Rightarrow$	$\not\Rightarrow$	[34] $\not\Rightarrow$	[18] $\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[19] $\not\Rightarrow$	[15],[19] $\not\Rightarrow$	[19] $\not\Rightarrow$	[18] $\Rightarrow$	•	[2] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[25] $\not\Rightarrow$	[24] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
15	[33] $\not\Rightarrow$	[28] $\not\Rightarrow$	$\not\Rightarrow$	[30] $\Rightarrow$	[30] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[28] $\not\Rightarrow$	[33],[22] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	•	[33] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
16	[11] $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[28] $\not\Rightarrow$	[35] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	def. $\Rightarrow$	•	[28] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
17	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[13] $\not\Rightarrow$	[5] $\not\Rightarrow$	[28] $\not\Rightarrow$	[28] $\not\Rightarrow$	def. $\Rightarrow$	def. $\Rightarrow$	•	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	[26] $\not\Rightarrow$	$\Rightarrow$
18	[25] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[26] $\not\Rightarrow$	[22],[25] $\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	[33],[25] $\Rightarrow$	$\not\Rightarrow$	•	def. $\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
19	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	[25] $\Rightarrow$	[5] $\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[25] $\not\Rightarrow$	•	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
20	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	•	$\not\Rightarrow$	[24] $\Rightarrow$	[24] $\not\Rightarrow$
21	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	•	$\not\Rightarrow$	[24] $\Rightarrow$
22	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	[24] $\not\Rightarrow$	$\not\Rightarrow$	•	$\not\Rightarrow$
23	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	•

TABLE 1

The results can be summarized in the following ways. Among the possible 506 implications, there are

- 136 implications (24 of them follow directly by the definitions), and
- 370 non-implications.

From another point of view there are

- 24 implications following directly from definitions, and
- 19 implications and 83 counterexamples *essential*, and
- 380 implications or non-implications generated by the transitivity.

There are two weakest (i.e., such that they do not imply any other property) mutually incomparable properties, namely "the trajectories of every two points are correlated" and "any iteration of the map restricted to any closed invariant set cannot be topologically almost conjugate to the shift"; on the other hand, there are three strongest properties (they are implied by no property), "a map restricted to the set of chain recurrent points is not chaotic in the sense of Li and Yorke," "a map is not distributionally chaotic in the weakest sense," and "the trajectory of every point is strongly approximable."

5. Comments on the Positive Implications

There are 136 positive results in Table 1: 24 of them follow by the definitions, 19 are *essential*, and the remaining are generated by the transitivity.

Some of the positive results are valid not only for triangular maps but for maps on general compact metric spaces as well, namely, the implications $10 \Rightarrow 1$, $11 \Rightarrow 2$, $16 \Rightarrow 1$, and $19 \Rightarrow 15$. The remaining 15 implications have been proved for triangular maps only. Let us recall the general results.

Theorem 5.1 (see [6]). *Let $f \in \mathcal{C}(X)$. If f has positive topological entropy, then it is Li-Yorke chaotic. (From this it immediately follows that $10 \Rightarrow 1$ and $11 \Rightarrow 2$.)*

Theorem 5.2 (see [11]). *Let $f \in \mathcal{C}(X)$. If f has positive topological entropy, then it is distributionally chaotic DC2, i.e., $16 \Rightarrow 1$.*

Theorem 5.3 (see [5]). *Let $f \in \mathcal{C}(X)$ and let $u, v \in X$ form a DC1 pair. Then the trajectory of u or the trajectory of v is not weakly approximable by compact periodic sets, i.e., $19 \Rightarrow 15$.*

Let us note that in the first two results chaoticity means the existence of an *uncountable* (not only two-points) Li-Yorke (distributionally, respectively) scrambled set. These results were proved in a very sophisticated way using a measure theoretical approach, in particular ergodic theory, while the proof of Theorem 5.3 is easy. Of course, Theorem 5.1 is a direct consequence of Theorem 5.2, but the first one was proved already in 2002, while the notion of DC2 was not introduced until 2004 in [33].

Note that positive topological entropy and chaos DC1 are for general triangular maps independent, while a triangular map with the base map f with $h(f) = 0$ cannot be DC1 (see [30]) but it can be DC2 (see [33]). On the other hand, in [33] a triangular map with positive topological entropy exhibiting DC2 but not DC1 was constructed. So a natural question arose as to whether positive topological entropy implies chaos DC2. As an open question it was published in 2006 in [32] with a conjecture that the answer is affirmative. The existence of the proof of Theorem 5.2 was announced by Tomasz Downarowicz around 2011 and published in 2014 [11]. Let us outline the proof as it was presented in the paper [12] by Downarowicz and Yves Lacroix where, in fact, a stronger version of Theorem 5.2 is proved in a more transparent way.

Let $(X, \mathcal{B}, \mu)$ be a measure space, $\mathcal{B}$ a complete $\sigma-$algebra on X, μ a probability measure on $\mathcal{B}$, and $T : X \to X$ a measure-preserving transformation. First, we have to define a new notion of chaos.

A sequence $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ of finite measurable partitions is *refining* if $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$ for every k and if $\mathcal{P}_k$ jointly generate $\mathcal{B}$ (i.e., $\mathcal{B}$ is the smallest complete σ-algebra containing all the partitions $\mathcal{P}_k$).

For a given refining sequence of finite measurable partitions $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$, a pair (x, y) is $\{\mathcal{P}_k\}$-*scrambled* if

(i) there exists a sequence n_i of upper density 1 such that, for every k and large enough i, $T^{n_i}(x)$ belongs to the same atom of $\mathcal{P}_k$ as $T^{n_i}(y)$; and

(ii) there exists a sequence m_i of positive upper density and k_0 such that, for every i, $T^{m_i}(x)$ and $T^{m_i}(y)$ belong to different atoms of $\mathcal{P}_{k_0}$.

A transformation T is called *measure-theoretically chaotic* (briefly, *MTC*) if, for every refining sequence of finite partitions $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$, there exists an uncountable $\{\mathcal{P}_k\}$-scrambled set.

Measure-theoretical chaos has, for example, the following properties:

• it is an isomorphism invariant;

• for a given refining sequence $\{\mathcal{P}_k\}$, the ergodic system $(X, \mathcal{B}, \mu, T)$ is MTC if and only if for any null set A, there exists an uncountable $\{\mathcal{P}_k\}$-scrambled set disjoint from A;

• $\{\mathcal{P}_k\}$-scrambled sets exist inside any set of positive measure, more precisely, let $(X, \mathcal{B}, \mu, T)$ be an ergodic MTC system, let $\{\mathcal{P}_k\}$ be a refining sequence of finite partitions, and let $B \in \mathcal{B}$ with $\mu(B) > 0$. Then there exists an uncountable $\{\mathcal{P}_k\}$-scrambled set contained in B;

• MTC $\Rightarrow$ DC2; more precisely, let (X,T) be a topological dynamical system and μ an ergodic T-invariant measure. If $(X,\mathcal{B},\mu,T)$ is MTC, then (X,T) is DC2;

• every ergodic system $(X,\mathcal{B},\mu,T)$ with positive Kolmogorov-Sinai entropy is MTC.

The fact that positive topological entropy implies DC2 is now a corollary of the last property.

6. Comments on Counterexamples

Recall that in Table 1, there are 370 non-implications and 82 of them are *essential*. The counterexample that initiated the Sharkovsky program is generally considered as the most important one. Note that a map is *of type* 2^∞ if the periods of its periodic orbits are 2^n for any $n \in \mathbb{N}$.

Theorem 6.1 (see [22]). *There exists a triangular map of type* 2^∞ *with positive topological entropy.*

Let us outline a construction of a map with these properties. The presented construction originates from [33]. The construction described in [33] was inspired by the construction presented in [22], but it contains more properties (namely it has positive entropy, it is of type 2^∞, and it is DC2 but not DC1) and uses simpler and more straightforward argumentation.

The base map f of the constructed triangular map F is $f(x) = \lambda x(1-x)$, where $\lambda = 3.569\ldots$ is such that f is of type 2^∞. It is well known that this map has exactly one infinite ω-limit set Q (which is perfect), two fixed points, and, for any $n \in \mathbb{N}$, a unique periodic orbit of period 2^n. For this set Q there is a sequence $\{I_n\}_{n\in\mathbb{N}}$ of minimal compact periodic intervals with the following properties: (i) for any $n \in \mathbb{N}$, I_n has period 2^n; (ii) $\bigcap_{n\in\mathbb{N}} Orb(I_n) = Q$; (iii) $\bigcap_{n\in\mathbb{N}} I_n = \{c\}$ for some $c \in Q$. Let $p_n \in I_n$ be the unique periodic point of period 2^n.

For the definition of the fiber maps, let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers and let $\tau(x) = 1 - |1-x|$, $x \in I$, be the standard tent map. The fiber maps are either zero maps or "diminished" tent maps,

$$g_x = \begin{cases} 0 & \text{for } \; x = 0,\; x = c,\; \text{or } x = p_n, n \in \mathbb{N}, \\ 2^{-k}\tau & \text{for } \; x \in Q \cap (I_{n_k} \setminus I_{n_{k+1}}). \end{cases}$$

Moreover, let $g_x(0) = 0$ for any $x \in I$. Finally, the triangular map F is a continuous extension of this map to the whole $I \times I$.

By an easy argument, F is of type 2^∞. Using a much more complex argument, it can be shown that an appropriate choice of the sequence

$\{n_k\}$ leads to positivity of topological entropy of F (and also to the above mentioned properties concerning distributional chaos).

Some of the counterexamples were constructed for a special class of triangular maps, namely the ones *monotone on the fibers* (i.e., such that the fiber maps g_x are monotone for any $x \in I$). Such maps have easier behavior than general triangular maps. For instance, some properties from the classification program (e.g., zero topological entropy and being of type 2^∞) are, for such maps, equivalent, but this is no longer true for general triangular maps (see Theorem 6.1). Another important property is that for triangular maps monotone on the fibers positive topological entropy implies DC1, the strongest type of distributional chaos (see [29]), but for general triangular maps positive entropy implies only DC2. Let us note that for maps of this kind there are some other important properties, for example, the topological entropy of such a map equals the topological entropy of its base map by Bowen's formula

$$h(f) + \sup_{x\in I} h(F, I_x) \geq h(F) \geq \max\{h(f),\ \sup_{x\in I} h(F, I_x)\},$$

where $h(F, I_x)$ is the topological entropy calculated from trajectories starting from the fiber I_x (see [8]).

Let us note that in order to construct required counterexamples, completely new methods were used which had not been known in the early stages of this program. In [35] a triangular map F was constructed which was nondecreasing on the fibers and without DC2 pairs such that $F|UR(F)$ was Li-Yorke chaotic (i.e., $16 \not\Rightarrow 11$). An analogous problem with DC3 instead of DC2 was still open at that time. I tried to change the construction in such a way that there are no DC3 pairs in order to show that $17 \not\Rightarrow 11$, but I did not succeed. A few months later, a general technique of embedding any zero-dimensional almost one-to-one extension of the dyadic odometer (in particular any dyadic Toeplitz system) in a triangular system of type 2^∞ as a unique non periodic minimal set was developed (see [13]) and using it, the property $17 \not\Rightarrow 11$ was proved. Finally, in [10], a method of embedding a special class of zero-dimensional almost two-to-one extension of the odometer was presented and used for proving the last problem, namely $3 \not\Rightarrow 2$. Let us point out that using these methods we obtain maps that are *not* monotone on the fibers.

Taking into account all the above mentioned arguments, it may be of some interest to give the classification just for triangular maps monotone on the fibers. Let us note that the full classification for properties 1 to 14 has been already given in [19].

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∗ Indicates papers devoted to Sharkovsky classification.

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