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Jaspreet Kaur, Hemant Kumar Singh, and Tej Bahadur Singh

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Department of Mathematics & Statistics
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CIRCLE GROUP ACTION ON THE PRODUCT OF TWO PROJECTIVE SPACES

JASPREET KAUR, HEMANT KUMAR SINGH, AND TEJ BAHADUR SINGH

Abstract. Let $G = \mathbb{S}^1$ act freely on a finitistic space X with mod 2 cohomology ring isomorphic to the product of two projective spaces. In this paper, we determine the possible cohomology algebra of the orbit space X/G when G acts freely on the product of two real projective spaces. We also show that the group G cannot act freely on the product of two complex projective spaces.

1. Introduction

Let G be a topological group acting continuously on a topological space X. An intricate problem associated with the transformation group (G, X) is to determine the topological or the homotopy type of the orbit space X/G. The first such question was raised by H. Hopf in 1925-26, for the orbit spaces of free actions of finite cyclic groups on spheres. Because of the complexity in resolving such problems P. A. Smith [7] introduced the study of homological relationships among the space X, the fixed point set X^G, and the orbit space X/G of a periodic homeomorphism on X. Since then, several authors have contributed to such problems. For instance, Ronald M. Dotzel et al. [3] determined the cohomology algebra of the orbit spaces of free circle group action on finitistic spaces having mod $\mathbb{Q}$ cohomology algebra of the product of two spheres. More recently, in [5] the possible mod p cohomology algebra of the orbit space of any free circle group action on a finitistic space having mod p cohomology algebra of a lens space or $\mathbb{S}^1 \times \mathbb{C}P^m$ has been investigated. Continuing this thread of research, in this paper we study free $G = \mathbb{S}^1$ action on a finitistic space

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with mod 2 cohomology algebra of the product of two real (complex) projective spaces. Using the Leray–Serre spectral sequence, we show that the group G cannot act freely on the product of two complex projective spaces. If G acts freely on a finitistic space X with mod 2 cohomology algebra of $\mathbb{R}P^m \times \mathbb{R}P^n$, the product of two real projective spaces, then we observe that at least one of m or n should be odd. In this case, we also determine the possible cohomology algebra of the orbit space.

2. Preliminaries

In this section, we review some basic definitions and results that are essential for the work done in this paper. We first recall the definition of a class of topological spaces which has been found most suitable for the study of the relationship between the cohomology structure of the total space and that of the orbit space of a transformation group [1].

Definition 2.1. A *finitistic space* is a paracompact Hausdorff space whose every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially.

Two main classes of finitistic spaces are the compact spaces and the finite-dimensional spaces (spaces with finite covering dimension). It is known that if a compact Lie group G acts freely on a finitistic space X, then the orbit space X/G is also finitistic [2].

We next recall some results about the Leray–Serre spectral sequence associated with the Borel fibrations. Let G be a compact Lie group acting (not necessarily) freely on a finitistic space X and let $G \hookrightarrow E_G \longrightarrow B_G$ be the universal principal G-bundle. Then, with the diagonal action of G on $X \times E_G$, the projection map $X \times E_G \to E_G$ is equivariant and hence, it induces a fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ called the *Borel fibration* [9], [10], where X_G is the orbit space $(X \times E_G)/G$. With a field as the coefficient group for cohomology, we have the following.

Proposition 2.2. *Let $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ be the Borel fibration, where X is connected. Suppose that the system of local coefficients (cohomology of the fibers) on B_G is simple. Then the edge homomorphisms*

$$H^k(B_G) = E_2^{k,0} \longrightarrow E_3^{k,0} \longrightarrow \cdots \longrightarrow E_k^{k,0} \longrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(X_G)$$

$$\text{and} \quad H^l(X_G) \longrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \cdots \subset E_2^{0,l} = H^l(X)$$

are the homomorphisms $\pi^* : H^k(B_G) \to H^k(X_G)$ and $i^* : H^l(X_G) \to H^l(X)$, respectively.

For details about spectral sequences, see [4]. The following proposition is well known.

Proposition 2.3. *Let G be a compact Lie group acting freely on a finitistic space X. Then the Borel space X_G is homotopy equivalent to the orbit space X/G.*

If G is connected, then B_G is simply connected, and so the system of local coefficients on B_G is simple. Hence, the E_2-term of the Leray–Serre spectral sequence of π has the form $E_2^{p,q} = H^p(B_G) \otimes H^q(X)$ and it converges to $H^*(X_G)$, the cohomology ring of X_G.

Theorem 2.4 (Künneth formula [8]). *Let F be a field. Then the cross product $(H^*(X;F) \otimes_R H^*(Y;F))^k \to H^k(X \times Y;F)$ is an isomorphism if $H_k(Y;F)$ is finite-dimensional over F.*

By the above theorem we have,

(1) $H^*(\mathbb{R}P^m \times \mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[a,b]/ < a^{m+1}, b^{n+1} >$, deg $a = 1$ and deg $b = 1$,

(2) $H^*(\mathbb{C}P^m \times \mathbb{C}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[a,b]/ < a^{m+1}, b^{n+1} >$, deg $a = 2$ and deg $b = 2$.

Further, we recall that for $G = \mathbb{S}^1$, the classifying space B_G is the infinite dimensional complex projective space, $\mathbb{C}P^\infty$, and $H^*(B_G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]$, deg $x = 2$.

In this paper, we shall use Čech cohomology with $\mathbb{Z}_2$ coefficients. By $X \sim_2 Y$, we will mean that the cohomology rings $H^*(X;\mathbb{Z}_2)$ and $H^*(Y;\mathbb{Z}_2)$ are isomorphic. The following result, which is an analogue of [1, chapter III, Proposition 10.7], has been proved in [5].

Proposition 2.5. *Let $G = \mathbb{S}^1$ act freely on a finitistic space X with $H^i(X;\mathbb{Z}_2) = 0$, for all $i > n$. Then $H^i(X/G;\mathbb{Z}_2) = 0$, for all $i > n$.*

3. Main Results

Proposition 3.1. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times \mathbb{R}P^n$. Then both m and n cannot be even.*

Proof. The Euler–Poincaré characteristic $\chi(\mathbb{R}P^k)$ is 0 or 1 according to whether k is odd or even. Also, for finite CW complexes X and Y, it is well known that $\chi(X \times Y) = \chi(X) \times \chi(Y)$. Now, suppose to the contrary, both m and n are even. Then the Euler–Poincaré characteristic of the space X is 1. If $\phi : \mathbb{S}^1 \times X \to X$ is a free action, then the restricted action $\phi' : \mathbb{Z}_2 \times X \to X$ must also be free. So by Floyd's formula (see [1, chapter III, Theorem 7.10]), we have $1 = \chi(X) \equiv 2\chi(X/\mathbb{Z}_2)$. Since this is not possible, therefore, at least one of m or n must be odd. $\square$

We now give an example of a free action of $\mathbb{S}^1$ on $\mathbb{R}P^m \times \mathbb{R}P^n$ when n is odd.

Example 3.2. Put $n = 2k-1$ and consider $\mathbb{S}^n \subset \mathbb{C}^k$. There is a natural free action of $\mathbb{S}^1$ on $\mathbb{S}^n$ given by

$$(\xi, (z_1, z_2, \cdots, z_k)) \xrightarrow{\theta} (\xi z_1, \xi z_2, \cdots, \xi z_k), \quad \xi \in \mathbb{S}^1.$$

The action θ induces a free action $\overline{\theta}$ of the group $\mathbb{S}^1/N \cong \mathbb{S}^1$ on the orbit space $\mathbb{R}P^n = \mathbb{S}^n/N$, where $N = \{+1, -1\} \cong \mathbb{Z}_2$. Taking any action of $\mathbb{S}^1$ on $\mathbb{R}P^m$ and the free action $\overline{\theta}$, the diagonal action gives a free action of $\mathbb{S}^1$ on $\mathbb{R}P^m \times \mathbb{R}P^n$.

We now exploit the Leray–Serre spectral sequence associated with the Borel fibration $X \hookrightarrow X_{\mathbb{S}^1} \longrightarrow B_{\mathbb{S}^1}$ to classify completely mod 2 cohomology algebra of the orbit space of a free circle group action on a finitistic $X \sim_2 \mathbb{R}P^m \times \mathbb{R}P^n$. We remark that the cohomology algebra of any free involution on finitistic spaces X having the mod 2 cohomology algebra of the product of two projective spaces has been completely determined in [6].

Theorem 3.3. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times \mathbb{R}P^n$, $1 \leq m \leq n$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded algebras:*

(1) $\mathbb{Z}_2[x, y]/\langle x^{\frac{m+1}{2}}, y^{n+1}\rangle$, *where* $\deg x = 2$, $\deg y = 1$, *and m is odd.*
(2) $\mathbb{Z}_2[x, y]/\langle x^{m+1}, y^{\frac{n+1}{2}}\rangle$, *where* $\deg x = 1$, $\deg y = 2$, *and n is odd.*
(3) $\mathbb{Z}_2[x, y, z]/\langle x^{\frac{m+1}{2}}, y^{\frac{n+1}{2}}, z^2 - \alpha x - \beta y\rangle$, *where* $\deg x = 2$, $\deg y = 2$, $\deg z = 1$ *and $\alpha, \beta \in \mathbb{Z}_2$ and m and n are odd.*

Proof. Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times \mathbb{R}P^n$, $1 \leq m \leq n$. Let a, $b \in H^1(X)$ be generators of the cohomology algebra $H^*(X)$. We note that

$$H^i(X) = \begin{cases} \oplus\mathbb{Z}_2 \ (i+1 \text{ copies}) & i \leq m \\ \oplus\mathbb{Z}_2 \ (m+1 \text{ copies}) & m+1 \leq i \leq n \\ \oplus\mathbb{Z}_2 \ (n+m+1-i \text{ copies}) & n+1 \leq i \leq m+n \\ 0 & \text{otherwise.} \end{cases}$$

Since the fundamental group of the space B_G is trivial, the system of local coefficients associated with the spectral sequence of the fibration $X \hookrightarrow X_G \xrightarrow{\pi} B_G$ is simple. So the spectral sequence has the form $E_2^{p,q} \cong H^p(B_G) \otimes H^q(X)$. Clearly, for p odd, $E_2^{p,q} = 0$. Thus,

- A basis of $E_2^{2s,q}$, $s \geq 0$, consists of $\{t^s \otimes a^q, t^s \otimes a^{q-1}b, ..., t^s \otimes ab^{q-1}, t^s \otimes b^q\}$, for $q \leq m$.

- A basis of $E_2^{2s,q}$ consists of $\{t^s \otimes a^m b^{q-m}, t^s \otimes a^{m-1}b^{q-m+1}, ..., t^s \otimes ab^{q-1}, t^s \otimes b^q\}$, for $m+1 \leq q \leq n$.
- A basis of $E_2^{2s,q}$ consists of $\{t^s \otimes a^m b^{q-m}, t^s \otimes a^{m-1}b^{q-m+1}, ..., t^s \otimes a^{q-n+1}b^{n-1}, t^s \otimes a^{q-n}b^n\}$, for $n+1 \leq q \leq n+m$.

We further note that there are four possibilities for the homomorphism $d_2 : E_2^{0,1} \to E_2^{2,0}$:

(i) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = 0$.
(ii) $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$.
(iii) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes 1$.
(iv) $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = t \otimes 1$.

We consider each case separately.

Case (i). $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = 0$.
By the derivation property of differentials, we see that $d_2(t^s \otimes a^k b^l) = 0$, for every $s \geq 0$, $k \geq 1$ and $l \geq 1$. Thus, the spectral sequence degenerates at E_2-term. This implies that $H^i(X/G) \neq 0$ for infinitely many i, contrary to the Proposition 2.5.

Case (ii). $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$. We have

$$d_2(t^s \otimes a^k b^l) = \begin{cases} t^{s+1} \otimes a^{k-1}b^l & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

If m is even, then $(1 \otimes a^m)(1 \otimes a) = 0$ gives $0 = d_2((1 \otimes a^m)(1 \otimes a)) = t \otimes a^m$, a contradiction. Hence, m must be odd. Now,

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$, $s \geq 0$, consists of $\frac{q+1}{2}$ elements $\{t^s \otimes a^{q-1}b, t^s \otimes a^{q-3}b^3, ..., t^s \otimes b^q\}$ for q odd and $q \leq m$ and a basis of the image of d_2 consists of $\frac{q+1}{2}$ elements $\{t^{s+1} \otimes a^{q-1}, t^{s+1} \otimes a^{q-3}b^2, ..., t^{s+1} \otimes b^{q-1}\}$. And a basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{q}{2}+1$ elements $\{t^s \otimes a^q, t^s \otimes a^{q-2}b^2, ..., t^s \otimes b^q\}$ for q even and $q \leq m$ and a basis of the image consists of $\frac{q}{2}$ elements $\{t^{s+1} \otimes a^{q-2}b, t^{s+1} \otimes a^{q-4}b^3, ..., t^{s+1} \otimes b^{q-1}\}$.

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{m+1}{2}$ elements $\{t^s \otimes a^{m-1}b^{q-m+1}, t^s \otimes a^{m-3}b^{q-m+3}, ..., t^s \otimes b^q\}$ for $m+1 \leq q \leq n$, and a basis of the image consists of $\frac{m+1}{2}$ elements $\{t^{s+1} \otimes a^{m-1}b^{q-m}, t^{s+1} \otimes a^{m-3}b^{q-m+2}, ..., t^{s+1} \otimes b^{q-1}\}$.

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{n+m-q}{2}$ elements $\{t^s \otimes a^{m-1}b^{q-m+1}, t^s \otimes a^{m-3}b^{q-m+3}, ..., t^s \otimes a^{q-n+1}b^{n-1}\}$ when $n+1 \leq q \leq n+m$ and (n is odd and q is even) or (n is even and q is odd). In this case, a basis of the image of d_2 consists of $\frac{n+m-q}{2}+1$ elements

$\{t^{s+1} \otimes a^{m-1}b^{q-m}, t^{s+1} \otimes a^{m-3}b^{q-m+2}, ..., t^{s+1} \otimes a^{q-n-1}b^{n}\}$. Now, a basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{n+m-q+1}{2}$ elements $\{t^s \otimes a^{m-1}b^{q-m+1}, t^s \otimes a^{m-3}b^{q-m+3}, ..., t^s \otimes a^{q-n}b^n\}$ if $n+1 \leq q \leq n+m$ and (both n and q are even) or (both n and q are odd). A basis of the image in this case consists of $\frac{n+m-q+1}{2}$ elements $\{t^{s+1} \otimes a^{m-1}b^{q-m}, t^{s+1} \otimes a^{m-3}b^{q-m+2}, ..., t^{s+1} \otimes a^{q-n}b^{n-1}\}$.

It follows that, for every $r \geq 3$, $E_r^{k,l} = \ker d_r/\operatorname{im} d_r = 0$ for all $k \geq 1$ and for all l. Also, $E_3^{0,q}$ is the kernel of $d_2 : E_2^{0,q} \to E_2^{2,q-1}$ for all q. Consequently, $d_r = 0$ for all $r \geq 3$. Hence, $E_3^{*,*} \cong E_\infty^{*,*}$. We thus obtain

$$H^j(X_G) \cong \begin{cases} \oplus\mathbb{Z}_2 \ (\frac{j+1}{2} \text{ copies}) & j \leq m \ j \text{ odd} \\ \oplus\mathbb{Z}_2 \ (\frac{j}{2}+1 \text{ copies}) & j \leq m \ j \text{ even} \\ \oplus\mathbb{Z}_2 \ (\frac{m+1}{2} \text{ copies}) & m+1 \leq j \leq n \\ \oplus\mathbb{Z}_2 \ (\frac{n+m-j}{2} \text{ copies}) & n+1 \leq j \leq n+m \\ & n \text{ even } j \text{ odd or } n \text{ odd } j \text{ even} \\ \oplus\mathbb{Z}_2 \ \frac{n+m+1-j}{2} \text{ copies} & n+1 \leq j \leq n+m \\ & n \text{ even } j \text{ even or } n \text{ odd } j \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and therefore it determines an element $u \in E_\infty^{0,2}$. Choose $x \in H^2(X_G)$ such that $i^*(x) = a^2$. Then x determines u and satisfies $x^{\frac{m+1}{2}} = 0$. Also, $1 \otimes b \in E_2^{0,1}$ is a permanent cocycle and determines an element $v \in E_\infty^{0,1}$. We choose $y \in H^1(X_G)$ such that $i^*(y) = b$. Then y determines v and $y^{n+1} = 0$. Therefore,

$$H^*(X_G) \cong \mathbb{Z}_2[x, y]/\langle x^{\frac{m+1}{2}}, y^{n+1}\rangle,$$

where deg $x = 2$, deg $y = 1$, and m is odd. Since group G acts freely on X, $H^*(X/G) \cong H^*(X_G)$, by Proposition 2.3. Thus, we have possibility (1) of the theorem.

Case (iii). $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes 1$.
As in Case (ii), we see that n must be odd and

$$H^*(X/G) \cong \mathbb{Z}_2[x, y]/\langle x^{m+1}, y^{\frac{n+1}{2}}\rangle,$$

where deg $x = 1$ and deg $y = 2$.

Case (iv). $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = t \otimes 1$.
By the multiplicative structure of the spectral sequence, it is easy to see

that

$$d_2(1 \otimes a^k b^l) = \begin{cases} (t \otimes a^{k-1} b^l) + (t \otimes a^k b^{l-1}) & \text{if } k \text{ and } l \text{ are odd} \\ t \otimes a^{k-1} b^l & \text{if } k \text{ is odd and } l \text{ is even} \\ t \otimes a^k b^{l-1} & \text{if } k \text{ is even and } l \text{ is odd} \\ 0 & \text{if } k \text{ and } l \text{ are even.} \end{cases}$$

As in the previous two cases, we see that both m and n are odd. We also observe that,

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{q+1}{2}$ elements, $\{t^s \otimes (a^q + a^{q-1}b), t^s \otimes (a^{q-2}b^2 + a^{q-3}b^3), ..., t^s \otimes (ab^{q-1} + b^q)\}$ for q odd and $q \leq m$ and a basis of the image of d_2 consists of $\frac{q+1}{2}$ elements $\{t^{s+1} \otimes a^{q-1}, t^{s+1} \otimes a^{q-3}b^2, ..., t^{s+1} \otimes b^{q-1}\}$. For q even and $q \leq m$, a basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{q}{2} + 1$ elements $\{t^s \otimes a^q, t^s \otimes a^{q-2}b^2, ..., t^s \otimes b^q\}$ and a basis of the image consists of $\frac{q}{2}$ elements $\{t^{s+1} \otimes (a^{q-2}b + a^{q-1}), t^{s+1} \otimes (a^{q-4}b^3 + a^{q-3}b^2), ..., t^{s+1} \otimes (b^{q-1} + ab^{q-2})\}$.

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{m+1}{2}$ elements $\{t^s \otimes (a^m b^{q-m} + a^{m-1}b^{q-m+1}), t^s \otimes (a^{m-2}b^{q-m+2} + a^{m-3}b^{q-m+3}), ..., t^s \otimes (ab^{q-1} + b^q)\}$, for q odd and $m+1 \leq q \leq n$ and a basis of the image of d_2 consists of $\frac{m+1}{2}$ elements $\{t^{s+1} \otimes a^{m-1}b^{q-m}, t^{s+1} \otimes a^{m-3}b^{q-m+2}, ..., t^{s+1} \otimes b^{q-1}\}$. For q even and $m+1 \leq q \leq n$, a basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$, $s \geq 0$, consists of $\frac{m+1}{2}$ elements $\{t^s \otimes a^{m-1}b^{q-m+1}, t^s \otimes a^{m-3}b^{q-m+3}, ..., t^s \otimes b^q\}$ and a basis of the image consists of $\frac{m+1}{2}$ elements $\{t^{s+1} \otimes (a^{m-1}b^{q-m} + a^m b^{q-m}), t^{s+1} \otimes (a^{m-3}b^{q-m+2} + a^{m-2}b^{q-m+1}), ..., t^{s+1} \otimes (b^{q-1} + ab^{q-2})\}$.

A basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{m+n-q+1}{2}$ elements $\{t^s \otimes (a^m b^{q-m} + a^{m-1}b^{q-m+1}), t^s \otimes (a^{m-2}b^{q-m+2} + a^{m-3}b^{q-m+3}), ..., t^s \otimes (a^{q-n+1}b^{n-1} + a^{q-n}b^n)\}$ for q odd, $n+1 \leq q \leq n+m$ and a basis of the image consists of $\frac{m+n-q+1}{2}$ elements $\{t^{s+1} \otimes a^{m-1}b^{q-m}, t^{s+1} \otimes a^{m-3}b^{q-m+2}, ..., t^{s+1} \otimes a^{q-n}b^{n-1}\}$. For q even and $n+1 \leq q \leq n+m$, a basis of the kernel of $d_2 : E_2^{2s,q} \to E_2^{2s+2,q-1}$ consists of $\frac{n+m-q}{2}$ elements $\{t^s \otimes a^{m-1}b^{q-m+1}, t^s \otimes a^{m-3}b^{q-m+3}, ..., t^s \otimes a^{q-n+1}b^{n-1}\}$ and a basis of the image of d_2 consists of $\frac{n+m-q}{2} + 1$ elements $\{t^{s+1} \otimes (a^{m-1}b^{q-m} + a^m b^{q-m-1}), t^{s+1} \otimes (a^{m-3}b^{q-m+2} + a^{m-2}b^{q-m+1}), ..., t^{s+1} \otimes (a^{q-n-1}b^n + a^{q-n}b^{n-1})\}$.

Now, it is clear that $E_r^{k,l} = 0$ for all l, $r \geq 3$, and $k \geq 1$. Hence, $E_\infty^{*,*} = E_3^{*,*}$. We see that the cohomology groups $H^j(X_G)$ are the same as in Case (ii).

In this case also we note that $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and therefore yields an element $u \in E_\infty^{0,2}$. Choose $x \in H^2(X_G)$ such that $i^*(x) = a^2$. Then x determines u and satisfies $x^{\frac{m+1}{2}} = 0$. For the same reason, $1 \otimes b^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $v \in E_\infty^{0,2}$. We choose $y \in H^2(X_G)$ such that $i^*(y) = b^2$. Then y determines v and $y^{\frac{n+1}{2}} = 0$. Also $1 \otimes (a+b) \in E_2^{0,1}$ is a permanent cocycle and yields an element $s \in E_\infty^{0,1}$. Let $z \in H^1(X_G)$ such that $i^*(z) = a + b$. Then z determines s. Observe that

$$z^2 = \alpha x + \beta y$$

for some $\alpha, \beta \in \mathbb{Z}_2$. Hence,

$$H^*(X/G) \cong H^*(X_G) \cong \mathbb{Z}_2[x, y, z]/\langle x^{\frac{m+1}{2}}, y^{\frac{n+1}{2}}, z^2 - \alpha x - \beta y\rangle$$

where deg $x = 2$, deg $y = 2$, and deg $z = 1$. □

Remark 3.4. With the action of $\mathbb{S}^1$ on $\mathbb{R}P^m$, m odd, as described in Example 3.2, and the trivial action of $\mathbb{S}^1$ on $\mathbb{R}P^n$, we find that the orbit space $(\mathbb{R}P^m \times \mathbb{R}P^n)/\mathbb{S}^1$ is $\mathbb{C}P^{\frac{m-1}{2}} \times \mathbb{R}P^n$, which realizes case (1) of Theorem 3.3. Similarly, case (2) can be realized.

Corollary 3.5. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^m \times \mathbb{R}P^n$. Then the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$ is zero.*

Proof. The first few terms of the Gysin sequence of the bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$ are

$$0 \to H^1(X/G) \xrightarrow{\nu^*} H^1(X) \to H^0(X/G) \xrightarrow{\psi^*} H^2(X/G) \xrightarrow{\nu^*} \cdots .$$

The characteristic class of the bundle is $\psi^*(1) \in H^2(X/G)$, where 1 is the unity of $H^0(X/G)$. Now, from Theorem 3.3, it is clear that $H^i(X/G) = \mathbb{Z}_2$, for $i = 0, 1$, and $H^2(X/G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the Gysin sequence takes the following form

$$0 \to \mathbb{Z}_2 \xrightarrow{\nu^*} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{\psi^*} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\nu^*} \cdots .$$

We note that the map $Z_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$, in the above sequence, is an epimorphism. So the map

$$\psi^* : H^0(X/G) \to H^2(X/G)$$

is the trivial homomorphism and thus, $\psi^*(1) = 0$. □

Remark 3.6. If $m = 0$, then $X \sim_2 \mathbb{R}P^n$. In this case, $G = \mathbb{S}^1$ acts freely on X when n is odd and $H^*(X/G) \cong \mathbb{Z}_2[x]/ < x^n >$, deg $x = 2$. This result has already been derived in [5].

Taking $m = 1$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.7. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{S}^1 \times \mathbb{R}P^n$, $n \geq 1$. Then $H^*(X/G; \mathbb{Z}_2)$ is isomorphic to one of the following graded algebras:*

(1) $\mathbb{Z}_2[y]/ < y^{n+1} >$ *where* $\deg y = 1$.
(2) $\mathbb{Z}_2[x, y]/ < x^2, y^{\frac{n+1}{2}} >$ *where* $\deg x = 1$, $\deg y = 2$, *and n is odd.*
(3) $\mathbb{Z}_2[y, z]/ < y^{\frac{n+1}{2}}, z^2 - \beta y >$ *where* $\deg y = 2$, $\deg z = 1$, $\beta \in \mathbb{Z}_2$, *and n is odd.*

Further, taking $n = 1$ in the above corollary, we obtain the following.

Corollary 3.8. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_2 \mathbb{S}^1 \times \mathbb{S}^1$. Then $H^*(X/G)$ is isomorphic to $\mathbb{Z}_2[x]/ < x^2 >$, where* $\deg x = 1$.

Remark 3.9. The significance of the above corollary is that the orbit space of any free circle group action on a space having mod 2 cohomology algebra as that of the product of two circles is a mod 2 cohomology circle. In particular, if ϕ is a free circle group action on $\mathbb{S}^1$ and θ the trivial action on another copy of $\mathbb{S}^1$, then the orbit space of the action $\phi \times \theta$ is mod 2 cohomology $\mathbb{S}^1$. This realizes Corollary 3.8.

Finally, we observe that $G = \mathbb{S}^1$ cannot act freely on the product of two complex projective spaces.

Theorem 3.10. *Let $X \sim_2 \mathbb{C}P^m \times CP^n$ be a finitistic space. Then the group $G = \mathbb{S}^1$ cannot act freely on X.*

Proof. Assume on the contrary that G acts freely on X. As $\pi_1(B_G)$ acts trivially on $H^*(X)$, so the spectral sequence has the following form

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X) \cong H^k(\mathbb{C}P^\infty) \otimes H^l(\mathbb{C}P^m \times CP^n).$$

It is clear that $E_2^{k,l} = 0$ when either k or l is odd. So, for all $r \geq 1$, the differentials

$$d_{2r} : E_{2r}^{k,l} \to E_{2r}^{k+2r, l-2r+1}$$

are the trivial homomorphism because $E_{2r}^{k,l} = 0$ when l is odd and $E_{2r}^{k+2r,l-2r+1} = 0$ when l is even. Also, for all $r \geq 1$, the differentials

$$d_{2r+1} : E_{2r+1}^{k,l} \to E_{2r+1}^{k+2r+1, l-2r}$$

are the trivial homomorphism because $E_{2r+1}^{k,l} = 0$ when k is odd and $E_{2r+1}^{k+2r+1,l-2r+1} = 0$ when k is even. Thus, $E_\infty^{*,*} = E_2^{*,*}$ and the spectral sequence degenerates at E_2-term. Hence, there are fixed points of G on X, contrary to our hypothesis. $\square$

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(Kaur, Singh, Singh) Department of Mathematics; University of Delhi; Delhi – 110007, India

E-mail address, Kaur: jasp.maths@gmail.com

E-mail address, H. K. Singh: hksinghdu@gmail.com

E-mail address, T. B. Singh: tbsingh@maths.du.ac.in