Half-Turn Linked Pairs of Isometries of Hyperbolic 4-Space

by

Andrew E. Silverio

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ISOMETRIES OF HYPERBOLIC 4-SPACE

ANDREW E. SILVERIO

ABSTRACT. In this paper we develop a complete theory of factorization for isometries of hyperbolic 4-space. Of special interest is the case where a pair of isometries is linked, that is, when a pair of isometries can be expressed each as compositions of two involutions, one of which is common to both isometries.

Here we develop a new theory of hyperbolic pencils and twisting planes involving a new geometric construction, their half-turn banks. This enables us to give complete results about each of the pair-types of isometries and their simultaneous factorization by half-turns. That is, we provide geometric conditions for each such pair to be linked by half-turns. The main result gives a necessary and sufficient condition for any given pair of isometries to be linked. We also provide a procedure for constructing a half-turn linked pair of isometries of $\mathbb{H}^4$ that do not restrict to lower dimensions, yielding an example that gives a negative answer to a question raised by Ara Basmajian and Karan Puri.

1. INTRODUCTION

A pair of isometries A and B of the hyperbolic space $\mathbb{H}^n$ is said to be linked if there are involutions α, β, and γ such that $A = \alpha\beta$ and $B = \beta\gamma$. If, furthermore, α, β, and γ have a $(n-2)$-dimensional fixed-point set, then the pair is said to be linked by half-turns. In dimensions 2 and 3, every pair of isometries is linked. Factoring by half-turns is used to determine the discreteness of the group $\langle A, B \rangle$ using the Gilman-Maskit algorithm in dimension 2, the non-separating-disjoint-circles condition, and the compact-core-geodesic-intersection condition in dimension 3 [4],

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[5], [6]. Unfortunately, in dimension 4, not all pairs are linked. Hence, we develop a complete theory of half-turn factorization of orientation preserving isometries of $\mathbb{H}^4$.

To each plane P in $\mathbb{H}^4$ corresponds a unique orientation preserving involution H_P called the half-turn about P whose fixed-point set is P. A space of planes can be associated to each isometry that is constructed from a new concept of hyperbolic pencils. This space is called the half-turn bank of an isometry, and it locates all the planes in which half-turn factoring is possible.

Theorem 1.1. *A pair of orientation preserving isometries of $\mathbb{H}^4$ is linked by half-turns if and only if they have a common element in their half-turn banks.*

Note that the half-turn bank of an isometry is a set of circles defined by its fixed points and invariants, not by its factorization. It is the introduction of the concept of a half-turn bank and the analysis of the half-turn bank for each type of isometry and the fact that the half-turn banks are constructible that make the theorem useful.

We also provide geometric criteria for a plane to be an element of the half-turn bank of an isometry. These criteria allow geometric conditions for various combinations of pairs of isometries to be linked. They are listed in section 7.

In section 8.2, Theorem 8.1 is proven to show that a pair of isometries can be linked even if the group it generates does not restrict to lower dimensions, giving a negative answer to the question raised by Ara Basmajian and Karan Puri [2]. A specific example is constructed.

2. Organization

This paper is organized as follows. In section 3, the classification of types of isometries in $\mathbb{H}^4$ are listed and defined in terms of reflection and other factoring. The half-turn is also defined. Some types are collected into so-called *atomic* types to consolidate a few definitions, while others are broken further into subtypes. In section 4, the new definitions of permuted, invariant and twisting (hyperbolic) pencils are defined for each type and subtype of isometry. The properties of hyperbolic pencils relevant to the main results are also listed. In section 5, the half-turn bank of each type of isometry is introduced and defined. Theorem 5.5 is proved, yielding the fact that the order of the linked pairs is negligible. Geometric criteria for a plane to be in a half-turn bank are listed and proven in section 5.1. In section 6, planes are combined to show the result of composing two half-turns. The main theorem is restated and proven in Corollary 6.11.

In section 7, some conditions for a pair to be linked are listed. The pairs of isometries are arranged according to the type or class.

In section 8, the hyperboloid model of hyperbolic space is used to find invariant subspaces of an isometry which relate to its hyperbolic pencils. The invariant spaces help in constructing a linked pair that does not restrict to lower dimensions, implying Theorem 8.1.

3. Types of Isometries in $\mathbb{H}^4$

In general dimensions, the isometries of $\mathbb{H}^n$ classify into elliptic, hyperbolic, and parabolic. Any isometry can be expressed as a composition of at most $n + 1$ reflections. In dimension 4, the classification of nontrivial isometries can subdivided into six types: type-I elliptic, type-II elliptic, pure hyperbolic, pure loxodromic, pure parabolic, and screw parabolic (see [8], [9]).

Pure parabolic and pure hyperbolic isometries are translations generalized from dimension 2. A type-I elliptic isometry has a fixed-point set that forms a plane. Together, type-I elliptic, pure parabolic, and pure hyperbolic are called *atomic* isometries since they can be expressed as compositions of two reflections. A screw parabolic or pure loxodromic isometry can be decomposed uniquely into two commuting atomic isometries called its translation part and its rotational part. The fixed-point set of its rotational part is called its twisting plane. A type-II elliptic isometry ρ is conjugate to a matrix with two diagonal blocks of 2×2 matrices in SO(2). Hence, there is an unordered pair of angles α and β associated with ρ. If $\alpha = \beta = \pi$, then ρ is conjugate to the antipodal map and ρ is an involution. Otherwise, if both α and β are not integer multiples of π, then there is a unique unordered pair of planes left invariant by ρ.

Using the upper half-space model and restricting the action to the boundary, a parabolic isometry τ is conjugate to a Möbius transformation of the form $x \mapsto Ax + b$ where $b \neq 0$, $A \in$ SO(3), and $Ab = b$ (see [9]). Assume for the moment that $\tau(x) = Ax + b$. Then b is called the *direction of* τ. The Euclidean line connecting 0 to b in $\widehat{\mathbb{R}^3}$ is left invariant by τ and so is the plane P_τ in $\mathbb{H}^4$ bounded by this Euclidean line. Note that if A is nontrivial, then τ is screw parabolic and P_τ is its twisting plane. The map $x \mapsto Ax$ is the rotational part of τ, while $x \mapsto x + b$ is the translation part.

The following are definitions of the six types of isometries of $\mathbb{H}^4$ and half-turn in terms of the number of reflection factors or atomic isometry factors (see [8], [10]). The advantage of these definitions is to locate the half-turn and reflection factors.

Definition 3.1. A *type-I elliptic isometry of* $\mathbb{H}^4$ is a composition of two reflections across distinct hyperplanes that intersect in a plane.

Let ρ be a type-I elliptic isometry of $\mathbb{H}^4$. The *twisting plane of* ρ is the set of fixed points of ρ in $\mathbb{H}^4$.

Let P be a plane in $\mathbb{H}^4$. The *half-turn* H_P *about* P is the elliptic isometry H_P that is a composition of reflections across an orthogonal pair of hyperplanes that intersects in P.

An elliptic isometry ρ of $\mathbb{H}^4$ is called *of type-II* if ρ fixes a unique point in $\mathbb{H}^4$.

A pure *hyperbolic isometry* δ of $\mathbb{H}^4$ is a composition of exactly two reflections across ultra-parallel hyperplanes.

A *pure loxodromic isometry* γ is a hyperbolic isometry of $\mathbb{H}^4$ that can be expressed as $\delta\rho$ where δ is a pure hyperbolic isometry leaving the axis of γ invariant and ρ is a type-I elliptic isometry whose twisting plane contains the axis of γ.

A *pure parabolic isometry* τ of $\mathbb{H}^4$ is a composition of exactly two reflections across hyperplanes that are tangent to a unique point at infinity.

A *screw parabolic isometry* γ is a parabolic isometry of $\mathbb{H}^4$ that can be expressed as $\tau\rho$ where τ is a pure parabolic isometry fixing the fixed point of γ and ρ is a type-I elliptic isometry whose twisting plane is left invariant by τ.

The ρ factor in the definitions of either pure loxodromic or screw parabolic isometries is called its *rotational part*, while the other factor (δ or τ) is called its *translation part*. These factors commute and are unique. The fixed-point set of the rotational part of γ is left invariant by γ and is called the *twisting plane* of γ.

An orientation preserving type-II elliptic isometry can be further classified into either an involution (antipodal map) or having a unique pair of invariant planes that intersect orthogonally to a single point. The latter subtype also has a pair of associated angles to the invariant planes; one such angle is not an integer multiple of π.

4. Hyperbolic Pencils

The concept of pencils can be found in complex analysis, inversive geometry, and projective geometry. Pencils are sometimes called coaxial families/systems of circles. In this section, new constructions and types of pencils are introduced to aid in the process of factoring isometries into half-turns. The permuted, invariant, dual, and twisting pencils of an isometry for each type are defined in this section. All types, except type-II elliptic, have permuted and dual pencils. The atomic isometries have invariant pencils, while those with rotational parts have twisting pencils.

Definition 4.1. Let ρ be a type-I elliptic isometry of $\mathbb{H}^4$ with a twisting plane P. The (elliptic) *permuted pencil of ρ* is the set

$$\mathcal{F}_\rho = \left\{ \partial h \subset \partial \mathbb{H}^4 : h \text{ is a hyperplane containing } P \right\}.$$

The (elliptic) *invariant pencil of ρ* is the set

$$\mathcal{T}_\rho = \left\{ \partial h \subset \partial \mathbb{H}^4 : h \text{ is a hyperplane orthogonal to } P \right\}.$$

The *dual pencil of ρ* is the set

$$\mathcal{D}_\rho = \big\{ \partial Q \subset \partial \mathbb{H}^4 : Q \text{ is a plane orthogonally intersecting}$$
$$P \text{ in a unique point} \big\}.$$

Definition 4.2. Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with axis L. The *permuted pencil $\mathcal{F}_\gamma$ of γ* is the set

$$\mathcal{F}_\gamma = \big\{ \partial h_x \subset \partial \mathbb{H}^4 : x \in L \text{ and } h_x \text{ is the orthogonal}$$
$$\text{complement of } L \text{ through } x \big\}.$$

The *invariant pencil of γ* is the set

$$\mathcal{T}_\gamma = \left\{ \partial t \subset \partial \mathbb{H}^4 : t \text{ is a hyperplane containing } L \right\}.$$

The *dual pencil of γ* or *pencil dual to $\mathcal{F}_\gamma$* is the set

$$\mathcal{D}_\gamma = \left\{ \partial P \subset \partial \mathbb{H}^4 : P \text{ is a plane containing } L \right\}.$$

It is possible to define the permuted, dual, and invariant pencils of any parabolic isometry that fixes any point other than ∞ (see [10]). For easy reading, we define the pencils by conjugating its fixed point to ∞.

Let τ be the map sending x to $Ax + b$ in $\widehat{\mathbb{R}^3} = \partial \mathbb{H}^4$ where $A \in \mathrm{SO}(3)$, $Ab = b$, and $b \neq 0$. The Euclidean line connecting 0 to b is left invariant by τ. Moreover, if A is trivial, then any other Euclidean line parallel to $\mathrm{span}\{b\}$ is also left invariant by τ and so are the Euclidean planes parallel to b.

Definition 4.3. Let τ be the map of $\widehat{\mathbb{R}^3}$ that sends x to $Ax + b$ in $\widehat{\mathbb{R}^3}$ where $A \in \mathrm{SO}(3)$, $Ab = b$, and $b \neq 0$. The *permuted pencil of τ* is the set

$$\mathcal{F}_\tau = \{(\text{Euclidean planes orthogonal to } b) \cup \{\infty\}\}.$$

The *invariant pencil of τ* is the set

$$\mathcal{T}_\tau = \{(\text{Euclidean planes parallel to } b) \cup \{\infty\}\}.$$

The *dual pencil of τ* is the set

$$\mathcal{D}_\tau = \{(\text{Euclidean lines parallel to } b) \cup \{\infty\}\}.$$

Definition 4.4. Let γ be either a pure loxodromic or a screw parabolic isometry of $\mathbb{H}^4$ with rotational part ρ. The *twisting pencil of γ* is defined to be the permuted pencil of ρ. The twisting pencil of γ is denoted $\mathcal{R}_\gamma$.

4.1. **Visualizing pencils.**

While all the pencils can be defined without assuming special fixed points, it may be difficult to imagine. Hence, we give examples of how they can possibly look. The formulas here are restrictions to the boundary model $\widehat{\mathbb{R}^3}$. By Poincaré extension, they extend uniquely to isometries of $\mathbb{H}^4$.

Let δ be the map $x \mapsto \lambda x$ where $\lambda > 0$. Then δ fixes 0 and ∞. Its pencils are as follows:

$$\mathcal{F}_\delta = \{\text{Euclidean spheres centered at the origin}\}.$$
$$\mathcal{T}_\delta = \{(\text{Euclidean planes through the origin}) \cup \{\infty\}\}.$$
$$\mathcal{D}_\delta = \{(\text{Euclidean lines through the origin}) \cup \{\infty\}\}.$$

Let ρ be the elliptic isometry that fixes the z-axis of $\mathbb{R}^3 \subset \widehat{\mathbb{R}^3}$ pointwise and rotating by angle $\pi/4$. Its pencils are as follows:

$$\mathcal{F}_\rho = \{(\text{Euclidean planes containing the } z\text{-axis}) \cup \{\infty\}\}.$$
$$\mathcal{T}_\rho = \{(\text{Euclidean planes orthogonal to the } z\text{-axis}) \cup \{\infty\}\} \cup$$
$$\{\text{spheres centered at the } z\text{-axis}\}.$$
$$\mathcal{D}_\rho = \{\text{circles centered at the } z\text{-axis but contained in a plane in } \mathcal{T}_\rho\}.$$

The composition $\delta\rho = \rho\delta$ has ρ as its rotational part and δ as its translation part. If $b = (0,0,1)$, then $x \mapsto \rho(x) + b$ has ρ as its rotational part, while $x \mapsto x + b$ is its translation part.

4.2. **Properties of pencils.**

For an atomic type of isometry, the permuted pencil serves as the source for reflection factoring. The rest of the nontrivial isometries need four factors of reflections. If h is a sphere at infinity or a hyperplane inside $\mathbb{H}^4$, denote the reflection across h with R_h. The following theorem is well known in many books about hyperbolic geometry, such as [3], [7], [8], [9].

Theorem 4.5. *Let τ be an atomic isometry of $\mathbb{H}^4$. Then, for each $h \in \mathcal{F}_\tau$, there exist $h_1, h_2 \in \mathcal{F}_\tau$ such that $\tau = R_{h_1} R_h = R_h R_{h_2}$.*

The elements of permuted pencils fill up the boundary while the hyperplanes that they bound fill up the hyperbolic 4-space. If there is a point x outside the axis, twisting plane, or fixed points of γ, there is a unique element of $\mathcal{F}_\gamma$ that contains x.

The rest of the properties are listed as follows.

Let ρ be an atomic isometry.

(1) If $t \in \mathcal{T}_\rho$, then $\rho(t) = t$ and t is orthogonal to each $s \in \mathcal{F}_\rho$.

(2) If $t, u \in \mathcal{T}_\rho$ such that $u \cap t$ has more than one point but $u \neq t$, then $u \cap t \in \mathcal{D}_\rho$.

(3) If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some $c \in \mathcal{D}_\rho$, then $s \in \mathcal{T}_\rho$.

(4) For each $d \in \mathcal{D}_\rho$, there are $t, u \in \mathcal{T}_\rho$ such that $d = t \cap u$.

(5) For each pair $p, q \in \mathcal{D}_\rho$ with $p \neq q$, there is a unique $t \in \mathcal{T}_\rho$ containing $p \cup q$.

Let γ be either a pure loxodromic or a screw parabolic isometry of $\mathbb{H}^4$ with twisting plane P.

(1) If $t \in \mathcal{R}_\gamma$, then $\gamma(t) \in \mathcal{R}_\gamma$, $\gamma(t) \cap t = \partial P$, and t is orthogonal to each $s \in \mathcal{F}_\gamma$.

(2) If $t, u \in \mathcal{R}_\gamma$ with $u \neq t$, then $u \cap t = \partial P$.

(3) If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains ∂P, then $s \in \mathcal{R}_\gamma$.

(4) For each $d \in \mathcal{D}_\gamma \setminus \{\partial P\}$, there is a unique $t \in \mathcal{R}_\gamma$ containing $d \cup (\partial P)$.

(5) If ρ is the rotational part of γ, then $\mathcal{F}_\rho \subset \mathcal{T}_\gamma$.

5. Half-Turn Banks

In order to find half-turns or planes in which involution factoring is possible, a space of planes is collected similarly to the elements of the pencils. The space $\mathcal{K}_\gamma$, called the half-turn bank of γ, is a collection of circles derived from the pencils, but not prevalent enough to be called another pencil. The goal is to show that all half-turn factorizations of γ come from $\mathcal{K}_\gamma$.

Whenever there is no confusion, we refer to the elements of $\mathcal{K}_\gamma$ as either planes in $\mathbb{H}^4$ or circles in $\partial \mathbb{H}^4$.

Definition 5.1. Let ρ be an atomic isometry of $\mathbb{H}^4$. The *half-turn bank of ρ* is the set

$$\mathcal{K}_\rho = \left\{ s \cap t \subset \partial \mathbb{H}^4 : s \in \mathcal{F}_\rho \text{ and } t \in \mathcal{T}_\rho \right\}.$$

The *half-turn bank of the identity map of $\mathbb{H}^4$* is the set $\mathcal{K}_{\mathrm{id}}$ consisting of the circles that bound all planes in $\mathbb{H}^4$.

Definition 5.2. Let γ be either a pure loxodromic or a screw parabolic isometry of $\mathbb{H}^4$. The *half-turn bank of γ* is the set

$$\mathcal{K}_\gamma = \left\{ s \cap t \subset \widehat{\mathbb{R}^3} : s \in \mathcal{F}_\gamma \text{ and } t \in \mathcal{R}_\gamma \right\}.$$

As previously mentioned, an orientation preserving elliptic isometry ρ is conjugate to an element of $\mathrm{SO}(4)$ that has two 2×2 blocks of matrices in $\mathrm{SO}(2)$. The associated angles α and β tell whether ρ is type-I or type-II. The condition for type-II elliptic is that α and β are nonzero. If exactly one of them is neither π nor 0, then ρ can be factored uniquely

into two commuting type-I elliptic isometries ρ_1 and ρ_2 whose twisting planes are orthogonally intersecting in a single point. Thus, ρ also leaves these two twisting planes invariant. If $\alpha = \beta = \pi$, then ρ is conjugate to the antipodal map of the 3-sphere and is also an involution.

Definition 5.3. Let $\rho = \rho_1 \rho_2$ be an orientation preserving type-II elliptic isometry of $\mathbb{H}^4$ with an associated angle not an integer multiple of π so that ρ_1 and ρ_2 are the unique type-I elliptic elements whose twisting planes intersect orthogonally in the fixed point. The *half-turn bank of ρ* is the set
$$\mathcal{K}_\rho = \left\{ s \cap t \subset \partial \mathbb{H}^4 : s \in \mathcal{F}_{\rho_1} \text{ and } t \in \mathcal{F}_{\rho_2} \right\}.$$

Definition 5.4. Let ρ be an isometry of $\mathbb{H}^4$ that is conjugate to the antipodal map with fixed point x. The *half-turn bank of ρ* is the set
$$\mathcal{K}_\rho = \left\{ \partial P \subset \partial \mathbb{H}^4 : P \text{ is a plane containing } x \right\}.$$

If k is the boundary at infinity of a plane P, let H_k be the half-turn about P as well. The following shows that the half-turn bank is a source for the half-turn factoring of an orientation preserving isometry.

Theorem 5.5. *Let γ be an orientation preserving isometry of $\mathbb{H}^4$. Then, for each $k \in \mathcal{K}_\gamma$, there exist $k_1, k_2 \in \mathcal{K}_\gamma$ such that $\gamma = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. If γ has a rotational part, let τ be the translation part of γ and ρ be the rotational part of γ. If $k \in \mathcal{K}_\gamma$, there are $s \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\tau = \mathcal{F}_\gamma$ such that $\tau = R_{s_1} R_s = R_s R_{s_2}$ and there are $t_1, t_2 \in \mathcal{F}_\rho = \mathcal{R}_\gamma$ such that $\rho = R_{t_1} R_t = R_t R_{t_2}$. Each sphere in $\{s, s_1, s_2\}$ is orthogonal to each sphere in $\{t, t_1, t_2\}$. The following relations hold.

$$\begin{aligned} R_s R_{t_1} &= R_{t_1} R_s & H_k &= R_s R_t \\ R_{t_2} R_s &= R_s R_{t_2} & H_k &= R_t R_s. \end{aligned}$$

Let $k_1 = s_1 \cap t_1$ and $k_2 = s_2 \cap t_2$. Then $H_{k_1} = R_{s_1} R_{t_1} = R_{t_1} R_{s_1}$ and $H_{k_2} = R_{s_2} R_{t_2} = R_{t_2} R_{s_2}$. Since $\gamma = \tau \rho = \rho \tau$,

$$\begin{aligned} \gamma &= \tau \rho & \gamma &= \rho \gamma \\ &= R_{s_1} R_s R_{t_1} R_t & &= R_t R_{t_2} R_s R_{s_2} \\ &= R_{s_1} R_{t_1} R_s R_t & &= R_t R_s R_{t_2} R_{s_2} \\ &= H_{k_1} H_k & &= H_k H_{k_2}. \end{aligned}$$

If γ is atomic and $k \in \mathcal{K}_\gamma$, there are $s \in \mathcal{F}_\gamma$ and $t \in \mathcal{T}_\gamma$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\gamma$ such that $\gamma = R_{s_1} R_s = R_s R_{s_2}$. Let $k_1 = s_1 \cap t$ and $k_2 = s_2 \cap t$. Since s, s_1, and s_2 are orthogonal to t,

$H_{k_1} = R_{s_1}R_t = R_tR_{s_1}$ and $H_{k_2} = R_{s_2}R_t = R_tR_{s_2}$. Then

$$\begin{aligned}
\delta &= R_{s_1}R_s & \delta &= R_sR_{s_2} \\
&= R_{s_1}R_tR_tR_s & &= R_sR_tR_tR_{s_2} \\
&= H_{k_1}H_k & &= H_kH_{k_2}.
\end{aligned}$$

If γ is a type-II elliptic involution, let P be the plane bounded by k. Then there is a unique plane Q orthogonal to P at the fixed point of γ. Let $k_1 = \partial Q$. Then $\gamma = H_kH_{k_1} = H_{k_1}H_k$ as shown in Corollary 6.8. Let $k_2 = k_1$ so that $\gamma = H_kH_{k_2}$.

If γ is type-II elliptic but not an involution, let $\gamma = \rho_1\rho_2$ be its factorization in Definition 5.3. Then there are $s \in \mathcal{F}_{\rho_1}$ and $t \in \mathcal{F}_{\rho_2}$ such that $k = s \cap t$. There are also $s_1, s_2 \in \mathcal{F}_{\rho_1}$ such that $\rho_1 = R_{s_1}R_s = R_sR_{s_2}$ and $t_1, t_2 \in \mathcal{F}_{\rho_2}$ such that $\rho_2 = R_{t_1}R_t = R_tR_{t_2}$. Since $s, s_1, s_2 \in \mathcal{T}_{\rho_2}$ and $t, t_1, t_2 \in \mathcal{T}_{\rho_1}$, each sphere in $\{s, s_1, s_2\}$ is orthogonal to each sphere in $\{t, t_1, t_2\}$. Let $k_1 = s_1 \cap t_1$ and $k_2 = s_2 \cap t_2$. Similar to the case where γ has a rotational part, we have $\gamma = H_{k_1}H_k = H_kH_{k_2}$. $\qquad\square$

5.1. ALTERNATIVE GEOMETRIC DEFINITIONS.

The elements of half-turn banks are defined as intersections of spheres. In lower dimensions, involution-factoring is constructed from the lines perpendicular to the axis of an isometry [6], [7]. The following statements are the analogous geometric definitions.

Theorem 5.6. *Let ρ be a type-I elliptic isometry of $\mathbb{H}^4$ with twisting plane P. Then Q is a plane orthogonal to P through a line if and only if $\partial Q \in \mathcal{K}_\rho$.*

Proof. If Q intersects P in a line ℓ, $P \cup Q$ forms a unique hyperplane h. There is a hyperplane t orthogonal to h through Q. Then P is orthogonal to t through ℓ since $P \subset h$. It follows that $\partial t \in \mathcal{T}_\rho$ and $\partial h \in \mathcal{F}_\rho$. Since $Q = t \cap h$, $\partial Q \in \mathcal{K}\rho$.

Conversely, if $\partial Q \in \mathcal{K}_\rho$, let $h \in \mathcal{F}_\rho$ and $t \in \mathcal{T}_\rho$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplanes bounded by h and t, respectively. The hyperplane $\hat{t}$ and the plane P are orthogonal through a line. Then $(\partial P) \cap t$ is a set of two points contained in ∂Q since $\partial P \subset h$. Both P and Q are planes in the hyperplane bounded by h; they intersect in the line $P \cap \hat{t}$. Since $\hat{t}$ is orthogonal to P and $Q \subset \hat{h}$, P and Q are orthogonal through the line $P \cap \hat{t}$. $\qquad\square$

Theorem 5.7. *Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$ with axis L. Then P is a plane orthogonal to L if and only if $\partial P \in \mathcal{K}_\delta$.*

Proof. Suppose P is a plane orthogonal to L. Let b be the intersection point of P and L. There are orthogonal lines $v_P, w_P \subset P$ through b. Let

y be the unique line orthogonal to v_P, w_P, and L through b. There is a unique hyperplane h_b orthogonal to L through b. Then $y \subset h_b$. Let $\hat{t}$ be the unique hyperplane spanned by L, w_P, and v_P. Then $h_b \cap \hat{t} = P$ since $w_P, v_P \subset h_b \cap \hat{t}$. Let $t = \partial \hat{t}$ and $s = \partial h_b$. Then $t \cap s = \partial P$. Since h_b is orthogonal to L, s is in $\mathcal{F}_\delta$. Similarly, the fact that $\hat{t}$ contains L implies that $t \in \mathcal{T}_\delta$.

Conversely, if $\partial P \in \mathcal{K}_\delta$, there are $h \in \mathcal{F}_\delta$ and $t \in \mathcal{T}_\delta$ such that $\partial P = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplanes bounded by h and t, respectively. Then $\hat{h}$ is orthogonal to L through a point x. Since $\hat{t}$ contains L and P, $x \in L \cap P$. As a subset of $\hat{h}$ containing x, P is orthogonal to L. $\qquad\square$

Theorem 5.8. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$. Then c is a circle in $\widehat{\mathbb{R}^3}$ such that $v \in c \subset h$ for some $h \in \mathcal{F}_\tau$ if and only if $c \in \mathcal{K}_\tau$.*

Proof. Let c be a circle in $\widehat{\mathbb{R}^3}$ such that $v \in c \subset h$ for some $h \in \mathcal{F}_\tau$. It suffices to show that $c \in \mathcal{K}_\tau$ in the case where $v = \infty$. Then h is a Euclidean plane and c is a Euclidean line. Let t be the set $\{x + \lambda\tau(0) : x \in c \setminus \{\infty\}, \lambda \in \mathbb{R}\} \cup \{\infty\}$. Then t is orthogonal to h through c so $t \in \mathcal{T}_\tau$ and $c = h \cap t$. Hence, $c \in \mathcal{K}_\tau$.

Conversely, if $c \in \mathcal{K}_\tau$, there are $h \in \mathcal{F}_\tau$ and $t \in \mathcal{T}_\tau$ such that $c = h \cap t$. Then $c \subset h$. Both h and t contain v, so $v \in c$. $\qquad\square$

Theorem 5.9. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with axis L and twisting plane P. Then $Q \subset \mathbb{H}^4$ is a plane orthogonal to both P and L through a line in P if and only if $\partial Q \in \mathcal{K}_\gamma$.*

Proof. Suppose Q is a plane orthogonal to both P and L such that $Q \cap P$ is a line. Let $\ell = Q \cap P$. If Q is orthogonal to L, then ℓ and L are perpendicular lines in P. Let x be the intersection point Q and L. Let ℓ' be the unique line in Q perpendicular to ℓ through x. Likewise ℓ' is perpendicular to L. There is a unique line ℓ_4 perpendicular to all L, ℓ, and ℓ'. In particular, ℓ_4 is the unique line orthogonal to the hyperplane spanned by P and Q passing through x. Let h be the hyperplane spanned by ℓ, ℓ', and ℓ_4. Since L is perpendicular to all ℓ, ℓ', and ℓ_4, h is the orthogonal complement of L through x. Then $\partial h \in \mathcal{F}_\gamma$. Let t be the hyperplane spanned by L, ℓ, and ℓ'. Therefore, $P \subset t$ and $t \cap h = Q$. Hence, $\partial t \in \mathcal{R}_\gamma$ and $\partial Q \in \mathcal{K}_\gamma$.

Conversely, if $\partial Q \in \mathcal{K}_\gamma$, there are $h \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplanes bounded by h and t, respectively. Then $\hat{h}$ is orthogonal to P through a line since ∂P is orthogonal to every element of $\mathcal{F}_\gamma$. Whereas $\hat{t}$ contains P, so $Q = \hat{h} \cap \hat{t}$ must also intersect P in the line $P \cap \hat{h}$ since $Q \cap P = (\hat{h} \cap \hat{t}) \cap P = \hat{h} \cap (\hat{t} \cap P) = \hat{h} \cap P$.

Then P is orthogonal to $\hat{h}$ through $P \cap Q$, so $Q \subset \hat{h}$ implies that P is also orthogonal to Q through the line $P \cap Q$.

Since L is a subset of $\hat{t}$ and L is orthogonal to $\hat{h}$ through x, Q intersects L at x. But $Q \subset \hat{h}$, so Q is orthogonal to L. $\qquad\square$

Theorem 5.10. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with fixed point v and twisting plane P. Then $Q \subset \mathbb{H}^4$ is a plane orthogonal to P through a line bounded by v if and only if $\partial Q \in \mathcal{K}_\gamma$.*

Proof. Suppose Q is a plane orthogonal to P through a line bounded by v. There is a hyperplane t containing $Q \cup P$ since they intersect in a line. Let $\ell = Q \cap P$. There is a point $x \in \partial \ell \setminus \{v\}$. Then there is a unique $h_x \in \mathcal{F}_\gamma$ that contains x. Since Q is orthogonal to P, Q is contained in the hyperplane bounded by h_x. But $Q, P \subset t$, so $\partial t \in \mathcal{R}_\gamma$ and $\partial Q = (\partial t) \cap h_x$. Hence, $\partial Q \in \mathcal{K}_\gamma$.

Conversely, if $\partial Q \in \mathcal{K}_\gamma$, there are $h \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplane bounded by h and t, respectively. Then $\hat{h}$ is orthogonal to P through a line bounded by v. Also $\hat{t}$ contains P, so $Q = \hat{h} \cap \hat{t}$ implies $Q \cap P = (\hat{h} \cap \hat{t}) \cap P = \hat{h} \cap (\hat{t} \cap P) = \hat{h} \cap P$. Then P is orthogonal to $\hat{h}$ through $P \cap Q$ and hence to Q. Since $v \in h \cap t$, $v \in \partial Q$; $v \in \partial P$ implies that v bounds the line $P \cap Q$. $\qquad\square$

6. Properties of Half-Turn Banks

The half-turn bank of an isometry of hyperbolic space serves as a collection of circles in which the isometry can be factored into a product of two half-turns. In this section, we show that any half-turn factorization of an orientation preserving isometry comes from the defined set called the half-turn bank.

In section 6.1, the common perpendicular line across a pair of ultra-parallel $(n-2)$-dimensional planes is constructed. In section 6.2 all possible combinations of pairs of planes in $\mathbb{H}^4$ are enumerated to show the result of composing half-turns about those planes. Together with classification of isometries, the combinations tell how the involved planes intersect whenever an isometry is factored into a product of two half-turns. Section 6.3 has the details of proving that all half-turn factorizations come from half-turn banks. The theorem implies that a pair of isometries is linked by a half-turn if and only if they have a common circle in their half-turn banks.

6.1. CODIMENSION-2 VERSION OF COMMON PERPENDICULAR.

If P and Q are ultra-parallel hyperplanes in $\mathbb{H}^4$, there is a unique line perpendicular to them (see [3], [6], [7]). If P and Q are ultra-parallel lines in $\mathbb{H}^4$, there is also a unique line perpendicular to them. The same result is shown for a pair of ultra-parallel $(n-2)$-dimensional subplanes of $\mathbb{H}^n$.

Theorem 6.1. *For each pair of ultra-parallel planes* $\alpha, \beta \in \mathbb{H}^4$*, there is a unique line orthogonal to both* α *and* β*.*

The definitions of half-turn banks can be extended to orientation preserving isometries of $\mathbb{H}^n$, but to justify its name, a more general statement is stated as follows.

Theorem 6.2. *Let* $n \geq 2$ *be an integer. For each pair of ultra-parallel* $(n-2)$*-dimensional planes* $\alpha, \beta \in \mathbb{H}^n$*, there is a unique line orthogonal to both* α *and* β*.*

Proof. Theorem 6.2 implies Theorem 6.1, so constructing the common orthogonal line in a general setting is enough for a case in the proof of Theorem 6.10. Let P and Q be ultra-parallel $(n-2)$-dimensional planes of $\mathbb{H}^n$. If $n = 2$, then P and Q are distinct points, and the unique line connecting them are vacuously orthogonal to P and Q. If $n = 3$, then P and Q are ultra-parallel lines, so there is a unique line perpendicular to both of them (see [3], [6], [7]). If $n \geq 4$, we use the hyperboloid model of $\mathbb{H}^n$ embedded in the Lorentzian space $\mathbb{R}^{n,1}$. Then P and Q extend to $(n-1)$-dimensional vector subspaces P' and Q', respectively. Since P and Q are ultra-parallel, the span of $P' \cup Q'$ is at least of dimension n.

Therefore, the dimension of the space-like subspace $P' \cap Q'$ is either $n-2$ or $n-3$. If it is $n-2$, then $\mathrm{span}(P' \cup Q')$ is n-dimensional and has P' and Q' as its hyperplanes with a unique common perpendicular line (see [9]). If $\dim(P' \cap Q')$ is $n-3$, then $N = (P' \cap Q')^L$, the Lorentz orthogonal complement of $P' \cap Q'$, is a 4-dimensional time-like vector subspace nontrivially intersecting both P' and Q' due to their large dimensions. Since N contains P'^L and Q'^L, which, respectively, intersect P' and Q' trivially, $N \cap P'$ and $N \cap Q'$ are 2-dimensional subspaces. If they are time-like, then there is a unique line in $\mathbb{H}^n$ perpendicular to both $\mathbb{H}^n \cap N \cap P'$ and $\mathbb{H}^n \cap N \cap Q'$. We must show that $N \cap P'$ and $N \cap Q'$ are time-like vector subspaces.

Pick a time-like vector $x \in N$. Then $x \notin P'^L \cup Q'^L$ since P'^L and Q'^L are space-like. Let $w \in P'^L$ and $v \in Q'^L$ be nontrivial elements so they are linearly independent with x. Since P'^L and Q'^L are subsets of N, the spans of $\{x, w\}$ and $\{x, v\}$ are subspaces of N. Then the vector $-\frac{\langle x,w \rangle_L}{\langle w,w \rangle_L} w + x$ is an element of both N and P' since it is Lorentz orthogonal

to $w \in P'^L$, and is a linear combination of w and x. Then its Lorentz norm can be computed as follows.

$$\left\| -\frac{\langle x, w \rangle_L}{\langle w, w \rangle_L} w + x \right\|_L^2 = \frac{\langle x, w \rangle_L^2}{\|w\|_L^2} - 2\frac{\langle x, w \rangle_L^2}{\|w\|_L^2} + \|x\|_L^2$$
$$= -\frac{\langle x, w \rangle_L^2}{\|w\|_L^2} + \|x\|_L^2 \, .$$

Since x is time-like and w is space-like, the quantity above is negative, making the vector time-like. Hence, $N \cap P'$ is a 2-dimensional time-like subspace of P'. By replacing the vector w with v, the same arguments show that $N \cap Q'$ is also a 2-dimensional time-like subspace of Q'. $\square$

6.2. **COMBINATIONS OF A PAIR OF PLANES.**

There are a few ways for two planes in $\mathbb{H}^4$ to intersect. They can intersect in a line, in a point, or in a point at infinity. The last case is not technically an intersection, but points in the two planes can be arbitrarily near.

Let P and Q be distinct planes in $\mathbb{H}^4$. There are four ways they can intersect or not intersect: ultra-parallel, both tangent at a unique point at infinity, in a line, or in a single point. Each of these combinations forms a unique type of isometry of $H_P H_Q$. The following are the results.

Lemma 6.3. *Let P and Q be ultra-parallel planes in $\mathbb{H}^4$. Then $H_P H_Q$ is a hyperbolic isometry.*

Proof. If P and Q are ultra-parallel, then there is a unique line N in $\mathbb{H}^4$ that is orthogonal to both P and Q. Both H_P and H_Q leave N invariant, so $H_P H_Q$ must also leave N invariant. That leaves two options for $H_P H_Q$: either hyperbolic or elliptic. Suppose $H_P H_Q$ has a fixed point y inside $\mathbb{H}^4$. Then $H_P(y) = H_Q(y)$, so the midpoint between y and $H_P(y)$ is an element of both P and Q. This contradicts the hypothesis that P and Q are ultra-parallel. $\square$

Lemma 6.4. *Let P and Q be tangent planes in $\mathbb{H}^4$. Then $H_P H_Q$ is a parabolic isometry.*

Proof. Let P and Q be distinct planes in $\mathbb{H}^4$ so that their boundaries meet at one point x at infinity. By conjugation, we may assume $x = \infty$ of $\widehat{\mathbb{R}^3}$. Then P' and Q' are Euclidean lines. If P' and Q' form a Euclidean plane, there are several Euclidean lines perpendicular to both P' and Q', but all of them identify a unique direction or vector $v \in \mathbb{R}^3$. If they do not form a plane, there is a unique Euclidean line N orthogonal to P' and Q', which identifies a direction $v \in \mathbb{R}^3$. In either case, all Euclidean

lines with same direction as v are left invariant by $H_P H_Q$. Furthermore, $H_P H_Q$ is a parabolic translation ($x \mapsto x + 2v, \quad x \in \mathbb{R}^3$) if P' and Q' are coplanar or a screw parabolic isometry with twisting plane bounded by N if not. So if P and Q are tangent, $H_P H_Q$ is definitely parabolic. $\qquad \square$

Lemma 6.5. *Let P and Q be distinct planes in $\mathbb{H}^4$ intersecting in a line. Then $H_P H_Q$ is a type-I elliptic isometry whose twisting plane is orthogonal to the hyperplane containing $P \cup Q$ through $P \cap Q$.*

Proof. Let P and Q be planes of $\mathbb{H}^4$ intersecting in a line L. Then they form a unique hyperplane h whose boundary is an element of both permuted pencils of H_P and H_Q. It follows that h is left invariant and L is fixed pointwise by $H_P H_Q$. The plane τ orthogonal to h through L is also left by $H_P H_Q$, but since L is fixed pointwise, $H_P H_Q$ also fixes τ pointwise. To illustrate it with the model $\widehat{\mathbb{R}^3}$ as the boundary at infinity, assume P' is a Euclidean line and Q' is a circle intersecting P' in two points y_1 and y_2 . Then $Q' \cup P'$ forms a unique Euclidean plane $\hat{h}$. There is a unique circle $\hat{\tau}$ passing through y_1 and y_2 centered at their midpoint and inside the Euclidean plane orthogonal to $\hat{h}$ through P'. Then $\hat{\tau}$ is orthogonal to both P' and Q' through $P' \cap Q'$. Both half-turns H_P and H_Q flip $\hat{\tau}$ across $P' \cap Q'$ so the composition $H_P H_Q$ fixes $\hat{\tau}$ pointwise. Since $\hat{\tau}$ is the unique plane orthogonal to h through $P \cap Q$, $\hat{\tau}$ is the only fixed-point set of $H_P H_Q$. It follows that $H_P H_Q$ is a type-I elliptic isometry. Moreover, its twisting plane is orthogonal to $\hat{h}$. $\qquad \square$

Lemma 6.6. *Let P and Q be distinct planes in $\mathbb{H}^4$ intersecting in a single point. Then $H_P H_Q$ is a type-II orientation preserving elliptic isometry.*

Proof. Suppose P and Q are planes $\mathbb{H}^4$ intersecting in a unique point p. Then for each $x \in \mathbb{H}^4 \setminus \{p\}$, we show that $H_P H_Q(x) \neq x$. Suppose first that $x \in Q$. Then $H_P H_Q(x) = H_P(x)$, which is not equal to x since $x \notin P$. For the rest of this combination of P and Q, assume that $x \in \mathbb{H}^4 \setminus Q$. If it happens that $H_Q(x) \in P$, then $H_P H_Q(x) = H_Q(x) \neq x$. If $H_Q(x) \notin P$ but $x \in P$, then $H_P H_Q(x) \notin P$ so $H_P H_Q(x) \neq x$.

The last possibility is when both $H_Q(x)$ and x are outside $Q \cup P$. We prove that H_P cannot map $H_Q(x)$ back to x. Suppose $H_P H_Q(x) = x$. Let m be the midpoint between $H_Q(x)$ and x, and let L be the line connecting x to $H_Q(x)$. Then m is in Q since it is the midpoint between x and $H_Q(x)$. Likewise, m must also be in P as $H_P(x) = H_Q(x)$. There is only one point in $P \cap Q$ so this midpoint must be p. Furthermore, L is orthogonal to both P and Q through p. It follows that the hyperplane orthogonal to L through p contains both P and Q. Since intersecting planes within a hyperplane must meet in at least a line, $P \cap Q$ must have

at least a line which has more points other than p. This contradicts that P and Q intersect only in a point.   □

Since $H_P H_Q$ is type-II elliptic, one might wonder where the invariant planes or lines are located. If P and Q are orthogonal complements of each other, then H_Q and H_P, respectively, rotate them half-way around. Otherwise, $H_P H_Q$ has a canonical pair of invariant planes. To locate these invariant planes, we can use the ball model of $\mathbb{H}^4$ embedded in $\mathbb{R}^4$ which is conformal to both the Euclidean geometry and the spherical geometry of S^3. The problem simplifies further if P and Q are conjugated to intersect at the origin. The following lemma locates the canonical invariant planes relative to P and Q.

Lemma 6.7. *Let P and Q be 2-dimensional vector subspaces of $\mathbb{R}^4$ intersecting trivially. Then there are 2-dimensional vector subspaces τ_1 and τ_2 orthogonal complements of each other such that τ_1 and τ_2 are orthogonal to P and Q through two separate lines.*

Proof. Let $U_P = S^3 \cap P$ and $U_Q = S^3 \cap Q$ be unit (Euclidean) circles. The Euclidean inner product restricted to $U_P \times U_Q$ has a maximum value realized by a pair (v_P, v_Q) since $U_P \times U_Q$ is compact and the inner product is continuous. The vectors v_P and v_Q can be augmented by $w_P \in P$ and $w_Q \in Q$ so that the sets $\{v_P, w_P\}$ and $\{v_Q, w_Q\}$ are orthonormal bases. Let $\tau_1 = \text{span}\{v_P, v_Q\}$ and $\tau_2 = \text{span}\{w_P, w_Q\}$. The dimension of τ_i is 2 since $v_P \neq v_Q$. It follows that $\tau_1 \cap \tau_2 = \{0\}$ and they intersect P and Q through two separate lines. What is left to show is that τ_1 and τ_2 are orthogonal to P, Q, and each other. It is sufficient to show that $\langle v_P, w_Q \rangle = \langle v_Q, w_P \rangle = 0$.

The function $\theta \mapsto \langle (\cos\theta)v_P + (\sin\theta)w_P, v_Q \rangle$ is continuous and smooth with a maximum at $\theta = 0$. Its derivative $\theta \mapsto \langle -(\sin\theta)v_P + (\cos\theta)w_P, v_Q \rangle$ therefore has a zero value at $\theta = 0$. Hence, $\langle v_Q, w_P \rangle = 0$. Similarly, the function $\theta \mapsto \langle (\cos\theta)v_Q + (\sin\theta)w_Q, v_P \rangle$ has derivative $\theta \mapsto \langle -(\sin\theta)v_Q + (\cos\theta)w_Q, v_P \rangle$ with a zero value at $\theta = 0$, implying that $\langle w_Q, v_P \rangle = 0$.   □

Corollary 6.8. *Let P and Q be orthogonal planes in $\mathbb{H}^4$ intersecting in a unique point. Then $H_P H_Q$ is a type-II elliptic involution that leaves every line through $P \cap Q$ invariant.*

Proof. Since P and Q are orthogonal, they are left invariant by both H_P and H_Q. The half-turns are involutions themselves so applying $H_P H_Q$ twice to $P \cup Q$ is the identity map on $P \cup Q$. The conformal ball model can be used to conjugate $H_P H_Q$ into an element of $SO(4)$, with p corresponding to the origin. Then P and Q form vector spaces that are orthogonal complements of each other. Any Euclidean orthonormal bases of them can be combined into an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^4$. If $x \in \mathbb{H}^4 = B_4$

is outside $P \cup Q$, it can be expressed as a linear combination of vectors in $\mathcal{B} \subset \partial(P \cup Q)$. The composition $(H_P H_Q)^2$ as an element of SO(4) therefore maps x back to itself. Hence, $H_P H_Q$ is an involution.

A matrix of SO(n) has its inverse and transpose equal, but if it is also an involution, then $H_P H_Q$ as a matrix is also symmetric and thus diagonalizable (Spectral Theorem). There are four linearly independent eigenvectors that correspond to lines in $\mathbb{H}^4$ that are pairwise perpendicular through p. The diagonal entries are all -1 since a 1 value would make $H_P H_Q$ have more than one fixed point and other values would make the matrix not in SO(4). Each vector is hence mapped into its opposite. In the conformal ball model of hyperbolic space, the action of $H_P H_Q$ on lines passing through p is a reflection across p. $\square$

Corollary 6.9. *Let P and Q be non-orthogonal planes in $\mathbb{H}^4$ intersecting in a unique point. Then $H_P H_Q$ is a type-II elliptic isometry with a unique unordered pair of invariant planes orthogonal to each other through the fixed point of $H_P H_Q$.*

Proof. If P and Q intersect only in one point, then $H_P H_Q$ is a type-II elliptic isometry. There are planes τ_1 and τ_2 that are orthogonal to each other through the fixed point of $H_P H_Q$ and also orthogonal to both P and Q through separate lines. Both H_P and H_Q leave τ_1 and τ_2 invariant since they are orthogonal to the half-turns' fixed point sets. Then τ_1 and τ_2 are also left invariant by $H_P H_Q$.

It must be shown that τ_1 and τ_2 are the unique invariant planes. In order to show their uniqueness, $H_P H_Q$ can be conjugated to a 4×4 matrix in SO(4) so that the upper-left and lower-right 2×2 blocks are elements of SO(2). Since $H_P H_Q$ has only one fixed point, either both these blocks are diagonal matrices with -1 in their entries or one of these blocks is non-diagonalizable.

Recall that the construction of τ_1 allows it to have $v_P \in P$ and $v_Q \in Q$ so that the angle between v_P and v_Q is at minimum. If P and Q are not orthogonal, this angle is less than $\pi/2$. The action of $H_P H_Q$ on τ_1 is a composition of reflections across span$\{v_P\}$ and span$\{v_Q\}$. Thus, one of the non-diagonalizable blocks corresponds to the rotation of τ_1 in an angle other than 0 and π. Then τ_1 is the unique invariant plane of $H_P H_Q$ with minimum angle or rotation. The orthogonal complement of τ_1 is τ_2, which is also unique. $\square$

6.3. The half-turn bank is exhaustive.

In this section, we show that every half-turn factorization of an orientation preserving isometry comes from its half-turn bank. The proof still

uses cases, but the lemmas from section 6.2 restrict the possibilities for how a pair of planes intersect.

Theorem 6.10. *Let γ be an orientation preserving isometry of $\mathbb{H}^4$. Then for every half-turn factorization $H_P H_Q$ of γ, the circles ∂P and ∂Q are elements of $\mathcal{K}_\gamma$.*

Proof. The proofs depend on the class of isometry of γ.

- γ is hyperbolic.

    If γ is hyperbolic, let L be its axis. The only combination for P and Q is that they are ultra-parallel. The common perpendicular line of P and Q is left invariant by $H_P H_Q$ so it must be L. Then both P and Q are orthogonal to the axis of γ. Let h_P be the hyperplane spanned by L and P. Similarly, let h_Q be the hyperplane spanned by L and Q. Then $\partial h_P \in \mathcal{F}_{H_P}$ and $\partial h_Q \in \mathcal{F}_{H_Q}$ with $P \subset h_P$; $Q \subset h_Q$.

    If $h_P = h_Q$, there are $f_P \in \mathcal{F}_{H_P}$ and $f_Q \in \mathcal{F}_{H_Q}$ such that $H_P = R_{f_P} R_{h_P}$ and $H_Q = R_{h_Q} R_{f_Q}$ are reflection factorizations. If $h_P = h_Q$, then $\gamma = H_P H_Q = R_{f_P} R_{f_Q}$ is a pure hyperbolic isometry with $\partial f_P, \partial f_Q \in \mathcal{F}_\gamma$ and $\partial h_P = \partial h_Q \in \mathcal{T}_\gamma$. So $\partial P = \partial(h_P \cap f_P) \in \mathcal{K}_\gamma$ and $\partial Q = \partial(h_Q \cap h_Q) \in \mathcal{K}_\gamma$.

    If $h_P \cap h_Q$ is a plane τ, then τ intersects P and Q in two different lines. The half-turns H_P and H_Q reflect τ across these lines and thus leave τ invariant. If γ is pure loxodromic, then $h_P \neq h_Q$, so the twisting plane of γ matches τ. Still, the hyperplanes $h_Q^\perp$ orthogonal to h_Q through Q and $h_P^\perp$ orthogonal to h_P through P are also orthogonal to L so $\partial h_P^\perp, \partial h_Q^\perp \in \mathcal{F}_\gamma$. Both h_P and h_Q contain τ, so $\partial h_P, \partial h_Q \in \mathcal{R}_\gamma$. Since $P = h_P \cap h_P^\perp$ and $Q = h_Q \cap h_Q^\perp$, we have $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is parabolic.

    If γ is parabolic, then P and Q are tangent at infinity. For a simpler illustration, assume that the fixed point of γ is ∞. Then the boundaries $(\partial P, \partial Q)$ of P and Q are straight non-crossing Euclidean lines in $\mathbb{R}^3$. Any Euclidean line commonly perpendicular to the boundaries of P and Q forms the same direction exactly equal to that of γ. Recall that the direction of the map $x \mapsto Ax + b$ is the Euclidean line spanned by b in $\mathbb{R}^3$. Let B be the direction of γ. Then B is either Euclidean-parallel or equal to any common perpendicular between ∂P and ∂Q. Let h_P and h_Q be the Euclidean planes orthogonal to B through ∂P and ∂Q, respectively. Let f_P be the Euclidean plane orthogonal to h_P through ∂P and let f_Q be Euclidean plane orthogonal to h_Q through ∂Q. Then $h_P, f_P \in \mathcal{F}_{H_P}$ and $h_Q, f_Q \in \mathcal{F}_{H_Q}$. Since (h_P, f_P) and (h_Q, f_Q)

are pairwise orthogonal, $H_P = R_{h_P} R_{f_P}$ and $H_Q = R_{f_Q} R_{h_Q}$ are reflection factorizations.

If $f_P = f_Q$, then $\gamma = H_P H_Q = R_{h_P} R_{h_Q}$ is pure parabolic, so $f_P = f_Q \in \mathcal{T}_\gamma$ and $h_P, h_Q \in \mathcal{F}_\gamma$. If $f_P \cap f_Q$ is a Euclidean line τ, it is left invariant by H_P and H_Q and hence, by γ. This implies that τ bounds the twisting plane of γ and therefore, $f_P, f_Q \in \mathcal{R}_\gamma$. Still, $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is type-I elliptic.

    If γ is type-I elliptic, then P and Q intersect in a line. Let τ be the twisting plane of γ. There is also a unique hyperplane h spanned by P and Q. Then τ is orthogonal to h through $P \cap Q$. The planes P, Q, and τ pairwise intersect at $P \cap Q$, while τ is orthogonal to both P and Q through $P \cap Q$. Let h_P be the hyperplane spanned by τ and P; let h_Q be the hyperplane spanned by Q and τ. It follows that $\partial h \in \mathcal{T}_\gamma$ and $\partial h_P, \partial h_Q \in \mathcal{F}_\gamma$. Since $P = h_P \cap h$ and $Q = h_Q \cap h$, we have $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is type-II elliptic.

    If γ is type-II elliptic, then P and Q intersect in a unique point p. Using the conformal ball model of $\mathbb{H}^4$ inside $\mathbb{R}^4$, we may assume that p is the origin. Then P and Q extend to Euclidean planes that intersect only at the origin. There are planes τ_1 and τ_2 that are orthogonal complements of each other and also orthogonal to both P and Q through separate lines (Lemma 6.7). Let h_P, h_Q, h_1, and h_2 be hyperplanes defined as follows.

$$
\begin{aligned}
h_P &= \mathrm{span}(\tau_1 \cup P) & h_Q &= \mathrm{span}(\tau_2 \cup Q) \\
h_1 &= \mathrm{span}(Q \cup \tau_1) & h_2 &= \mathrm{span}(P \cup \tau_2).
\end{aligned}
$$

The configuration of planes and hyperplanes yields to $\tau_1 \subset h_P \cap h_1$, $\tau_2 \subset h_Q \cap h_2$, $P = h_P \cap h_2$, and $Q = h_Q \cap h_1$.

There are two options for γ and the planes τ_1 and τ_2. Either γ has an associated angle that is not an integer multiple of π (Corollary 6.9) or γ is an involution (Corollary 6.8). In the former case, γ has a unique pair of invariant planes that must match τ_1 and τ_2. Then $\gamma = \rho_1 \rho_2$ where ρ_1 and ρ_2 are type-I elliptic isometries whose respective twisting planes are τ_1 and τ_2. Since $\partial h_2, \partial h_P \in \mathcal{F}_{\rho_2}$ and $\partial h_1, \partial h_Q \in \mathcal{F}_{\rho_1}$, we have $\partial P, \partial Q \in \mathcal{K}_\gamma$. In the latter case, ∂P and ∂Q are already in $\mathcal{K}_\gamma$. $\qquad\square$

Corollary 6.11. *Let A and B be orientation preserving isometries of $\mathbb{H}^4$. Then the pair A and B is linked by half-turns if and only if $\mathcal{K}_A \cap \mathcal{K}_B$ is nonempty.*

Proof. If A and B are linked by half-turns, then there are planes bounded by α, β, and δ such that $A = H_\alpha H_\beta$ and $B = H_\beta H_\delta$. But β is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$.

If $\mathcal{K}_A \cap \mathcal{K}_B$ is nonempty, let β be one of its elements. Then there are $\alpha \in \mathcal{K}_A$ and $\beta \in \mathcal{K}_B$ such that $A = H_\alpha H_\beta$ and $B = H_\beta H_\delta$. $\square$

7. Conditions for Linking in Dimension 4

In this section, some conditions for a pair of isometries in $\mathbb{H}^4$ to be linked are stated and proved. The main idea is to find geometric or computational requirements for a given pair to be linked. The conditions are divided into cases depending on which type of isometry is given. Note that in the definition of linked pairs, the order of the isometries matters, whereas in this paper, the order of the isometries does not matter once they are linked (see Theorem 5.5).

The conditions for linking pairs with twisting planes are quite demanding, justifying the theorem of Ara Basmajian and Bernard Maskit [1] that linked pairs are of measure zero. Still, pencils can be a useful tool for finding a common perpendicular plane to factorize orientation preserving isometries.

Theorem 7.1. *If A and B are pure hyperbolic isometries of $\mathbb{H}^4$ with ultra-parallel axes, then there is a plane P orthogonal to both axes of A and B. Hence, A and B are linked.*

Proof. Suppose both A and B are pure hyperbolic with ultra-parallel axes. Let L be the common perpendicular line between the axes of A and B. If the axes of A and B lie in a plane C, then $L \subset C$ and there are plenty of planes that are orthogonal to C containing L. Pick P to be any of those planes. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$. If the axes of A and B do not lie in the same plane, the axis of A and L still lies in a plane P_1. As L and the axis of B intersect in a single point, so do P_1 and the axis of B and they are contained in a unique hyperplane C. There is a unique plane P orthogonal to C through L. Since both axes of A and B lie in the hyperplane C, P must be orthogonal to both of them. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$. It follows that there are $P_1, P_2 \in \mathcal{K}_A$ and $P_3, P_4 \in \mathcal{K}_B$ such that $A = H_{P_1} H_P = H_P H_{P_2}$ and $B = H_{P_3} H_P = H_P H_{P_4}$. $\square$

Theorem 7.2. *Let A and B be pure parabolic isometries of $\mathbb{H}^4$. Then A and B are linked.*

Proof. Suppose their fixed points x_A and x_B are not equal. Then there is a unique line L connecting x_A to x_B. Let $h_A \in \mathcal{F}_A$ be the element containing x_B and $h_B \in \mathcal{F}_B$ be the element containing x_A. Then $h_B \cap h_A$ contains the line L. Either $h_A \cap h_B$ is a circle or $h_A = h_B$. Let P be

a circle in $h_A \cap h_B$ through x_A and x_B. Thus, P is a circle in different or same spheres in $\mathcal{F}_A$ and $\mathcal{F}_B$. Then $P \in \mathcal{K}_A \cap \mathcal{K}_B$. Hence, there are $P_1, P_2 \in \mathcal{K}_A$ and $P_3, P_4 \in \mathcal{K}_B$ such that $A = H_{P_1} H_P = H_P H_{P_2}$ and $B = H_{P_3} H_P = H_P H_{P_4}$.

If $x_A = x_B$, there are still $h_A \in \mathcal{F}_A$ and $h_B \in \mathcal{F}_B$ that have three possibilities: $h_A = h_B$; $P = h_A \cap h_B$ is a circle; or $h_A \cap h_B = \{x_A\}$. In the first case, let $t \in \mathcal{T}_A$. Then $h_B \cap t \in \mathcal{K}_A \cap \mathcal{K}_B$ and so A and B are linked. In the second case, $P \in \mathcal{K}_A \cap \mathcal{K}_B$ and links A and B. In the last case, let $\mathcal{F}_A = \mathcal{F}_B$ which implies that $\mathcal{K}_B = \mathcal{K}_A$ whose elements make A and B linked. $\qquad\square$

Theorem 7.3. *Let A and B be isometries of $\mathbb{H}^4$. Suppose A is pure hyperbolic and B is pure parabolic and $\mathrm{fix}(A) \cap \mathrm{fix}(B) = \emptyset$. Then A and B are linked.*

Proof. Let v be the fixed point of B and let L be the axis of A. Then there is an $h_x \in \mathcal{F}_A$ containing v. If $\hat{h}$ is the hyperplane bounded by h_x, there is a unique point $a \in \hat{h} \cap L$. Then there is a unique $h_a \in \mathcal{F}_B$ which bounds a hyperplane that contains a. Since the hyperplanes bounded by h_x and h_a intersect in a, their intersection is at least a plane orthogonal to L and bounded by v. Otherwise, $h_x = h_a$ and one can pick a plane $P \subseteq \hat{h}_x \cap \hat{h}_a$ bounded by v and that passes through a. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$, so A and B are linked. $\qquad\square$

Let A be a pure parabolic isometry of $\mathbb{H}^4$ and let B be a pure loxodromic isometry of $\mathbb{H}^4$. Suppose the fixed points of A and B in $\widehat{\mathbb{R}^3}$ are disjoint. The following are conditions for A and B to be linked.

Theorem 7.4 (Condition 1)**.** *Suppose there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$. Then A and B are linked.*

Proof. Let L be the boundary of the twisting plane of B and let x be the fixed point A. Since $L \in \mathcal{D}_B$, h intersects L in two points y and z. If x is equal to either y or z, then L is the unique element of $\mathcal{D}_B$ containing x. It allows any plane or sphere t in $\widehat{\mathbb{R}^3}$ containing L to be an element of both $\mathcal{T}_A$ and $\mathcal{R}_B$. Hence, $h \cap t$ is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$ that links A and B. If x, y, and z are three distinct points, they form a unique circle L_2 that must be a subset of h. Then L_2 is perpendicular to L, so it is in $\mathcal{K}_B$. It also passes through x, so $L_2 \in \mathcal{K}_A$. Thus, $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$, so A and B are linked. $\qquad\square$

Theorem 7.5 (Condition 2)**.** *If the fixed point of A is in the boundary of the twisting plane of B, then A and B are linked.*

Proof. Let x be the fixed point of A. Since x is not the fixed point of B, there is a unique $h_x \in \mathcal{F}_B$ containing x. Let L be the boundary of the twisting plane of B. Then $L \in \mathcal{D}_B$ intersects h in two points, one of which is x. Let m be the other intersection point. There is also a unique $h_m \in \mathcal{F}_A$ containing m. Either $h_m = h_x$, which is done in the first condition, or $h_m \cap h_x$ is a circle c since both h_m and h_x contain x and m. We must show $c \in \mathcal{K}_A \cap \mathcal{K}_B$. The intersection points of c and L are x and m, but $c \subset h_x \in \mathcal{F}_B$, so $c \in \mathcal{K}_B$. As A is pure parabolic, any circle inside any element of $\mathcal{F}_A$ and passing through x is an element of $\mathcal{K}_A$. Hence, $c \in \mathcal{K}_A$ as $x \in c \subset h_m \in \mathcal{F}_A$ and $x \in c$. Then other planes can be found so that A and B are linked. $\square$

Theorem 7.6 (Condition 3). *Let $L \subset \widehat{\mathbb{R}^3}$ be the boundary of the twisting plane of B. If three points in L form a subset of h for some $h \in \mathcal{F}_A$, then A and B are linked.*

Proof. The three points determine L and are sufficient for L to fit inside h. Let $x \in \widehat{\mathbb{R}^3}$ be the fixed point of A. The case where $x \in L$ is handled in the previous condition. We may assume $x \notin L$. Then there is a unique $c_x \in \mathcal{K}_B$ containing x. We must show $c_x \in \mathcal{K}_A$. As $c_x \in \mathcal{K}_B$, it must intersect L in two points which are in h. But h contains x since A is parabolic, so it has three points in common with c_x. They are sufficient for c_x to be a subset of h. A circle or line in h containing x is an intersection of h with some element of $\mathcal{T}_A$. Hence, c_x, having this property, must be in $\mathcal{K}_A$. This implies that A and B are linked. $\square$

Let A be a screw parabolic and B be a pure hyperbolic isometry of $\mathbb{H}^4$. Assume both have disjoint fixed points in the boundary. Throughout this case, let x be the fixed point of A and let L be the boundary of its twisting plane. Likewise, let d_x be the unique element of $\mathcal{D}_B$ containing x.

Theorem 7.7 (Condition 1). *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$, then A and B are linked.*

Proof. Let a and b be points in $\widehat{\mathbb{R}^3}$ so that $d_x \cap h = \{x, b\}$ and $L \cap h = \{x, a\}$. If $a = b$, then $d_x = L$, so any sphere c containing L is an element of both $\mathcal{T}_B$ and $\mathcal{R}_A$. It follows that $c \cap h$ is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$. If $a \neq b$, the points a, b, and x form a unique circle L_2 in $\widehat{\mathbb{R}^3}$. Since $a, b, x \in h$, $L_2 \subset h$. We claim that $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$. First, L_2 intersects L in two points, a and x, while sitting in $h \in \mathcal{F}_A$. Hence, $L_2 \in \mathcal{K}_A$. Next, L_2 intersects d_x in two points, b and x, while sitting in $h \in \mathcal{F}_B$. Thus, $L_2 \in \mathcal{K}_B$. The half-turn around L_2 is the desired common factor of A and B linking them. $\square$

Theorem 7.8 (Condition 2)**.** *If $L \subset h$ for some $h \in \mathcal{F}_B$, then A and B are linked.*

Proof. The hypothesis implies that h is the unique element of $\mathcal{F}_B$ containing x. Let b be the intersection of d_x and h in $\widehat{\mathbb{R}^3}$ other than x. There is a unique $h_b \in \mathcal{F}_A$ containing b. Let a be the intersection of h_b and L other than x. Connect a, b, and x with the unique circle L_2. Then $L_2 = h_b \cap h$ while $h \in \mathcal{R}_A$ and $h_b \in \mathcal{T}_B$. It also connects L and d_x in two distinct points. Hence, $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$, so A and B are linked.   □

Theorem 7.9 (Condition 3)**.** *If either $m \in L$ or d_x is orthogonal to k_m, then A and B are linked.*

The definitions of m and k_m are as follows. There is a unique element $h_x \in \mathcal{F}_B$ containing x, and it is orthogonal to d_x through two intersection points, one of which is x. Let m be the other intersection point. If $m \notin L$, there are unique elements $h_m \in \mathcal{F}_A$ and $k_m \in \mathcal{K}_A$ containing m.

Proof. Suppose $m \in L$. It is possible that $d_x = L$ which implies that $h_m = h_x$, reducing the case to the previous condition. If $d_x \neq L$, $h_m \cap h_x$ is a circle c containing m and x. It follows that $c \subset \mathcal{K}_A \cap \mathcal{K}_B$ since it intersects d_x and L orthogonally through two points. Hence, A and B are linked.

If $m \notin L$, the hypothesis requires that d_x is orthogonal to k_m. It follows that $k_m \subset h_x$ since all circles orthogonal to d_x at x and m are contained in h_x. Therefore, $k_m \in \mathcal{K}_B$, so A and B are linked.   □

Theorem 7.10 (Computational Condition)**.** *Let A be a screw parabolic isometry of $\mathbb{H}^4$ leaving the z-axis of $\widehat{\mathbb{R}^3}$ invariant and fixing ∞. Let B be a pure hyperbolic isometry of $\mathbb{H}^4$ fixing v and w in $\mathbb{R}^3$. If v and w are equidistant to the z-axis, then A and B are linked.*

If $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, we say that v and w are equidistant to the z-axis if $v_1^2 + v_2^2 = w_1^2 + w_2^2$.

Proof. The unique element $d_\infty \in \mathcal{D}_B$ containing ∞ is the Euclidean line connecting v and w. Let m be the midpoint of v and w. Thus,

$$m = \left(\frac{v_1 + w_1}{2}, \frac{v_2 + w_2}{2}, \frac{v_3 + w_3}{2} \right).$$

Let $(m_1, m_2, m_3) = m$. Then the horizontal Euclidian line k_m connecting m and the z-axis through $(0, 0, m_3)$ is an element of $\mathcal{K}_A$. The angle between k_m and d_∞ can be computed from the inner product $(-m_1, -m_2, 0) \cdot (v - m)$, which simplifies to $-\frac{1}{4}\left(v_1^2 - w_1^2\right) - \frac{1}{4}\left(v_2^2 - w_2^2\right)$. The hypothesis implies that this is zero so k_m and d_∞ are perpendicular and hence, $k_m \in \mathcal{K}_B$. Then A and B are linked.   □

Let A be a screw parabolic and B be a pure parabolic isometry of $\mathbb{H}^4$ with different fixed points. Throughout this case, let x be the fixed point of A and let y be the fixed point of B. Suppose L is the boundary of the twisting plane of A. There are unique elements $d_x \in \mathcal{D}_B$ containing x and $d_y \in \mathcal{D}_A$ containing y. If $y \notin L$, denote the unique element of $\mathcal{K}_A$ containing y by k_y.

Theorem 7.11 (Condition 1). *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$, then A and B are linked.*

Proof. As A and B are both parabolic, h contains both x and y. Since $L \in \mathcal{D}_A$, it intersects h orthogonally in a point a other than x. If $a = y$, then $L = d_x$, so any circle in h connecting x and a is an element of $\mathcal{K}_A \cap \mathcal{K}_B$, making A and B linked. If $a \neq y$, then the points a, x, and y form a circle c in h. The circle c intersects L in two points and also d_x in x and y. Hence, c is in both $\mathcal{K}_A$ and $\mathcal{K}_B$, so A and B are linked. $\qquad\square$

Theorem 7.12 (Condition 2). *If $y \in L$, then A and B are linked.*

Proof. There are spheres $h_x \in \mathcal{F}_B$ and $h_y \in \mathcal{F}_A$ containing both x and y. If $h_x = h_y$, then A and B are linked as per the previous condition. Otherwise, $h_x \cap h_y$ is a circle k passing through x and y. If $y \in L$, $k \in \mathcal{K}_A$, but k is also in $\mathcal{K}_B$, so A and B are linked. $\qquad\square$

Theorem 7.13 (Condition 3). *If $L \subset h$ for some $h \in \mathcal{F}_B$, then A and B are linked.*

Proof. We may assume $y \notin L$. Otherwise, the linking of A and B is implied by the previous condition. Then there is a unique $k_y \in \mathcal{K}_A$ containing y. It intersects L in a point p other than x. The circle k_y is uniquely determined by the points $x, p, y \in h$. Hence, $k_y \subset h$, so $k_y \in \mathcal{K}_B$ since B is pure parabolic. The half-turn about k_y links A and B. $\qquad\square$

Theorem 7.14 (Condition 4). *If $k_y \perp d_x$, then A and B are linked.*

If $y \notin L$, there is a unique $k_y \in \mathcal{K}_A$ containing y. If $y \in L$, there are many choices for $k_y \in \mathcal{K}_A$, but choosing $k_y \perp d_x$ is not necessary.

Proof. There is a unique $h_x \in \mathcal{F}_B$ containing x. It intersects d_x in x and y which are also in k_y. Every circle orthogonal to d_x through x and y must be a subset of h_x, so $k_y \subset h_x$. The span of $k_y \cup d_x$ is an element of $\mathcal{T}_B$, so $k_y \in \mathcal{K}_B$. Recall that $k_y \in \mathcal{K}_A$. Therefore, A and B are linked. $\quad\square$

Linking pairs of hyperbolic isometries are better expressed inside $\mathbb{H}^4$. The hyperplanes and planes in $\mathbb{H}^4$ bounded by the pencils can intersect the invariant planes or lines of an isometry.

Let A be a pure loxodromic isometry of $\mathbb{H}^4$ with axis Ax_A and twisting plane τ_A. Let B be a pure hyperbolic isometry of $\mathbb{H}^4$ with axis Ax_B. Assume Ax_A and Ax_B are disjoint. Then there is a unique line N perpendicular to both Ax_A and Ax_B. It intersects A in a point a and B in a point b. There is a unique hyperplane h_a orthogonal to Ax_A through a. Likewise, there is a unique hyperplane h_b orthogonal to Ax_B through b. Since $\mathrm{Ax}_A \subset \tau_A$, there is a unique line $L_a \subset \tau_A$ perpendicular to Ax_A through a. Throughout this case, A, B, Ax_A, Ax_B, τ_A, N, a, b, and L_a are used.

Theorem 7.15 (Condition 1). *If $L_a = N$, then A and B are linked.*

Proof. The hypothesis implies that τ_A intersects Ax_B at least at b. If $\mathrm{Ax}_B \subset \tau_A$, there are plenty of planes orthogonal to τ_A through N. Any of them is orthogonal to Ax_B. If Ax_B and τ_A intersect only at b, they span a hyperplane h. There is a plane k orthogonal to h through N. Since h contains τ_A and Ax_B, k is also orthogonal to τ_a and Ax_B. (By dimension count, $k \subset h_b$.) Hence, k is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$, so A and B are linked. $\qquad\square$

Theorem 7.16 (Condition 2). *If $L_a \subset h_b$, then A and B are linked.*

Proof. If $L_a \subset h_b$, it is possible that $L_a = N$, which is the previous case. We may assume that $L_a \neq N$. Then L_a and N span a plane k which sits in h_a. It follows that $k \subset h_b$ since both L_a and N lie in h_b. Thus, k is orthogonal to Ax_B and τ_A, so $k \in \mathcal{K}_B$ linking A and B. $\qquad\square$

Let A and B be screw parabolic isometries of $\mathbb{H}^4$ with disjoint fixed points x and y, respectively. Let τ_A and τ_B be their respective twisting planes. Define $L_A = \partial \tau_A$ and $L_B = \partial \tau_B$. As $x \neq y$, there are unique elements $h_x \in \mathcal{F}_B$ and $h_y \in \mathcal{F}_A$ such that $x \in h_x$ and $y \in h_y$. The conditions in which a common orthogonal plane exists are quite restrictive, so A and B are highly unlikely to be linked.

Theorem 7.17 (Condition 1). *If there is $h \in \mathcal{F}_A \cap \mathcal{F}_B$ and the points x, y, x_h, and y_h form a circle, then A and B are linked.*

Any $h \in \mathcal{F}_A$ intersects L_A in two points. Let x_h be the intersection point other than x. If $h \in \mathcal{F}_B$, let y_h be the element of $h \cap L_B$ other than y.

Proof. Let c be the circle formed by x, y, x_h, and y_h as allowed by the hypothesis. We must show $c \in \mathcal{K}_A \cap \mathcal{K}_B$. Since $\{x, y, x_h, y_h\} \subset h$, $c \subset h$. Also c is orthogonal to L_A through $\{x, x_h\}$, so $c \in \mathcal{K}_A$. Similarly, c is orthogonal to L_B through $\{y, y_h\}$, so $c \in \mathcal{K}_B$. Hence, A and B are linked. $\qquad\square$

Theorem 7.18 (Condition 2)**.** *If $y \in L_A$ and $x \in L_B$, then A and B are linked.*

Proof. We may assume $h_x \neq h_y$; otherwise $L_A = L_B$, making any circle in h_x that connects x to y an element of $\mathcal{K}_A \cap \mathcal{K}_B$. Let c be $h_x \cap h_y$ which is a circle containing $\{x, y\}$. The hypothesis implies that $h_y \cap L_A = \{x, y\} = h_x \cap L_B$, so c is orthogonal to both L_A and L_B. Therefore, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. The existence of c makes A and B linked. $\square$

Theorem 7.19 (Condition 3)**.** *If $h_y \cap L_A \subset h_x$, $h_x \cap L_B \subset h_y$, and $h_x \neq h_y$, then A and B are linked.*

Proof. Let c be $h_x \cap h_y$. Then c is a circle containing $(h_y \cap L_A) \cup (h_x \cap L_B)$. It is orthogonal to L_B and L_A through two distinct points each. Hence, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. It follows that A and B are linked. $\square$

Theorem 7.20 (Condition 4)**.** *If $L_B \subset h_y$ and $L_A \subset h_x$, then A and B are linked.*

Proof. Since $L_B \subset h_y$, $h_y \in \mathcal{R}_B$. Let $c = h_y \cap h_x$. Then $c \in \mathcal{K}_B$. Similarly, $L_A \subset h_x$ implies that $h_x \in \mathcal{R}_A$, so $c \in \mathcal{K}_A$. Thus, A and B are linked. $\square$

Let A and B be pure loxodromic isometries of $\mathbb{H}^4$ with axes Ax_A and Ax_B and twisting planes τ_A and τ_B, respectively. Assume τ_A and τ_B are disjoint. There is a unique line N perpendicular to both Ax_A and Ax_B. There are hyperplanes h_a orthogonal to Ax_A through a, and h_b orthogonal to Ax_B through b. There are also lines $L_a \subset \tau_A$ perpendicular to Ax_A through a, and $L_b \subset \tau_B$ perpendicular to Ax_B through b. Like the previous case, A and B are rarely linked.

Theorem 7.21. *If L_a and L_b are coplanar, then A and B are linked.*

Proof. Since τ_A and τ_B are disjoint, the lines N, L_a, and L_b are distinct. The plane P containing L_a and L_b is unique. Since N connects a and b, N lies in P. The axis of A is perpendicular to both N and L_a so τ_A is orthogonal to P through N. Likewise, τ_B is orthogonal to P through N. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$. $\square$

Let A be a screw parabolic isometry of $\mathbb{H}^4$ fixing x and with twisting plane τ_A. Let B be a pure loxodromic isometry of $\mathbb{H}^4$ with axis Ax_B and twisting plane τ_B. Suppose x does not bound Ax_B. There is a unique $h_x \in \mathcal{F}_B$ that contains x. There is also $d_x \in \mathcal{D}_B$ containing x. Let x_2 be the intersection of d_x with h_x other than x. Since $x \neq x_2$, there are unique elements $d_2 \in \mathcal{D}_A$ and $h_2 \in \mathcal{F}_A$ containing x_2.

Let L_A be the boundary of τ_A and L_B be that of τ_B. Since $L_A \in \mathcal{D}_A$, $L_A \cap h_2$ has exactly two points. Let a be the element of $L_A \cap h_2$ other

than x. Similarly, let b_1 and b_2 be the elements of $L_B \cap h_x$. Throughout this case, A, B, τ_A, τ_B, Ax_B, x, h_x, x_2, d_x, d_2, h_2, L_A, L_B, b_1, and b_2 are used consistently.

Theorem 7.22 (Condition 1). *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$ and the points of $h \cap (L_A \cup L_B)$ form a circle, then A and B are linked.*

Proof. Since $x \in h$, h is the unique element of $\mathcal{F}_B$ containing x. That is $h = h_x$. It follows that $x_2 \in h$ and h is the unique element of $\mathcal{F}_A$ containing x_2. So $h = h_x = h_2$. Hence,

$$
\begin{aligned}
h \cap (L_A \cup L_B) &= (h \cap L_A) \cup (h \cap L_B) \\
&= (h_2 \cap L_A) \cup (h_x \cap L_B) \\
&= \{x, a, b_1, b_2\}.
\end{aligned}
$$

Let c be the circle containing $\{x, a, b_1, b_2\}$ according to the hypothesis. Then $c \subset h$ since $\{x, a, b_1, b_2\} \subset h$. It is orthogonal to L_A through $\{x, a\}$ and to L_B through $\{b_1, b_2\}$. Therefore, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. It implies that A and B are linked. $\qquad\square$

Theorem 7.23 (Condition 2). *If $a \in h_x$; $b_1, b_2 \in h_2$; and $h_x \neq h_2$, then A and B are linked.*

Proof. Let $c = h_x \cap h_2$. Then c is a circle since $h_x \neq h_2$. It is orthogonal to L_A through $\{a, x\}$ and to L_B through $\{b_1, b_2\}$. Thus, $c \in \mathcal{K}_A \cap \mathcal{K}_B$, so A and B are linked. $\qquad\square$

Theorem 7.24 (Condition 3). *If there are $p_A \in \mathcal{F}_A$ and $p_B \in \mathcal{F}_B$ such that $L_B \subset p_A$ and $L_A \subset p_B$, then A and B are linked.*

Proof. Since $L_A \subset p_B$, we have $p_B \in \mathcal{R}_A$, so $p_B \neq p_A$. Let $c = p_B \cap p_A$. Then $c \in \mathcal{K}_A$ as it is an intersection of a pair in $\mathcal{F}_A \times \mathcal{R}_A$. Likewise, $L_B \subset p_A$ implies that $p_A \in \mathcal{R}_B$, so c is also in $\mathcal{K}_B$. Hence, A and B are linked. $\qquad\square$

8. Pairs without Invariant Subplane

It is possible for a pair to be linked by half-turns without having invariant lower dimensional planes. This section shows how to construct a linked pair that does not have an invariant lower dimensional space. It can be started by investigating individual isometries then proving that the only invariant planes of an isometry are those that are bounded by the elements of its invariant pencil (if such exist), twisting pencil (if the rotational part is an involution), axis, twisting plane, or twisting hyperplane.

One way to find all invariant subplanes is to express an isometry of $\mathbb{H}^n$ as a matrix in $\mathrm{PSO}(n, 1)$. The eigenvectors correspond to fixed points if

they are not space-like in the Lorentzian space $\mathbb{R}^{n,1}$. The following are examples of hyperbolic isometries.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 1 \\
0 & 0 & 0 & 1 & \sqrt{2}
\end{bmatrix}
\text{ and }
\begin{bmatrix}
\cos\theta & -\sin\theta & 0 & 0 & 0 \\
\sin\theta & \cos\theta & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 1 \\
0 & 0 & 0 & 1 & \sqrt{2}
\end{bmatrix}.
$$

The upper left 3×3 entries can be replaced by any nontrivial element of $SO(3)$ and the resulting matrix is pure loxodromic. The construction shows that the invariant lower dimensional subspaces are exactly the axis, twisting plane, twisting hyperplane (the hyperplane orthogonal to the twisting plane through the axis), and the elements of the invariant and dual pencils. Moreover, the permuted and dual pencils can also be located.

For elliptic isometries, the upper left 4×4 entries of the identity matrix Id_5 can be replaced by an element of $O(4)$ and the resulting matrix is an elliptic element. Analyzing the invariant subspaces is a matter of studying $O(4)$ and $SO(4)$.

8.1. PARABOLIC ISOMETRIES OF $PSO(n, 1)$.

A parabolic element of $PSO(2, 1)$ that fixes the light-like vector $(0, 1, 1)$ has the form

$$
P_2 = \begin{bmatrix}
1 & -t & t \\
t & 1 - \frac{t^2}{2} & t^2/2 \\
t & -t^2/2 & 1 + \frac{t^2}{2}
\end{bmatrix}
\text{ where } t \neq 0.
$$

It can be decomposed into Jordan canonical form $P_2 = S_2 J_2 S_2^{-1}$ where

$$
S_2 = \begin{bmatrix}
0 & t & -t/2 \\
t^2 & 0 & -1 \\
t^2 & 0 & 0
\end{bmatrix}
\text{ and } J_2 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$

The form of P_2 can be extended to $PSO(3, 1)$ using $x, y \in \mathbb{R}$ with $x^2 + y^2 > 0$. Specifically, let

$$
P_3 = \begin{bmatrix}
1 & 0 & -x & x \\
0 & 1 & -y & y \\
x & y & 1 - \frac{x^2+y^2}{2} & \frac{x^2+y^2}{2} \\
x & y & -\frac{x^2+y^2}{2} & 1 + \frac{x^2+y^2}{2}
\end{bmatrix}.
$$

Then P_3 fixes $(0,0,1,1)$ and has Jordan decomposition $P_3 = S_3 J_3 S_3^{-1}$ where

$$S_3 = \begin{bmatrix} -\frac{xy}{x^2+y^2} & 0 & x & 0 \\ -\frac{x^2}{x^2+y^2} & 0 & y & 0 \\ -y/2 & x^2+y^2 & \frac{x^2+y^2}{2} & 0 \\ -y/2 & x^2+y^2 & \frac{x^2+y^2}{2} & 1 \end{bmatrix} \text{ and } J_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that P_3 has a space-like eigenvector, whereas P_2 has only light-like eigenvector. If v_1 and v_2 are the second and third column vectors of S_3, then P_3 maps any vector $v \in \mathbb{R}^{3,1}$ into a linear combination of v, v_1, and v_2. In particular, if v is time-like, then P_3 leaves span$\{v, v_1, v_2\}$ invariant. Hence, the 3-dimensional time-like vector subspaces that contain v_1 and v_2 project to the hyperbolic planes that form the invariant pencil of P_3.

The extension of P_3 to PSO$(4,1)$ is P_4 given by

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & -x & x \\ 0 & 1 & 0 & -y & y \\ 0 & 0 & 1 & -z & z \\ x & y & z & 1 - \frac{x^2+y^2+z^2}{2} & \frac{x^2+y^2+z^2}{2} \\ x & y & z & -\frac{x^2+y^2+z^2}{2} & 1 + \frac{x^2+y^2+z^2}{2} \end{bmatrix}.$$

The Jordan decomposition of P_4 is $S_4 J_4 S_4^{-1}$ where

$$S_4 = \begin{bmatrix} 0 & -\frac{xy}{x^2+y^2+z^2} & 0 & x & 0 \\ -z/y & \frac{x^2+z^2}{x^2+y^2+z^2} & 0 & y & 0 \\ 1 & -\frac{yz}{x^2+y^2+z^2} & 0 & z & 0 \\ 0 & -y/2 & x^2+y^2+z^2 & \frac{x^2+y^2+z^2}{2} & 0 \\ 0 & -y/2 & x^2+y^2+z^2 & \frac{x^2+y^2+z^2}{2} & 1 \end{bmatrix} \text{ and }$$

$$J_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then P_4 has two space-like eigenvectors which correspond to the first two columns of S_4. Let v_1 and v_2 be the third and fourth columns of S_4. Like P_3, any vector $v \in \mathbb{R}^{4,1}$ is mapped by P_4 into a linear combination of v, v_1, and v_2. Thus, the invariant pencil of P_4 is precisely the set of intersections of the hyperboloid $\mathbb{H}^4$ with the 4-dimensional time-like vector subspaces containing $\{v_1, v_2\}$. The dual pencil of P_4 can be located by the 3-dimensional vector time-like subspaces containing $\{v_1, v_2\}$.

Let v_3 be any time-like vector in $\mathbb{R}^{4,1}$ and let ρ be an element of SO(5) that fixes span$\{v_1, v_2, v_3\}$ pointwise. The Lorentz orthogonal complement of span$\{v_1, v_2, v_3\}$ is two dimensional space-like that is rotated by ρ. Then $\rho \in$ PSO(4, 1) and ρP_4 is a screw parabolic isometry of PSO(4, 1). If the rotation angle of ρ is not an integer multiple of π, then only the scalar multiples of $(0, 0, 0, 1, 1)$ are the eigenvectors of ρP_4. Hence, the only time-like subspace left invariant by ρP_4 is the span of $\{v_1, v_2, v_3\}$. The other invariant subspaces of ρP_4 are either light-like or space-like.

8.2. Mismatched invariant planes.

If two isometries have no common invariant line, plane, or hyperplane, they can still be linked. The elements of the half-turn bank of an isometry are not invariant circles/planes, so a pair of isometries can still have a common element in their half-turn banks.

Theorem 8.1. *There exists a half-turn linked pair of isometries of $\mathbb{H}^4$ that leaves neither a point, a line, a plane, nor a hyperplane invariant.*

Proof. Start with $\widehat{\mathbb{R}^3}$ which is a model for the boundary of upper half-space $\mathbb{H}^4$. Let A be a loxodromic isometry that fixes $(0, 0, 1)$ and $(0, 0, -1)$ but rotates the vertical line connecting them by $\pi/6$ angle. Let B be the Poincaré extension of the function sending x to $x + (1, 0, 1)$. Note that B does not have to be pure parabolic; it can be a hyperbolic isometry fixing $(1, 0, 1)$ and $(-1, 0, -1)$.

Then the horizontal line connecting $(0, 0, 0)$ to $(0, 1, 0)$ is an element of $\mathcal{K}_A \cap \mathcal{K}_B$. The invariant planes of B do not include the twisting plane of A. Also the unit sphere, which bounds the twisting hyperplane of A, is not left invariant by B since it does not contain the fixed point of B. There is no invariant hyperbolic line either, as the fixed points of A and B do not match. $\qquad\square$

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Department of Mathematical Sciences; University of Cincinnati; Cincinnati, Ohio 45221

E-mail address: andrew.silverio@uc.edu