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# TRAJECTORIES OF CHAOTIC INTERVAL MAPS

SHARAN GOPAL

Abstract. This paper proves the existence of an abundant number of sequences in $[0, 1]$ that can occur as trajectories for chaotic interval maps. It is proved that given an allowed sequence in $[0, 1]$ of certain kind, there always exists a chaotic interval map with this sequence as a trajectory.

## 1. Introduction

By a *dynamical system* $(X, f)$, we mean a topological space $X$ and a continuous self map $f$ on it. The *trajectory* of a point $x \in X$ is the sequence $(x, f(x), f^2(x), . . . )$, where $f^n = f \circ f \circ f... \circ f(n$ times) and the set $\{f^n(x) : n \in \mathbb{N}_0\}$ is called the *orbit* of $x$ ($\mathbb{N}_0$ is the set of non-negative integers and $f^0(x) = x$). If $f^n(x) = x$ for some $n \in \mathbb{N}$, then $x$ is called a *periodic point*. An *interval map* is a dynamical system, where the underlying topological space is $[0, 1]$, i.e., a system of the form $([0, 1], f)$.

**Definition 1.1** (See [3])**.** Let $(X, f)$ be a dynamical system, where $X$ is a metric space with metric $d$. $(X, f)$ is said to be *Devaney chaotic* if

(1) $f$ has sensitive dependence on initial conditions (i.e., there is an $r > 0$ such that for each point $x \in X$ and for each $\epsilon > 0$ there is a point $y \in X$ with $d(x, y) < \epsilon$ and a $k \geq 0$ such that $d(f^k(x), f^k(y)) \geq r$),

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(2) $(X, f)$ is topologically transitive (i.e., $(X, f)$ contains a dense orbit), and
(3) the set of periodic points is dense in $X$.

This definition is due to Robert L. Devaney. There are, however, other definitions of chaos which are not necessarily equivalent to this one. One of them, namely, topological chaos, implies the abundance of trajectories. By definition, a system is topologically chaotic if it has positive topological entropy and topological entropy is the exponential growth rate of the number of essentially different orbits of length $n$ (see [2]). In this paper, the existence of an abundant number of sequences in $[0,1]$ that can occur as trajectories for Devaney chaotic interval maps is shown, thus demonstrating the significant variety that chaotic interval maps exhibit in terms of trajectories. Hereafter, a Devaney chaotic system will be referred to simply as a chaotic system, as we will not be considering any other type of chaotic systems.

Chaotic interval maps have been well studied in the literature. There are several results providing some conditions on an interval map that ensure its chaoticity. It is proved in [1] that the sensitive dependence on initial conditions in the above definition is redundant if $X$ (not necessarily an interval) is infinite. Further, for interval maps, transitivity alone is sufficient to establish the chaoticity (see [7]). There are some conditions that imply transitivity for interval maps (see [4], [5], [6]). A concept called indecomposability is discussed in [8], where it is proved that indecomposability, together with dense periodicity, is equivalent to chaoticity.

Here, we prove that, given any (allowed) sequence $(x_n)$ in $[0,1]$ whose range $X = \{x_n : n \in \mathbb{N}\}$ has finite derived length, if each level of limit points of $X$ is invariant under the extension of the map $x_n \mapsto x_{n+1}$ to $\overline{X}$, then $(x_n)$ occurs as a trajectory of some chaotic map on $[0,1]$. The sufficient steepness of the graph of a function is used to establish its transitivity, an idea that is used in [4] also. Some other kinds of sequences which occur or do not occur as trajectories of chaotic systems are also discussed. It is interesting to note that there are "many" sequences as well which can occur as trajectories of interval maps in general but not of chaotic interval maps.

## 2. Main Theorem

Given a sequence $(p_k)$ in $[0,1]$ with some conditions, the current problem is to construct a chaotic map with $(p_k)$ as a trajectory. In other words, a sequence $(p_k)$ is given along with the map $f(p_k) = p_{k+1}$ on the set $\{p_k : k \in \mathbb{N}\}$ and $f$ has to be extended to a chaotic map on $[0,1]$. It is well known that $f$ should be necessarily uniformly continuous on

$\{p_k : k \in \mathbb{N}\}$ for $f$ to be at least continuous on $[0,1]$. So, we include this condition in the hypothesis of the theorem and such sequences $(p_k)$, where the map $p_k \mapsto p_{k+1}$ is uniformly continuous on $\{p_k : k \in \mathbb{N}\}$ are called *allowed sequences*.

The following notation and terms will be used in the sequel. If $(n_k)$ is increasing and converging to $n$, then we write $(n_k) \uparrow n$ and if $(n_k)$ is decreasing and converging to $n$, then it is written as $(n_k) \downarrow n$. The phrase *strictly increasing* (*strictly decreasing*, respectively) is used to imply that $n_k \lneqq n_{k+1}$ ($n_k \gneqq n_{k+1}$, respectively) for every $k \in \mathbb{N}$. For a subset $X \subset [0,1]$, denote by $D(X)$, the set of limit points of $X$ in $[0,1]$ and for every $k \in \mathbb{N}$, define inductively $D^{k+1}(X) = D(D^k(X))$. If $X \neq \emptyset$ and there is a non-negative integer $n$ such that $D^{n+1}(X) = \emptyset$ and $D^n(X) \neq \emptyset$, then $X$ is said to have finite derived length and $n$ is called the *derived length* of $X$. In such a case, owing to the compactness of $[0,1]$, $D^n(X)$ is a finite set. Then, for $k \in \mathbb{N}$, denote by $X_k$, the set $D^k(X) \setminus D^{k+1}(X)$ and let $X_0$ denote the set of isolated points of $X$. $X_k$ is called the set of $k^{th}$ level limit points of $X$ for every $k \in \mathbb{N}_0$. The notation $(a,b)$ is used to denote sometimes an ordered pair and sometimes an open interval. However, there is no ambiguity, as the context makes the matter clear.

**Theorem 2.1.** *If $(p_k)$ is a sequence in $[0,1]$ such that*

(i) *the set $X = \{p_k : k \in \mathbb{N}\}$ is of finite derived length, say $n$,*
(ii) *the map $f : p_k \mapsto p_{k+1}$ is uniformly continuous on $X$, and*
(iii) *after extending $f$ continuously to $\overline{X}$ (which will be denoted by $f$ again), for each $0 \leq j \leq n$, the set $X_j$ is $f$-invariant, then there exists a chaotic map $f$ on $[0,1]$ with $(p_k)$ as a trajectory.*

**Lemma 2.2.** *Let $S \subset [0,1]$ be a closed set and $f : S \to [0,1]$ be uniformly continuous. $f$ can be extended continuously to $[0,1]$ such that*
(a) *if $x, y \in S$ such that $(x,y) \cap S = \emptyset$, then $f$ is a linear map on $[x,y]$;*
(b) *$f$ is piecewise linear on $[0,\ inf\ S]$ and $[sup\ S,\ 1]$ such that the modulus of the slope of each piece is greater than 4.*

*Proof.* $f$ can be defined on $[inf\ S,\ sup\ S]$ satisfying (a) in a unique way. On $[0,\ inf\ S]$ and $[sup\ S,\ 1]$, $f$ can be defined with its graph having a finite number of line segments each with slope greater than 4 in absolute value. It can be easily proved that any such map $f$ is continuous on $[0,1]$. $\square$

**Lemma 2.3.** *If $X$ is a countable set with finite derived length, say $n$, then there is a countable set $Y \supset X$ with the same derived length $n$ such that*

(1) *for each $0 \leq k \leq n$, $X_k \subset Y_k$ and*

(2) *for any $k > 0$ and for each $y \in Y_k \setminus \{0, 1\}$, there is a strictly increasing sequence and a strictly decreasing sequence in $Y_{k-1}$ both converging to $y$.*

*Moreover, if $f$ is a function on $X$ satisfying* Theorem 2.1(iii)*, then $f$ can be extended to $Y$ satisfying the same condition on $Y$.*

*Proof.* Let $p \in X_k \setminus \{0\}$ for some $0 < k \leq n$ and suppose that there is no strictly increasing sequence in $X_{k-1}$ converging to $p$. Choose an interval $(a, b)$ containing $p$ and no other element of $X_k$ (if $p = 1$, choose an interval of the type $(a, 1]$ for a suitable $a$). Let $(u_l)$ be an arbitrary strictly increasing sequence in $(a, b) \setminus \overline{X}$ such that $(u_l) \uparrow p$. Let $F_k := \{u_l : l \in \mathbb{N}\}$. Now the aim is to make every element of $F_k$ a limit point of $(k-1)^{\text{th}}$-level in the new set $Y$. If $k \neq 1$, define another sequence $(a_0, a_1, a_2, ....)$ such that $a_l < u_{l+1} < a_{l+1}$ for every $l \geq 0$. In each interval $(a_l, a_{l+1})$, choose a strictly increasing sequence $(v_m^{(l)})$ and a strictly decreasing sequence $(w_m^{(l)})$, both converging to $u_{l+1}$. Denote by $F_{k-1}$ the union of ranges of all the sequences $(v_n^{(l)})$ and $(w_n^{(l)})$, $l$ running over all non-negative integers. The elements of $F_{k-1}$ will be made the limit points of $(k-2)^{\text{th}}$-level in $Y$. If $k - 1 \neq 1$, repeat this procedure taking each sequence in $F_{k-1}$ in place of $(u_l)$ and $F_{k-2}$ be the union of ranges of sequences obtained at the end of this step. Proceeding this way, we finally get $F_1$.

This can be performed for any such point $p$ to which no strictly increasing sequence in $X_{k-1}$ converges. A similar procedure can be followed for any point to which no strictly decreasing sequence in $X_{k-1}$ converges. Let $F$ be the union of $F_1$'s obtained for all such points. Then $Y = F \cup X$ is the required set.

Let $f$ be a continuous function on $X$ satisfying Theorem 2.1(iii). Let $k$ be the largest integer such that $Y_k \setminus X_k \neq \emptyset$ (note that $k \lneq n$). Then there exists a point $p \in X_{k+1}$ such that there is an increasing or a decreasing sequence in $Y_k \setminus X_k$ converging to $p$. Map this sequence under $f$ to that sequence in $Y_k$ which converges to $f(p)$; this is possible because $X_{k+1}$ is $f$-invariant. Thus, $f$ is defined from $Y_{k-1}$ to itself. Now $Y_{k-1} \setminus X_{k-1}$ consists of sequences converging to points in $Y_k$; so, as done in the above case (of $p \in X_{k+1}$), $f$ can be defined from $Y_{k-1} \setminus X_{k-1}$ to $Y_{k-1}$, corresponding to their limits. Proceeding this way, $f$ is extended to $Y$ as required. □

### 2.1. **Proof of Theorem 2.1 in a special case.**

Here, we prove Theorem 2.1 in the case, when $X$ has a unique limit point, say $p$ and $X$ is of the form $\{x_k : k \in \mathbb{N}\} \cup \{y_k : k \in \mathbb{N}\}$, where $x_k \uparrow p$ and $y_k \downarrow p$. Further, assume that $(x_k)$ is strictly increasing unless $p = 0$ (in which case, $x_k = 0$ for every $k$) and $(y_k)$ is strictly decreasing unless $p = 1$ (in which case, $y_k = 1$ for every $k$).

Choose a sequence $(a_k)$ in $[0,1]$ such that if $(x_k)$ is strictly increasing, then $x_k < a_k < x_{k+1}$; otherwise, take $a_k = x_k$ for every $k$. Similarly, define $(b_k)$ such that $y_{k+1} < b_k < y_k$ if it is strictly decreasing, or otherwise, $b_k = y_k$.

Now, we extend the map $f$ to $\overline{X} \cup Z$, where $Z = \{a_k : k \in \mathbb{N}\} \cup \{b_k : k \in \mathbb{N}\}$. Note that $f$ is defined at $p$ and $f(p) = p$. In case $p = 0$, $f(a_k)$ is already defined for every $k$. Otherwise, define $f(a_1) = 1$, $f(a_2) = 0$, and for $k > 2$,

$$f(a_k) = \begin{cases} a_{\frac{k}{2}} & \text{if } k \text{ is even} \\ b_k & \text{if } k \text{ is odd} \end{cases}.$$

Similarly, if $p = 1$, then $f(b_k)$ is already defined for every $k$. Otherwise, define $f(b_1) = 0$, $f(b_2) = 1$, and for $k > 2$,

$$f(b_k) = \begin{cases} b_{\frac{k}{2}} & \text{if } k \text{ is even} \\ a_k & \text{if } k \text{ is odd} \end{cases}.$$

Suppose $x$ and $y$ are consecutive numbers in $Z \cup X$ and the modulus of the slope of the line segment joining $(x, f(x))$ and $(y, f(y))$ is at most 4. Consider a particular case where $x = a_l$ for some $l$. Then $(a_k)$ is strictly increasing, because otherwise $a_k = x_k = 0$ for every $k$; in particular, 0 is the limit of all the sequences that constitute $Z \cup X$, contradicting the fact that $x(= 0)$ and $y$ are consecutive (distinct) numbers of $Z \cup X$. Choose $a'_l$ and $a'_{l+1}$ such that $a_l < a'_l < a'_{l+1} < y$ and define $f(a'_l) = f(a_{l+1})$ and $f(a'_{l+1}) = f(a_l)$. Since $f(a_l) \neq f(a_{l+1})$, the slope of each line segment joining the successive points among these four points is more than the slope of the earlier segment joining $(x, f(x))$ and $(y, f(y))$. Repeat this, if necessary, by replacing $x$ and $y$ with the new successive points, until the slope of the line segment joining any two consecutive points is greater than 4. A similar technique can be applied in other cases also. Doing this for all the pairs where the required slope is at most 4, we finally get a set $F$ such that if $x$ and $y$ are consecutive terms of $Z \cup X \cup F$, the slope of the line segment joining $(x, f(x))$ and $(y, f(y))$ is greater than 4 in absolute value. Extend this map to $[0,1]$ using Lemma 2.2.

Let $(a,b) \subset [0,1]$ be an open interval. We prove that $f^j(a,b) = [0,1]$ for some $j$, establishing the transitivity of the system and thus chaoticity.

Suppose $p \in (a,b)$. Choose the least positive integer $m$ such that $[a_{2^m},\ b_{2^m}] \subset (a,b)$. It follows by the definition of $f$ that $[a_{2^{m-1}},\ b_{2^{m-1}}] \subset f(a,b)$. Iterating further, we get $[a_1, b_1] \subset f^m(a,b)$. Since $p \in (a,b)$, $p \notin \{0,1\}$, and thus both $(a_k)$ and $(b_k)$ are strictly monotonic. Hence, $f(a_1) = 1$ and $f(b_1) = 0$. Therefore, $f^{m+1}(a,b) = [0,1]$. On the other hand, if $p = 0$ and $[0,u)$ is a neighborhood of $p$, choose the least positive integer $m$ such that $[0,\ b_{2^m}] \subset [0,u)$. Then $[0,b_1] \subset f^m([0,u))$, and thus it contains $b_2$ also. Since $f(b_1) = 0$ and $f(b_2) = 1$, we have $f^{m+1}([0,u)) =$

$[0,1]$. Similarly, if $p=1$ and $(v,1]$ is a neighborhood of $p$, then the same conclusion follows.

Suppose $p \notin (a,b)$ but there are two distinct points $x,y \in (Z \cup F) \cap (a,b)$. Then by the definition of $f$ on $Z \cup F$, $p$ lies between $f(x)$ and $f(y)$. Thus, $p \in f(a,b)$ and this falls under the previous case. So it is enough to prove that there are at least two distinct points of $Z \cup F$ in $f^j(a,b)$ for some $j \in \mathbb{N}$.

If $(a,b)$ contains at most one point of $Z \cup F$, then $(a,b)$ can be divided into at most four subintervals, on each of which $f$ is linear. Choose a maximal such subinterval, say $(c,d)$. We have $d-c \geq \frac{1}{4}(b-a)$. Since the modulus of the slope of the graph of $f$ on $(c,d)$ is greater than 4, we have $|f(d)-f(c)| > 4(d-c) \geq (b-a)$. Thus, the length of the interval $f(a,b)$ is greater than the length of $(a,b)$. By iterating, the length keeps on increasing until it contains at least two points of $Z \cup F$; i.e., the cardinality of the set $(Z \cup F) \cap f^j(a,b)$ is at least 2 for some $j \in \mathbb{N}$. □

## 2.2. A method of extending $f$.

Let $[a,b] \subset [0,1]$. Let $X \subset [a,b]$ with a unique limit point, say $p$ and $X = \{x_k : k \in \mathbb{N}\} \cup \{y_k : k \in \mathbb{N}\}$, where $x_k \uparrow p$ and $y_k \downarrow p$. Further, assume that $(x_k)$ is strictly increasing unless $p=a$ and $(y_k)$ is strictly decreasing unless $p=b$. Now, given a uniformly continuous function $f : X \to [0,1]$, we give here a particular method of defining a continuous map from $\overline{X} \cup Z$ to $[0,1]$ where $Z$ is a countable subset of $[0,1]$ defined based on the elements of $X$. This is similar to the proof in section 2.1.

Choose a sequence $(a_k)$ in [0,1] such that if $(x_k)$ is strictly increasing, then $x_k < a_k < x_{k+1}$; otherwise, take $a_k = x_k (= a)$ for every $k$. Similarly, define $(b_k)$ such that, for every $k \in \mathbb{N}$, $y_{k+1} < b_k < y_k$ or $b_k = y_k (= b)$ depending on whether $(y_k)$ is strictly decreasing or not. Let $Z = \{a_k : k \in \mathbb{N}\} \cup \{b_k : k \in \mathbb{N}\}$. We now define $f$ on $Z$ as follows. Note that $f$ is defined at $p$, say $f(p) = q$. Choose two sequences $(r_k)$ and $(s_k)$ such that $(r_k) \uparrow q$ and $(s_k) \downarrow q$. In case $p=a$, $f(a_k)$ is already defined for every $k$; otherwise, define $f$ on $Z$ as

$$f(a_k) = \begin{cases} r_{\frac{k}{2}} & \text{if } k \text{ is even} \\ s_k & \text{if } k \text{ is odd} \end{cases} .$$

Similarly, if $p=b$, $f(b_k)$ is already defined for every $k$; otherwise, define

$$f(b_k) = \begin{cases} s_{\frac{k}{2}} & \text{if } k \text{ is even} \\ r_k & \text{if } k \text{ is odd} \end{cases} .$$

This method will be used extensively in the proof of Theorem 2.1, where it will be referred to simply as "the method" and $Z$ will be referred as the "sequence of intermediate points."

2.3. **Proof of Theorem 2.1.**

$f$ is extended to $[0,1]$ in the following $n+1$ steps. The techniques used here are similar to those used in section 2.1. Following Lemma 2.3, we can assume that for every $0 < k \leq n$ and for every $x \in X_k \setminus \{0,1\}$, there is a strictly increasing sequence and a strictly decreasing sequence in $X_{k-1}$, both converging to $x$. Say $X_n = \{m_1, m_2, .., m_l\}$ such that $m_i < m_{i+1}$ for every $1 \leq i < l$. Choose $n_1, n_2, ...n_{l+1} \in [0,1]$ such that

(1) $n_1 = 0$, $n_{l+1} = 1$,
(2) $n_i \notin \overline{X}$ for $2 \leq i \leq l$, and
(3) $m_i \in [n_i, n_{i+1}]$ for $1 \leq i \leq l$.

The hypothesis that each $X_j$ is $f$-invariant will be used in each step. Further, observe that $X$ is a discrete set; otherwise, $X$ will have infinite derived length.

*Step 1:*

Each of the sets $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$ is a set with unique limit point, namely $m_i$. So, the method can be applied to each of these, taking $a = n_i$ and $b = n_{i+1}$. As done in the method, choose an increasing and a decreasing sequence in $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$ both converging to $m_i$. Then, for each $1 \leq i \leq l$, we define the sequences of intermediate points as done in the method with an additional condition that they are chosen from $[0,1] \setminus \overline{X}$. Let $X^{(1)}$ be the union of the ranges of these sequences of intermediate points.

Now fix an $i \in \{1, 2, ...l\}$. Let $f(m_i) = m_j$. Now apply the method to the set $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$ by choosing $r_1 = 0$, $s_1 = 1$, and $(r_k)_{k=2}^{\infty}$, $(s_k)_{k=2}^{\infty}$ to be the sequences that constitute $X^{(1)} \cap [n_j, n_{j+1}]$. Doing this for every $i$, $f$ is thus defined on the set $X^{(1)}$. Define $\mathfrak{X}^1 = \{(a,b) \in X^{(1)} \times X^{(1)} : a$ and $b$ are consecutive terms of one of these sequences such that $a \neq b$ $\}$. Note that every point of $X_{n-1}$ lies in some interval $[x,y]$ where $(x,y) \in \mathfrak{X}^1$, and, on the other hand, each such interval $[x,y]$ contains exactly one point of $X_{n-1}$.

*Step 2:*

Consider the collection: $\{\overline{X_{n-2}} \cap [x,y] : (x,y) \in \mathfrak{X}^1\}$. *The method* can be applied to each member of the collection. First, define the sequences of intermediate points (again choosing them from $[0,1] \setminus \overline{X}$ ) for all the sequences as done in the method and let $X^{(2)}$ be the union of the ranges of these sequences of intermediate points. Now consider a member of the above collection, say $\overline{X_{n-2}} \cap [x,y]$, $(x,y) \in \mathfrak{X}^1$. This contains a unique point of $X_{n-1}$, say $p$. Apply the method to this set, choosing the sequences $(r_k)$ and $(s_k)$ such that $r_1 = \frac{x'}{2}$, $s_1 = \frac{y'+1}{2}$ and $(r_k)_{k=2}^{\infty}$, $(s_k)_{k=2}^{\infty}$ are the sequences that constitute $X^{(2)} \cap [x', y']$, where $(x', y') \in \mathfrak{X}^1$ such that

$f(p) \in [x', y']$. Apply the method in the same way for each member of the collection. Thus, $f$ is defined on $X^{(2)}$. Also, define $\mathfrak{X}^2 = \{(a,b) \in X^{(2)} \times X^{(2)} : a$ and $b$ are consecutive terms of one of these sequences, i.e., the sequences defined here$\}$.

In general, for any $2 \leq k \leq n$, having completed *Step* $k-1$, we do the following in *Step* $k$. First, define the sequences of intermediate points (choosing them from $[0,1] \setminus \overline{X}$ ) for all the sequences as done in the method, considering separately each member of the collection $\{\overline{X_{n-k}} \cap [x,y] : (x,y) \in \mathfrak{X}^{k-1}\}$, where $\mathfrak{X}^{k-1} = \{(a,b) \in X^{(k-1)} \times X^{(k-1)} : a$ and $b$ are consecutive terms of one of the sequences defined in *Step* $k-1$ $\}$. Apply the method to each member of the collection, choosing the sequences $(r_k)$ and $(s_k)$, as done in Step 2. Thus, $f$ is defined on the closed set $Y = \overline{X} \bigcup (\cup_{i=1}^{n} X^{(i)})$.

Note that, in the beginning, $[0,1]$ is divided into $l$ intervals, each containing an element of $X_n$, and any two of these intersect, at most, in an end point. Each of these is further divided into smaller intervals, whose end points are represented as ordered pairs by the elements of $\mathfrak{X}^1$. This division goes on until *Step* $n$ and finally $[0,1]$ is the union of the intervals of type $[x,y]$, where $(x,y) \in \mathfrak{X}^n$. In one step earlier, i.e., for $(u,v) \in \mathfrak{X}^{n-1}$, the interval $[u,v]$ contains four sequences, two of which constitute $X \cap [u,v]$ and the other two constitute $X^{(n)} \cap [u,v]$.

Consider the set $(X \cup X^{(n)}) \cap [u,v]$ for some $(u,v) \in \mathfrak{X}^{n-1}$. $X^{(n)} \cap [u,v]$ consists of an increasing sequence, say $(a_k)$, and a decreasing sequence, say $(b_k)$. Suppose $x$ and $y$ are consecutive numbers in $(X \cup X^{(n)}) \cap [u,v]$ and the modulus of the slope of the line segment joining $(x, f(x))$ and $(y, f(y))$ is at most 4. Consider a particular case where $x = a_h$ for some $h$. Then $(a_k)$ is strictly increasing because otherwise, $a_k = u$ for every $k$, and thus $a_h$ is the limit of all the four sequences that constitute $(X \cup X^{(n)}) \cap [u,v]$, contradicting the fact that $a_h$ and $y$ are consecutive (distinct) numbers of $Z \cup X$. Now, choose $a'_h$ and $a'_{h+1}$ such that $a_h < a'_h < a'_{h+1} < y$ and define $f(a'_h) = f(a_{h+1})$ and $f(a'_{h+1}) = f(a_h)$. By the definition of $f$, $f(a_h) \neq f(a_{h+1})$; thus, the slope of each line segment joining the successive points in these four points is more than the slope of the earlier segment joining $(x, f(x))$ and $(y, f(y))$. Repeat this, if necessary, by replacing $x$ and $y$ with the new successive points until the slope of the line segment joining any two consecutive points is greater than 4. A similar technique can be applied in other cases also. Doing this for all the pairs where the required slope is at most 4, we finally get a set $F$ such that if $x$ and $y$ are consecutive terms of $X^{(n)} \cup X \cup F$, the slope of the line segment joining $(x, f(x))$ and $(y, f(y))$ is greater than 4 in absolute value.

In the final step, extend the function $f$ to $[0,1]$ as done in Lemma 2.2. Note that $(p_k)$ is a trajectory of $f$.

It is now claimed that $f$ is chaotic. Before proving this, following are four important observations that will be used in the proof later.

(1) The graph of $f$ consists of line segments, each of slope greater than 4 in absolute value.
(2) The set of critical points of $f$ is contained in $Y \cup F$.
(3) In *Step* $k$, we define $X^{(k)}$ and $\mathfrak{X}^k \subset X^{(k)} \times X^{(k)}$. Further, the terms of the sequences of intermediate points contained in $X^{(k)}$ lie between the terms of the sequences that constitute $X_{n-k}$.
(4) If $x, y \in X^{(1)}$ such that $x$ and $y$ are the first terms of the increasing and the decreasing sequences, respectively, in $X^{(1)} \cap [n_i, n_{i+1}]$ for some $i$, then $f(x) = 1$ and $f(y) = 0$. So, if $(a, b)$ is a subinterval containing $x$ and $y$, then $f(a, b) = [0, 1]$.

Let $(a, b) \subset [0, 1]$. We prove that $f^t(a, b) = [0, 1]$ for some $t$, which establishes the transitivity of the system and thus chaoticity.

Recall that $X_n = \{m_1, m_2, ..., m_l\}$ is a finite $f$-invariant set and $m_i$ is the unique element of $X_n$ in $[n_i, n_{i+1}]$. Observe that each set $X^{(1)} \cap [n_i, n_{i+1}]$ consists of an increasing and a decreasing sequence, say $(y_k^{(i)})$ and $(z_k^{(i)})$, respectively, both converging to $m_i$. Suppose $m_1 = 0$ and $[0, u)$ is a neighborhood of $m_1$. Choose the least integer $m$ such that $z_{2^m}^{(1)} \in [0, u)$. It follows from the definition of $f$ that $[f(0),\ z_{2^m-1}^{(i)}] \subset f([0, u))$, where $i$ is given by $m_i = f(0)$. Iterating further, we get $[f^m(0), z_1^{(j)}] \subset f^m([0, u))$ for some $j$. Since $f^m(0) \in X_n$, we have $z_2^{(j)} \in [f^m(0), z_1^{(j)}]$. Now, $f(z_1^{(j)}) = 0$ and $f(z_2^{(j)}) = 1$, and hence $f^{m+1}([0, u)) = [0, 1]$. Similarly, if $m_l = 1$ and $(v, 1]$ is a neighborhood of $m_l$, then the same conclusion follows.

Now consider the interval $(a, b)$. Suppose that $m_i \in (a, b)$ for some $i$. Choose the least positive integer $m$ such that $y_{2^m}^{(i)}, z_{2^m}^{(i)} \in (a, b)$. If neither 0 nor 1 occurs in the first $m + 1$ terms of the trajectories of $y_{2^m}^{(i)}$ and $z_{2^m}^{(i)}$ (i.e., until the $m^{\text{th}}$ iterate), then, by iterating as above, we get $[y_1^{(j)}, z_1^{(j)}] \subset f^m(a, b)$ for some $j$, and thus $f^{m+1}(a, b) = [0, 1]$. On the other hand, if $f^k(y_{2^m}^{(i)}) = 0$ for some $0 \le k \le m$, then we have $[0, z_{2^{m-k}}^{(i')}] \subset f^k(a, b)$ for some $i'$. However, this happens only if $i' = 1$ and $m_1 = 0$. It then immediately follows from the above discussion that $f^{m+1}(a, b) = [0, 1]$. Similarly, even if 1 occurs in one of the first $m + 1$ terms of the trajectory of $z_{2^m}^{(i)}$, we can arrive at the same conclusion.

Suppose that $X_n \cap (a, b) = \emptyset$, but $D(X) \cap (a, b) \neq \emptyset$. Choose the least positive integer $m$ such that $(a, b) \cap X_{m+1} = \emptyset$. Choose $q \in (a, b) \cap X_m$. There exists a unique ordered pair $(x, y) \in \mathfrak{X}^{n-m}$ such that $q \in [x, y]$. We applied the method to the set $\overline{X_{m-1}} \cap [x, y]$ in *Step* $(n - m + 1)$ to get the sequences of intermediate points, say $(u_k)$ and $(v_k)$, respectively, such that

$(u_k) \uparrow q$ and $(v_k) \downarrow q$ and the terms of these sequences constitute the set $X^{(n-m+1)} \cap (x, y)$. Choose the least integer $s$ such that $[u_{2^s}, v_{2^s}] \subset (a, b)$.

By a similar argument as done in the previous case, it can be proved that one of the following cases arises: $[f^i(0), v_1'] \subset f^s(a, b)$, $[u_1', f^i(1)] \subset f^s(a, b)$, or $[u_1', v_1'] \subset f^s(a, b)$ for some $i$, where $u_1'$ and $v_1'$ are the first terms of some sequences $(u_k')$ and $(v_k')$, whose ranges are contained in the set $X^{(n-m+1)}$. It follows from the definition of $f$ at $u_1'$, $u_2'$, $v_1'$, and $v_2'$ that, in any of the three cases, $[x', y'] \subset f^{s+1}(a, b)$ for some $(x', y') \in \mathfrak{X}^{n-m}$. Say $q'$ is the limit of the sequence in $X^{(n-m)}$, of which $x'$ and $y'$ are terms. Then $q' \in X_{m+1}$ and, by the definition of $f$, $f(q')$ lies between $f(x')$ and $f(y')$, and thus $f(q') \in f^{s+2}(a, b)$. Moreover, $f(q') \in X_{m+1}$. Thus, $f^{s+2}(a, b) \cap X_{m+1} \neq \emptyset$. Repeating the above argument by choosing a point in $f^{s+2}(a, b) \cap X_{m+1}$, we get $f^{s'}(a, b) \cap X_{m+2} \neq \emptyset$ for some $s' \in \mathbb{N}$, and thus, finally, $f^r(a, b) \cap X_n \neq \emptyset$ for some $r \in \mathbb{N}$. Then from the above case, it follows that $f^{r'}(a, b) = [0, 1]$ for some $r' \in \mathbb{N}$.

Now, let $D(X) \cap (a, b) = \emptyset$. If there are two distinct points $x, y \in (X^{(n)} \cup F) \cap (a, b)$, then by the definition of $f$ on $X^{(n)} \cup F$, $f(a, b) \cap X_1 \neq \emptyset$, and this falls under the previous case. So it is enough to prove that there are at least two distinct points of $X^{(n)} \cup F$ in $f^{t'}(a, b)$ for some $t' \in \mathbb{N}$.

If $(a, b)$ contains at most one point of $X^{(n)} \cup F$, then $(a, b)$ can be divided into at most four subintervals, on each of which $f$ is linear. Choose a maximal such subinterval, say $(c, d)$. We have $d - c \geq \frac{1}{4}(b - a)$. Since the modulus of the slope of the graph of $f$ on $(c, d)$ is greater than 4, we have $|f(d) - f(c)| > 4(d - c) \geq (b - a)$. Thus, the length of the interval $f(a, b)$ is greater than the length of $(a, b)$. By iterating, the length keeps on increasing, until it contains at least two points of $X^{(n)} \cup F$; i.e., the cardinality of the set $(X^{(n)} \cup F) \cap f^{t'}(a, b)$ is at least 2 for some $t' \in \mathbb{N}$. $\square$

The following proposition shows that there are several sequences which occur as trajectories of some interval maps in general, but not as trajectories of chaotic interval maps. A subset $Y$ of a topological space $X$ is said to be somewhere-dense in $X$ if the closure of $Y$ in $X$ has non-empty interior.

**Proposition 2.4.** *Let $(x_n)$ be a sequence in $[0, 1]$.*

(1) *If $\{x_n : n \in \mathbb{N}\}$ is dense in $[0, 1]$, then any interval map with $(x_n)$ as a trajectory is chaotic.*
(2) *If $\{x_n : n \in \mathbb{N}\}$ is somewhere-dense but not dense in $[0, 1]$, then no interval map with $(x_n)$ as a trajectory is chaotic.*

*Proof.* The proof of the first statement is obvious from the fact that $\{x_n : n \in \mathbb{N}\}$ is a dense orbit of the map.

Now suppose the set $X = \{x_n : n \in \mathbb{N}\}$ is somewhere-dense but not dense in $[0,1]$. Let $f$ be an interval map with $(x_n)$ as a trajectory. It can be easily seen that the closure of any orbit is $f$-invariant, so $\overline{X}$ is $f$-invariant. As $X$ is somewhere-dense, we can choose a non-empty open set $U \subset \overline{X}$. Then $f^k(U) \subset \overline{X}$ for every $k \in \mathbb{N}$. Observe that $[0,1] \setminus \overline{X}$ is a non-empty open set and $f^k(U) \cap ([0,1] \setminus \overline{X}) = \emptyset$ for every $k \in \mathbb{N}$, which shows that $f$ is not transitive and thus not chaotic. □

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