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by

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TREE-LIKE INVERSE LIMITS ON [0,1] WITH INTERVAL-VALUED FUNCTIONS

M. M. MARSH

Abstract. We investigate dimension one, tree-likeness, and dimension greater than one in inverse limits on $[0,1]$ with interval-valued bonding functions. Our investigation leads to some generalizations of results of W. T. Ingram and to necessary conditions and sufficient conditions for tree-likeness in this setting.

In recent papers [9], [10], [11], W. T. Ingram has determined a number of sufficient conditions for inverse limits with set-valued bonding functions to be 1-dimensional, and in many cases, to be tree-like. We generalize some of Ingram's results and establish necessary conditions and sufficient conditions for tree-likeness of inverse limits on $[0,1]$ with interval-valued bonding functions. We also establish necessary and sufficient conditions for the emergence of dimension greater than one in the sets $G'(f_1,\ldots,f_n)$. Our conditions involve the notion of flat spots for the bonding functions and whether the flat spots compose to nondegenerate values of earlier bonding functions in the inverse sequence. Ingram introduced these concepts in the papers referenced above and he states at the end of section 3 in [11] that it would be interesting to know if the only way that the graph of a composition of a sequence of interval-valued functions can have dimension greater than one is for some flat spot for a term of the inverse sequence to iterate to a point where an earlier term of the sequence has a nondegenerate value. Example 14 shows that this does not have to be the case. That is, the graph of a composition can have dimension two even though no flat spot composes to a nondegenerate value. However,

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our Corollary 23 shows that it is the only way for dimension greater than one to appear in the sets $G'(f_1, \dots, f_n)$.

In [3], Włodzimierz J. Charatonik and Robert P. Roe show that for inverse sequences on finite dimensional continua with trivial shape, where each continuum-valued bonding function has values with trivial shape, the inverse limit space must have trivial shape. One dimensionality is equivalent to tree-likeness for nondegenerate continua with trivial shape. Since our setting is inverse limits on $[0, 1]$ with upper semi-continuous interval-valued bonding functions, it follows that determining if an inverse limit is tree-like is equivalent to determining if its dimension is one. Along the way, we establish results involving the emergence of dimension two or greater in the set $G'(f_1, \dots, f_n)$ when the set $G'(f_1, \dots, f_{n-1})$ has dimension one.

It is known that having a flat spot of a bonding function compose to a nondegenerate value of an earlier bonding function can introduce dimension two or greater into the sets $G'(f_1, \dots, f_n)$, see, for example, Ingram's papers [10], [11]; in particular, see [11, Theorem 3.6] and [10, examples 5.1 and 5.3]. Also, see [2, §4] about inverse limits on straight quadrilaterals. Our Theorem 6 points out the sufficiency of this occurrence.

All spaces considered in this paper will be compact metric spaces, referred to as *compacta*. A *continuum* is a connected compactum. A continuous function with be referred to as a *mapping*. For a compactum X, $\dim(X)$ will denote covering dimension. In our setting, covering dimension is equivalent to inductive dimension (see [18, Theorem 15.2 and Corollary 15.3] or [7, p. 67, Theorem V 8 and Corollary]), so we use dimension theorems from both [7] and [18].

A function $f\colon X \to 2^Y$ is *upper semi-continuous at the point* $x \in X$ if, for each open set V in Y containing the set $f(x)$, there is an open set U in X such that $x \in U$ and $f(p) \subset V$ for each $p \in U$. If $f\colon X \to 2^Y$ is upper semi-continuous at each point of X, then f is said to be *upper semi-continuous.* A function $f\colon X \to 2^Y$ is *lower semi-continuous at* $x \in X$ provided that whenever $\{x_i\}$ converges to x in X and $y \in f(x)$, there exists a sequence $\{y_i\}$ converging to y in Y with $y_i \in f(x_i)$ for each $i \geq 1$.

It will be convenient and natural, in our setting, to take the historical point of view of considering a function $f\colon X \to 2^Y$ to be a multi-valued function (or transformation) from X to Y. This point of view is quite common in mathematics, particularly in fixed point theory for multi-valued functions, see [1], [4], [5], [20], [21], [22], [23], [24], [25], [26], [27], [28].

It is typical, in the references in the previous paragraph, to say "f is a multi-valued function from X to Y" and to use the notation $f\colon X \to Y$. Following definitions from these references, given a multi-valued function

$f\colon X \to Y$, $f(x)$ denotes a subset of Y. For $A \subset X$, let $f(A) = \{y \in Y \mid \text{there exists } x \in A \text{ such that } y \in f(x)\}$. The multi-valued function $f\colon X \to Y$ is *surjective* if $f(X) = Y$. The *graph* of f, which we denote by $G(f)$, is the set of points in $X \times Y$ consisting of points (x, y) with $y \in f(x)$.

A multi-valued function $f\colon X \to Y$ is *continuous* if, for each $x \in X$, $f(x)$ is closed in Y and, whenever a sequence $\{x_n\}$ converges to x_0 in X, the following two conditions are satisfied:

(1) If $y_n \in f(x_n)$ for $n \geq 1$, then the set $\{y_n \mid n \geq 1\}$ has a limit point in $f(x_0)$, and

(2) if $y_0 \in f(x_0)$, then, for each $n \geq 1$, there exists $y_n \in f(x_n)$ such that $\{y_n\}$ converges to y_0.

This definition of continuous is equivalent to that used by Robert L. Plunkett [22], Ronald H. Rosen [23], Wyman L. Strother [25], [26], L. E. Ward, Jr. [27], and others. In our setting with closed set values $f(x)$, condition (1) is equivalent to upper semi-continuity of the associated function $f\colon X \to 2^Y$ and condition (2) is equivalent to lower semi-continuity of $f\colon X \to 2^Y$. See [23, p. 168] for these equivalences as well as some others.

Several authors have observed that the multi-valued function $f\colon X \to Y$ being upper semi-continuous and having closed values $f(x)$ is equivalent to having $G(f)$ be closed. (See, for example, [4, §2, para. 1].) Ingram proves this fact in [8, Theorem 1.2].

Note 1. We will hereafter refer to a multi-valued function $f\colon X \to Y$ as a set-valued function. All set-valued functions in this paper will have closed values $f(x)$ and closed graphs $G(f)$ and thus will be upper semi-continuous.

Let $X_1, X_2, \ldots$ be a sequence of compacta. Our setting will be the product space $\prod_{n\geq 1} X_n$ with the usual metric. Our focus will be subcompacta of $\prod_{n\geq 1} X_n$ that are inverse limits of inverse sequences $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \ldots$, where the bonding functions are set-valued and surjective. This point of view allows that the sequence above is indeed an inverse sequence with generalized functions " bonding" points in the factor spaces. Ordinarily, inverse sequences have mappings that " bond" points in the factor spaces. Hereafter, we let $\{X_n, f_n\}$ denote an inverse sequence with set-valued bonding functions and its inverse limit is given by

$$\varprojlim\{X_n, f_n\} = \{\mathbf{x} = (x_1, x_2, \ldots) \in \prod_{n\geq 1} X_n \mid x_n \in f_n(x_{n+1}) \text{ for } n \geq 1\}.$$

For $j, m \in \mathbb{N}$ with $j \leq m$, we define the set below.

$$G_j^{m+1} = G'(f_j, \dots, f_m) = \{\mathbf{x} \in \prod_{i=j}^{m+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } j \le i \le m\}.$$

For $j = 1$, our set G_1^{m+1} is the same set as $G'(f_1, \dots, f_m)$ in [8, p. 17]. Some of our results and proofs gain clarity by using both superscripts and subscripts on G. See, for example, Theorem 5.

For consistency of notation in theorems that follow, if $1 \le k \le m+1$, we let $G_k^k = X_k$. The notation $X \overset{T}{\approx} Y$ will indicate that X is homeomorphic to Y.

Fix $1 \le j < m+1$. Since $\prod_{i=1}^{j} X_i \times \prod_{i=j+1}^{m+1} X_i \overset{T}{\approx} \prod_{i=1}^{m+1} X_i$ under the homeomorphism defined by $h((x_1, \dots, x_j), (x_{j+1}, \dots, x_{m+1})) = (x_1, \dots, x_j, x_{j+1}, \dots, x_{m+1})$, we will make no distinction between these spaces or between subsets M and $h(M)$ of the two.

For $1 \le j \le k < m+1$, we denote the set-valued composition function $f_k \circ f_{k+1} \circ \dots \circ f_m \colon X_{m+1} \to X_k$ by $f_{k,m+1}$, and we let $F_k \colon G_{k+1}^{m+1} \to G_j^k$ be the set-valued function defined by $(x_j, x_{j+1}, \dots, x_k) \in F_k(x_{k+1}, \dots, x_{m+1})$ if and only if $(x_j, \dots, x_{m+1})$ is in G_j^{m+1}. We note that $G(F_k^{-1}) \overset{T}{\approx} G_j^{m+1}$. For $j = 1$ and $k = m$, our function $F_m \colon X_{m+1} \to G_1^m$ is the same as Van Nall's function F_m [19, p. 8]. In order to have a simple notation for the map F_k, one must note, in application, the beginning and ending subscripts of the factor spaces involved in the definition. At the introduction of such a function F_k, we will always indicate the domain and range of F_k in order to clarify the factor spaces involved. We will often say, "Let $F_k \colon G_{k+1}^{m+1} \to G_j^k$ be given."

For $j \ge 1$, let $\pi_j \colon \prod_{i=1}^{\infty} X_i \to X_j$ denote j^{th}-coordinate projection and for $k \ne j$, let $\pi_{j,k} \colon \prod_{i=1}^{\infty} X_i \to X_j \times X_k$ denote projection into the j^{th} and k^{th} factors. It will be useful to also let $\pi_j \colon \prod_{i=k}^{m} X_i \to X_j$ denote j^{th}-coordinate projection for any finite subsequence $\{k, k+1, \dots, m-1, m\}$ of $\mathbb{N}$ with $k < m$ and $k \le j \le m$.

At this point, it is worth reiterating that f_i, $f_{i,j}$, and $F_k \colon G_{k+1}^{m+1} \to G_j^k$ denote set-valued functions with closed graphs and we have the set inclusions $G(f_i) \subset X_{i+1} \times X_i$, $G(f_{i,j}) \subset X_j \times X_i$, and $G(F_k^{-1}) \overset{T}{\approx} G_j^{m+1} \subset \prod_{i=j}^{m+1} X_i$.

If $A \subset X_{i+1}$ for some $i \ge 1$, let $f_i|_A$ be the set-valued function whose domain is A and such that $f_i|_A(x) = f_i(x)$ for $x \in A$. If $A \subset X_{m+1}$, let $G_j^{m+1}|_A = \{z \in G_j^{m+1} \mid \pi_{m+1}(z) \in A\}$. Also note that

$$(*) \qquad G_j^{m+1}|_A = G'(f_j|_{f_{j+1,m+1}(A)}, \dots, f_{m-1}|_{f_{m,m+1}(A)}, f_m|_A).$$

A set-valued function $f\colon [0,1] \to X$ has a *flat spot* (at $p \in X$) if there exists a point $p \in X$ and a nondegenerate interval $[a,b] \subset [0,1]$ such that $[a,b] \times \{p\} \subset G(f)$. We say that $[a,b] \times \{p\}$ is a *flat spot of* f. Suppose $\{f_i\colon [0,1] \to [0,1]\}_{i=1}^m$ is a finite inverse sequence of set-valued functions. A *flat spot at* p *of* f_j *composes to a nondegenerate value of* f_i in the composition $f_i \circ f_{i+1} \circ \ldots \circ f_j$ if $f_i(p)$ is nondegenerate for $i = j-1$ and if there exists a point q in $f_{i+1,j}(p)$ such that $f_i(q)$ is nondegenerate for $i < j-1$.

A set-valued function $f\colon X_2 \to X_1$ is *continuum-valued* if, for each $x \in X_2$, the set $f(x)$ is connected in X_1.

Observation 2 is [8, Theorem 2.5].

Observation 2. *If $f\colon X_2 \to X_1$ is a continuum-valued function and X_2 is connected, then $G(f)$ is connected.*

Observation 3. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are compacta and for each $1 \leq i \leq m$, $f_i\colon X_{i+1} \to X_i$ is a surjective set-valued function. Then for $1 \leq i < k \leq m+1$, $G(f_{i,k}^{-1}) = \pi_{i,k}(G_i^k) = \pi_{i,k}(G_1^{m+1})$.*

Proof. The proof is straightforward. □

Observation 4. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are finite dimensional continua with trivial shape, and for each $1 \leq i \leq m$, $f_i\colon X_{i+1} \to X_i$ is a surjective continuum-valued function, each of whose values has trivial shape. If $1 \leq i \leq k \leq m+1$ and $\dim(G_i^k) = 1$, then G_i^k is a tree-like continuum.*

Proof. That G_i^k is a continuum follows from [8, Theorem 2.6]. By [3, proof of Theorem 2], G_i^k has trivial shape. Since, by hypothesis, $\dim(G_i^k) = 1$, it follows that G_i^k is tree-like. □

Theorem 5. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are continua and for each $1 \leq i \leq m$, $f_i\colon X_{i+1} \to X_i$ is a surjective continuum-valued function. Then, for $1 \leq j \leq k < m+1$, the set-valued function $F_k\colon G_{k+1}^{m+1} \to G_j^k$ is continuum-valued.*

Proof. We use induction on the number of bonding functions in the sequence $f_j, \ldots, f_m$. If $j = m$, then $k = j$. So we have one bonding function f_m and we have that $F_m\colon G_{m+1}^{m+1} = X_{m+1} \to G_m^m = X_m$ is the same function as f_m, which is continuum-valued by assumption. Assume $j < m$ and the theorem is true whenever there are fewer than $m+1-j$ bonding functions.

For $j \leq k \leq m$ and $(x_{k+1}, \ldots, x_{m+1}) \in G_{k+1}^{m+1}$, we have that the set $F_k(x_{k+1}, \ldots, x_{m+1}) \overset{T}{\approx} G_j^k|_{f_k(x_{k+1})}$. Since $f_k(x_{k+1})$ is a continuum and $F_{k-1}\colon G_k^k = X_k \to G_j^{k-1}$ is continuum-valued by inductive assumption, it follows from Observation 2 that $G_j^k|_{f_k(x_{k+1})}$ is a continuum. So F_k is continuum-valued. $\square$

Corollary 6. *Suppose that $X_1, X_2, \ldots, X_{m+1}$ are finite dimensional continua with trivial shape, and for each $1 \leq i \leq m$, $f_i\colon X_{i+1} \to X_i$ is a surjective continuum-valued function, each of whose values has trivial shape. Suppose also that $\dim(G_1^{m+1}) = 1$. Then, for $0 \leq k < m+1$, G_{k+1}^{m+1} is tree-like. So if $\dim(G_1^n) = 1$ for all $1 \leq n \leq m+1$, then for all $1 \leq i \leq j \leq m+1$, G_i^j is tree-like.*

Proof. If $k = 0$, G_1^{m+1} is tree-like by Observation 4. So assume that $k \geq 1$.

By Theorem 5, $F_k\colon G_{k+1}^{m+1} \to G_1^k$ is continuum-valued. It follows that $\eta\colon G_1^{m+1} \to G_{k+1}^{m+1}$ is a monotone mapping, where η is projection onto the $(k+1)^{\text{th}}$ through $(m+1)^{\text{th}}$ coordinates. By [17, Theorem 2.1], the confluent image of a tree-like continuum is tree-like. Since monotone maps are confluent, it follows that G_{k+1}^{m+1} is tree-like.

It follows that if $\dim(G_1^n) = 1$ for all $1 \leq n \leq m+1$, then for all $1 \leq i \leq j \leq m+1$, G_i^j is tree-like. $\square$

Note 7. Hereafter, we consider only inverse sequences where each factor space is $[0,1]$ and each set-valued bonding function $f_i\colon [0,1] \to [0,1]$ is interval-valued. We consider a point to be a degenerate interval. We let $[0,1]_j^k = \prod_{i=j}^k [0,1]$. The notation $[0,1]_j$ will denote $[0,1]$ as the j^{th} factor in a product $[0,1]_1^{m+1}$ or $[0,1]_1^\infty$. In this setting, all sets $G_i^j = G'(f_i, \ldots, f_j)$ and inverse limits are continua; see [8, theorems 2.6 and 2.7].

Remark 8. The following lemmas, theorems, and corollaries preceding Theorem 22 remain valid if each factor space $[0,1]$ is replaced with an arbitrary real number interval, so long as the largest indexed factor (usually $m+1$) is nondegenerate.

Lemma 9. *Let $f\colon [0,1]_{m+1} \to [0,1]_1^m$ be a continuum-valued function. Let $M = \{t \mid f(t)$ is nondegenerate$\}$. If $\dim(\pi_{i,m+1}G(f^{-1})) = 1$ for each $1 \leq i \leq m$, then for $t \in M$, f is not lower semi-continuous at t.*

Proof. We think of $G(f^{-1})$ as a subset of $[0,1]_1^{m+1}$. The proof is similar to [11, proof of Theorem 3.2]. Let $t \in M$. Since $f(t)$ is a nondegenerate continuum, there exists $1 \leq i \leq m$ such that $\pi_i f(t)$ is a nondegenerate interval $[a,b]$. Let p be the midpoint of the interval $[a,b]$. By hypothesis, $\dim(\pi_{i,m+1}G(f^{-1})) = 1$. So no open set in $[0,1]_i \times [0,1]_{m+1}$ is a subset of $\pi_{i,m+1}G(f^{-1})$. Note that $[a,b] \times \{t\} \subset \pi_{i,m+1}G(f^{-1})$. Let $\{U_j\}$ be a sequence of open sets in $[0,1]_i \times [0,1]_{m+1}$ closing down on the point (p,t).

For each $j \geq 1$, let (z_j, t_j) be a point of U_j not in $\pi_{i,m+1}G(f^{-1})$. So, $z_j \notin \pi_i f(t_j)$ for each $j \geq 1$. Since $f(t_j)$ is a continuum (possibly degenerate) for each $j \geq 1$, so is $\pi_i f(t_j)$. It follows that either $\pi_i f(t_j) \subset (z_j, 1]$ or $\pi_i f(t_j) \subset [0, z_j)$. Assume, without loss of generality, that for all $j \geq 1$, $\pi_i f(t_j) \subset (z_j, 1]$. Since $\{z_j\}$ converges to p, it follows that there exists $k \in \mathbb{N}$ and $a < c \leq b$ such that for all $j \geq k$, $\pi_i f(t_j) \subset [c, 1]$. Let $y \in f(t)$ so that $\pi_i(y) = a$ and for each $j \geq 1$, let $y_j \in f(t_j)$. Since for each $j \geq k$, $\pi_i(y_j) \in [c, 1]$, it follows that $\{\pi_i(y_j)\}$ does not converge to $\pi_i(y)$. Hence, $\{y_j\}$ does not converge to y; i.e., f is not lower semi-continuous at t. □

Corollary 10. *Let $\{f_i\colon [0,1] \to [0,1]\}_{i=1}^m$ be a finite inverse sequence of surjective interval-valued functions. If $\dim(G(f_{i,m+1}^{-1})) = 1$ for each $1 \leq i \leq m$, then $F_m\colon X_{m+1} \to G_1^m$ is not lower semi-continuous at each point of $M = \{t \mid F_m(t)$ is nondegenerate$\}$.*

Proof. Corollary 10 follows immediately from Observation 3, Theorem 5, Lemma 9, and the fact that $G(F_m^{-1}) \overset{T}{\approx} G_1^{m+1}$. □

The following corollary follows from [12, p. 71, Corollary 1].

Corollary 11. *The set M in Lemma 9 and Corollary 10 is a 1^{st} Category set.*

Observation 12 will be used frequently in the theorems and lemmas that follow.

Observation 12. *Let $\{f_i\colon [0,1] \to [0,1]\}_{i=1}^m$ be a finite inverse sequence of surjective interval-valued functions, and let $F_m\colon X_{m+1} \to G_1^m$ be given. If $H \subset G_1^{m+1}$ such that $H \subset F_m(t) \times \{t\}$ for some $t \in [0,1]_{m+1}$, then H is homeomorphic to a subset of G_1^m. So if $\dim(G_1^m) \leq 1$, then $\dim(H) \leq 1$.*

Proof. The range of the function F_m is G_1^m. So $F_m(t) \times \{t\} \subset G_1^m \times \{t\}$, and the observation follows. □

Theorem 13 generalizes [11, Theorem 4.2].

Theorem 13. *Let $\{f_i\colon [0,1] \to [0,1]\}_{i=1}^m$ be an inverse sequence of surjective interval-valued functions with $\dim(G_1^m) = 1$. If, for each integer j with $1 \leq j \leq m$, $\dim(G(f_{j,m+1})) = 1$, then $\dim(G_1^{m+1}) = 1$.*

Proof. Let $F_m\colon X_{m+1} \to G_1^m$ and let $M = \{t \mid F_m(t)$ is nondegenerate$\}$. Let $G = G_1^{m+1}$. Recall that $G(F_m^{-1}) \overset{T}{\approx} G$. Since G is a nondegenerate continuum, $\dim(G) \geq 1$. By corollaries 10 and 11, M is a 1^{st} Category set. Let $C = [0,1]_{m+1} - M$. So C is dense in $[0,1]_{m+1}$.

Let $G_1 = \overline{G|_C}$ and $G_2 = G - G_1$. Since G_1 is closed, it follows from [7, p. 32, Corollary 1] that $\dim(G) \leq \max\{\dim(G_1), \dim(G_2)\}$.

Suppose $\dim(G_2) \geq 2$. We note that if $(x_1, \ldots, x_m, t) \in G_2$, then $t \in M$. Also, since G_1 is closed in $[0,1]_1^{m+1}$, G_2 is open relative to G. So $G_2 = U \cap G$ for some open set U in $[0,1]_1^{m+1}$. Since U is a countable union of closed sets K_n, we have that $G_2 = U \cap G = (\cup_{n\geq 1} K_n) \cap G = \cup_{n\geq 1}(K_n \cap G)$. By [7, p. 30, Theorem III 2], $\dim(K_n \cap G) \geq 2$ for some $n \geq 1$. Since M is 0-dimensional [7, p. 22, D)], the components of G_2 lie in the disjoint sets $G \cap ([0,1]_1^m \times \{t\}) = F_m(t) \times \{t\}$ for $t \in M$. So the components of $K_n \cap G$ lie in the disjoint sets $F_m(t) \times \{t\}$ for $t \in M$. By [7, p. 94, Theorem VI 8], there exists a continuum H in $K_n \cap G$ such that $\dim H \geq 2$. But $H \subset F_m(t) \times \{t\}$ for some $t \in M$. So, by Observation 12, $\dim(H) \leq 1$, a contradiction. Thus, $\dim(G_2) \leq 1$,

Suppose $\dim(G_1) \geq 2$. Again, applying [7, p. 94, Theorem VI 8] as in the previous paragraph, we get that some subcontinuum K of G_1 has dimension larger than one. If $\pi_{m+1}(K)$ is degenerate, then K is contained in the fiber $F_m(t) \times \{t\}$ for $\{t\} = \pi_{m+1}(K)$, contradicting Observation 12.

So we assume that $\pi_{m+1}(K) = [a,b]$ for $a \neq b$. Let $p = (x_1, \ldots, x_m, a)$ and $q = (y_1, \ldots, y_m, b)$ be points of K. We observe that K is irreducible between p and q. Suppose P is a subcontinuum of K that contains p and q. Since, for each $t \in C \cap [a,b]$, $(F_m(t), t)$ is a separating point of G, it follows that $(F_m(t), t) \in P$. So $G|_{C\cap[a,b]} \subset P$. Thus, we have that $K \subset \overline{G|_{C\cap[a,b]}} \subset \overline{P} = P$. So $P = K$.

By [13, Theorem 1], some fiber of K has dimension larger than one. Again, we have a contradiction. So $\dim(G_1) \leq 1$.

It follows that $\dim(G) = 1$. □

The following example shows that the dimension of the graph of a composition of bonding functions can be two, even though no flat spot composes to a nondegenerate value. It also shows that the converse of Theorem 13 does not hold and that having $\dim(G(f_{i,j})) = 1$ for all $i < j$ is not a necessary condition for tree-likeness of the inverse limit (see [11, Theorem 4.3]).

Example 14. Let C be the standard "middle thirds" Cantor set in $[0,1]$. Let $f_1 \colon [0,1] \to [0,1]$ be the interval-valued function defined by $f_1(t) = [0,1]$ for $t \in C$, and $f_1(t)$ is alternately 0 or 1 for t in components of the complement of C of decreasing lengths. That is, $f_1(t) = 0$ for $t \in (1/3, 2/3)$, $f_1(t) = 1$ for $t \in (1/9, 2/9) \cup (7/9, 8/9)$, etc. For a picture of the graph of f_1, see [12, p. 191, Figure 3].

Let $f_2 \colon [0,1] \to [0,1]$ be the inverse of the standard Cantor mapping $g \colon [0,1] \to [0,1]$; that is, $f_2 = g^{-1}$. For a picture of the graph of g, see [6, p. 131, Figure 3–19].

Clearly, $\dim(G(f_1)) = 1$ and $\dim(G(f_2)) = 1$. We point out that $\pi_{1,2}$ maps G_1^3 homeomorphically onto $G(f_1^{-1})$. Suppose (r, s, t) and (r, s, c) are in $(\pi_{1,2})^{-1}(r, s) \cap G_1^3$. Since $(f_2)^{-1}$ is a mapping, it follows that $t = c$. So $\pi_{1,2}|_{G_1^3}$ is injective. Surjectiveness is clear. It follows that $\pi_{1,2}|_{G_1^3}$ is a homeomorphism. Hence, $\dim(G_1^3) = 1$.

We now observe that $\pi_{1,3}(G_1^3) = [0,1]_1 \times [0,1]_3$. Let $(r,t) \in [0,1]_1 \times [0,1]_3$. There exists a point $s \in C$ such that $g(s) = t$. By definition, $f_1(s) = [0,1]$; so, $r \in f_1(s)$, $(r,s,t) \in G_1^3$, and $(r,t) \in \pi_{1,3}(G_1^3)$.

We note that although $\dim(G(f_{1,3}^{-1})) = 2$, no flat spot of f_2 composes to a nondegenerate value of f_1. By taking $f_i = \mathrm{id}\colon [0,1] \to [0,1]$ for each $i \geq 3$, we have that $\varprojlim\{[0,1], f_i\} \overset{T}{\approx} G_1^3$ is tree-like. Thus, the converses of Ingram's theorems 4.2 and 4.3 in [11] do not hold.

Lemma 15. *Let $\{f_i\colon [0,1] \to [0,1]\}_{i=1}^m$ be a finite inverse sequence of surjective interval-valued functions and let $F_m\colon X_{m+1} \to G_1^m$ be given. If $\dim(G_1^{m+1}) = 1$, then F_m is not lower semi-continuous at each point of $M = \{t \mid F_m(t)$ is nondegenerate$\}$.*

Proof. We use induction on the number of bonding functions. If $m = 1$, the result follows from [11, Theorem 3.2]. Assume $m > 1$ and the theorem is true for fewer than m bonding functions.

By Corollary 6, $\dim(G_2^{m+1}) = 1$. Thus, by inductive assumption, $F'_m\colon X_{m+1} \to G_2^m$ is not lower semi-continuous at each point $t \in [0,1]_{m+1}$ where $F'_m(t)$ is nondegenerate. We use F'_m notation here to distinguish from F_m in the statement of the lemma.

Let $t \in M$.

Case 1. Suppose $F'_m(t)$ is degenerate; let $F'_m(t) = \{x\}$. Since $t \in M$, $f_1\pi_2(x)$ is nondegenerate; say $f_1\pi_2(x) = [a,b]$. Let p be the midpoint of $[a,b]$. So $(p,x) \in F_m(t)$. Let $\{(a_j,b_j) \times U_j\}_{j=1}^\infty$ be a sequence of product open sets closing down on $(p,(x,t))$ in $[0,1] \times G_2^{m+1}$. Since $\dim(U_j) = 1$ for $j \geq 1$, by Hurewicz's Product Theorem (see [18, p. 127]) and by [18, Theorem 20.2], it follows that $\dim((a_j,b_j) \times U_j) = 2$ for $j \geq 1$. Since $\dim(G_1^{m+1}) = 1$, there exists, for each $j \geq 1$, a point $(s_j,(x_j,t_j))$ in $(a_j,b_j) \times U_j$ that is not in G_1^{m+1}. Also, $\{(x_j,t_j)\}$ converges to (x,t) and $\{s_j\}$ converges to p.

Hereafter, the argument follows analogously to the second half of the proof of Lemma 9. For clarity, we complete the argument. We have that, for $j \geq 1$, $(x_j,t_j) \in G_2^{m+1}$, but $s_j \notin f_1\pi_2(x_j)$. Since $f_1\pi_2(x_j)$ is an interval for each $j \geq 1$, we have that either $f_1\pi_2(x_j) \subset (s_j,1]$ or $f_1\pi_2(x_j) \subset [0,s_j)$ for each $j \geq 1$. Assume, without loss of generality, that for all $j \geq 1$, $f_1\pi_2(x_j) \subset (s_j,1]$. Since $\{s_j\}$ converges to p, it follows that there exist $k \in \mathbb{N}$ and $a < c \leq b$ such that for all $j \geq k$, $f_1\pi_2(x_j) \subset [c,1]$.

For $j \geq 1$, let $(r_j, x_j) \in F_m(t_j)$; so $r_j \geq c$ for each $j \geq k$. It follows that $\{(r_j, x_j)\}$ does not converge to (a, x) and $\{((r_j, x_j), t_j)\}$ does not converge to $((a, x), t)$. So F_m is not lower semi-continuous at t.

Case 2. Suppose $F'_m(t)$ is nondegenerate. Since F'_m is not lower semi-continuous at t, there exists a point y in $F'_m(t)$ and a sequence $\{t_j\}$ converging to t such that whenever $y_j \in F'_m(t_j)$ for $j \geq 1$, $\{y_j\}$ does not converge to y. Let $(s, y) \in F_m(t)$. Then whenever $(s_j, y_j) \in F_m(t_j)$ for $j \geq 1$, $\{(s_j, y_j)\}$ does not converge to (s, y). That is, F_m is not lower semi-continuous at t. $\square$

Corollary 16. *The set M in Lemma 15 is a 1^{st} Category set.*

A continuum is *hereditarily unicoherent* if the intersection of each pair of its subcontinua is connected. A continuum is *decomposable* if it can be written as the union of two proper subcontinua; otherwise, it is *indecomposable*. A continuum X is *hereditarily divisible by points* if each subcontinuum K of X contains a point that separates K. A continuum X is a λ*-dendroid* if X is hereditarily decomposable and hereditarily unicoherent.

Theorem 17. *If $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^m$ is a finite inverse sequence of surjective interval-valued functions and $\dim(G_1^n) = 1$ for each $1 \leq n \leq m+1$, then G_1^{m+1} is a λ-dendroid, as is G_1^n for each $1 \leq n \leq m+1$. In fact, G_i^j is a λ-dendroid for all $1 \leq i \leq j \leq m+1$.*

Proof. By T. Maćkowiak [14, (2.16)], if a continuum is hereditarily divisible by points, then it is a λ-dendroid. We use induction on m to show that G_1^{m+1} is hereditarily divisible by points. If $m = 0$, then by notational convention, $G_1^1 = [0,1]$. Clearly, $[0,1]$ is hereditarily divisible by points.

Assume true for $m \geq 0$. From Theorem 5, the map $F_m \colon [0,1] \to G_1^m$ is continuum-valued. By inductive assumption, G_1^m is hereditarily divisible by points.

Let H be a nondegenerate subcontinuum of G_1^{m+1}. If, for some t, $H \subset F_m(t) \times \{t\}$, then, by Observation 12, H is homeomorphic to a subcontinuum of G_1^m. By inductive assumption, H has a separating point.

So, assume that $s, t \in \pi_{m+1}(H)$ with $s \neq t$. By Corollary 16, there exists r such that $s < r < t$ and $F_m(r)$ is degenerate. So H meets $[0,1]_1^m \times \{r\}$ only at the point $(F_m(r), r)$. It follows that $(F_m(r), r)$ is a separating point of H. So G_1^{m+1} is hereditarily divisible by points and thus, G_1^{m+1} is a λ-dendroid. It follows that G_1^n is a λ-dendroid for each $1 \leq n \leq m+1$.

Lastly, let $1 \leq i \leq j \leq m+1$. By Theorem 5, $F_{i-1} \colon G_i^j \to G_1^{i-1}$ is continuum-valued. It follows that $\eta \colon G_1^j \to G_i^j$ is monotone, where

η is projection onto coordinates i through j. By [14, (7.24)], the semi-confluent image of a λ-dendroid is a λ-dendroid. Since G_1^j is a λ-dendroid and monotone maps are semi-confluent, it follows that G_i^j is also a λ-dendroid. □

A mapping $f\colon X \to Y$ has a *fixed point* if there exists a point $x \in X$ such that $f(x) = x$. A continuum X has the *fixed point property* if each mapping of X to itself has a fixed point.

Let $\{X_i, f_i\}_{i\geq 1}$ be an inverse sequence of compacta with surjective set-valued bonding functions. The following definition was introduced in [16, p. 244]. Suppose there exists $k \in \mathbb{N}$ such that, for $i \geq k$, there exists $Y_i \subset X_i$ and a set-valued function $g_i\colon Y_{i+1} \to X_i$ such that $G(g_i) \subset G(f_i)$, $Y_i \subset g_i(Y_{i+1})$, and $g_i^{-1}\colon g_i(Y_{i+1}) \to X_{i+1}$ is a mapping. Then we say that $\{g_i\}_{i\geq k}$ is a *k-tail sequence of inverse mappings.* If, for each $i \geq k$, $g_i(Y_{i+1}) = X_i$, then we say that $\{g_i\}_{i\geq k}$ is a *surjective k-tail sequence.* Simply stated, the existence of a surjective k-tail sequence implies that for all $i \geq k$, the graph of f_i contains the graph of the inverse of a mapping from X_i into X_{i+1}.

Corollary 18. *Suppose $X = \varprojlim\{[0,1], f_i\}$, where for each $i \geq 1$, f_i is a surjective interval-valued function and $\dim(G_1^i) = 1$. If, for some $k \geq 1$, there exists a surjective k-tail sequence $\{g_i\}_{i\geq k}$ of inverse mappings, then X has the fixed point property.*

Proof. By Theorem 17, for each $n \geq 1$, G_1^n is a λ-dendroid. Since λ-dendroids have the fixed point property [15], the result follows immediately from [16, Corollary 2.4]. □

Corollary 18 gives an answer to [8, Problem 6.54] and also shows that [10, Example 5.5] and [11, Example 6.1] have the fixed point property.

Question 19. Do all tree-like continua obtainable as inverse limits on $[0,1]$ with interval-valued functions have the fixed point property?

The following lemma is a generalization of [11, Theorem 3.5].

Lemma 20. *If X is a 1-dimensional, hereditarily unicoherent continuum, $f\colon [0,1] \to X$ is a continuum-valued function, and $\dim(G(f)) = 1$, then $\{x \in X \mid \dim(f^{-1}(x)) = 1\}$ is countable.*

Proof. The proof is similar to [11, proof of Theorem 3.5]. Let x_1 and x_2 be two points of X, and let J_1 and J_2 be nondegenerate intervals in $f^{-1}(x_1)$ and $f^{-1}(x_2)$, respectively. Since X is hereditarily unicoherent, there is a unique irreducible continuum L in X containing x_1 and x_2.

For each point t in $J_1 \cap J_2$, $f(t)$ is a continuum in X. Also, $x_1 \in f(t)$ and $x_2 \in f(t)$, so $L \subset f(t)$. If $J_1 \cap J_2$ is a nondegenerate interval, it follows that $\dim((J_1 \cap J_2) \times L) = 1 + \dim(L) = 2$ (see [18, p. 127] and [18, Theorem 20.2]). Also, $(J_1 \cap J_2) \times L \subset G(f)$, a contradiction. The result follows. $\square$

The following theorem is a generalization of [11, Theorem 4.2].

Theorem 21. *Let $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^{m}$ be an inverse sequence of surjective interval-valued functions. If $\dim(G_1^m) = 1$, $\dim(G_2^{m+1}) = 1$, and $\dim(G(f_{1,m+1})) = 1$, then $\dim(G_1^{m+1}) = 1$.*

Proof. The basic idea of the proof is similar to [11, proof of Theorem 4.2]. Let H be a nondegenerate subcontinuum of G_1^{m+1}.

Suppose $\pi_{m+1}(H) = \{t\}$ is degenerate. Let $F_m \colon [0,1]_{m+1} \to G_1^m$ be given. Then $H \subset F_m(t) \times \{t\}$, and, by Observation 12, $\dim(H) = 1$.

Suppose $\pi_{m+1}(H)$ is nondegenerate; let $\pi_{m+1}(H) = [a,b]$. We consider the three term inverse sequence below with continuum-valued bonding functions.

$$[0,1]_1 \xleftarrow{F_1} G_2^m \xleftarrow{F'_m} [0,1]_{m+1}$$

Note that $G'(F_1, F'_m) \overset{T}{\approx} G_1^{m+1}$ and $F_1 \circ F'_m = f_{1,m+1}$. Since the dimension of $G(f_{1,m+1})$ is one, it follows from [11, Corollary 3.3] that $\{t \mid f_{1,m+1}(t) \text{ is nondegenerate}\}$ is a 1st Category set. By hypothesis, $\dim(G_2^{m+1}) = 1$. So, by Corollary 16, $\{t \mid F'_m(t) \text{ is nondegenerate}\}$ is a 1st Category set.

So there exists a countable dense set of points $\{z_i\}$, with $a < z_i < b$ for each $i \geq 1$, such that each of $f_{1,m+1}(z_i)$ and $F'_m(z_i)$ is degenerate for each $i \geq 1$. Thus, for each $i \geq 1$, $\{x \in H \mid \pi_{m+1}(x) = z_i\}$ is a degenerate set that separates H. Let x_i be the point of H such that $\pi_{m+1}(x_i) = z_i$ for $i \geq 1$.

We will now show that each pair of points of H can be separated by a 0-dimensional subset of H. Let p and q be two points of H.

Suppose $\pi_{m+1}(p) < \pi_{m+1}(q)$. Pick a point z_i such that $\pi_{m+1}(p) < z_i < \pi_{m+1}(q)$. Then x_i separates H between p and q.

Suppose $\pi_{m+1}(p) = t = \pi_{m+1}(q)$. Referring to the second paragraph of this proof, we see that $H_t = H \cap (F_m(t) \times \{t\})$ is one dimensional, and hence, there exists a 0-dimensional subset B of H_t that separates p from q in H_t [18, Theorem 8.4]. It follows that $B \cup \{x_i\}_{i \geq 1}$ separates p from q in H. Since $B \cup \{x_i\}_{i \geq 1}$ is 0-dimensional, $\dim(H) = 1$.

Since G_1^{m+1} does not contain a 2-dim subcontinuum, $\dim(G_1^{m+1}) = 1$ [7, p. 94, Theorem VI 8]. $\square$

Theorem 22. *Let $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^{m}$ be an inverse sequence of surjective interval-valued functions. If $\dim(G_1^m) = 1$, $\dim(G_2^{m+1}) = 1$, and*

$\dim(G_1^{m+1}) \geq 2$, *then there exist a 1-dimensional continuum A in G_2^{m+1} and a nondegenerate interval $[a,b]$ such that $[a,b] \times A \subset G_1^{m+1}$.*

Proof. Let $F'_m \colon [0,1]_{m+1} \to G_2^{m+1}$. Let $M = \{t \mid F'_m(t) \text{ is nondegenerate}\}$ and let $G = G_2^{m+1}$. Recall that $G \overset{T}{\approx} G((F'_m)^{-1})$. By Corollary 16, M is a 1st Category set. Let $C = [0,1]_{m+1} - M$. So C is dense in $[0,1]_{m+1}$.

Let $G_1 = \overline{G|_C}$ and $G_2 = G - G_1$. By hypothesis, $\dim(G) = 1$, so both G_1 and G_2 have dimension less than or equal to one.

Let $H = G_1^{m+1} \subset [0,1]_1 \times G$. Let $H_1 = H \cap ([0,1] \times G_1)$ and $H_2 = H \cap ([0,1] \times G_2)$. So $H = H_1 \cup H_2$ and H_1 is closed in $[0,1]_1^{m+1}$. By [7, p. 32, Corollary 1], either $\dim(H_1) \geq 2$ or $\dim(H_2) \geq 2$.

Suppose $\dim(H_2) \geq 2$. Since H_1 is closed in $[0,1]_1^{m+1}$, H_2 is open relative to H. So $H_2 = U \cap H$ for some open set U in $[0,1]_1^{m+1}$. Since U is a countable union of closed sets K_n, we have that $H_2 = U \cap H = (\cup_{n\geq 1} K_n) \cap H = \cup_{n \geq 1}(K_n \cap H)$. By [7, p. 30, Theorem III 2], $\dim(K_n \cap H) \geq 2$ for some $n \geq 1$. Since M is 0-dimensional, the components of H_2 lie in the disjoint sets $[0,1]_1 \times (G \cap ([0,1]_2^m \times \{t\})) = [0,1]_1 \times F'_m(t) \times \{t\}$ for $t \in M$. So the components of $K_n \cap H$ lie in the disjoint sets $F_m(t) \times \{t\}$ for $t \in M$. As we have previously seen, there exists a subcontinuum K of $K_n \cap H$ such that $\dim(K) \geq 2$. But by Observation 12, $\dim(K) \leq 1$, a contradiction.

Suppose $\dim(H_1) \geq 2$. Clearly, $\pi_{m+1}(G_1) = [0,1]_{m+1}$. Once again, some subcontinuum K of H_1 has dimension greater than one. If $\pi_{m+1}(K)$ is degenerate, then K is contained in a fiber $F_m(t) \times \{t\}$ for $\{t\} = \pi_{m+1}(K)$, contradicting Observation 12.

So we assume that $\pi_{m+1}(K) = [u,v]$ for $u \neq v$. Let $\rho \colon H \to G$ denote projection onto the second through $(m+1)^{\text{th}}$ coordinates. By definition, $\rho(H_1|_{[u,v]}) = G_1|_{[u,v]}$ and $K \subset H_1|_{[u,v]}$, so $\rho(K) \subset G_1|_{[u,v]}$. Since $\rho(K)$ is a continuum and meets both $F'_m(u) \times \{u\}$ and $F'_m(v) \times \{v\}$, $\rho(K)$ must contain $G((F'_m)^{-1})|_{C \cap [u,v]}$. Thus, $G_1|_{[u,v]} \subset \rho(K)$. We have that $\rho(K) = G_1|_{[u,v]}$. So $G_1|_{[u,v]}$ is a continuum, and since C is dense in $[u,v]$, $\pi_{m+1}(G_1|_{[u,v]}) = [u,v]$. It follows that $\dim(G_1|_{[u,v]}) = 1$. The continuum $G_1|_{[u,v]}$ will be the 1-dimensional continuum A in the statement of the theorem.

We also have that $K \subset H_1|_{[u,v]} \subset H|_{[u,v]} = G_1^{m+1}|_{[u,v]}$. If $G_1^m|_{f_m([u,v])}$ is degenerate, then $f_m([u,v])$ is degenerate. So $\dim(G_1^{m+1}|_{[u,v]}) = 1$ and $K \not\subset G_1^{m+1}|_{[u,v]}$. Thus, $G_1^m|_{f_m([u,v])}$ is nondegenerate and has dimension one. So, by $(*)$ (see page 4 at bottom), Remark 8, and Theorem 21, we have that $\dim(\pi_{1,m+1}(H|_{[u,v]})) = 2$. Hence, there exist nondegenerate intervals $[a,b]$ and $[c,d]$ such that $[a,b] \times [c,d] \subset \pi_{1,m+1}(H|_{[u,v]})$. We assume, without loss of generality, that $[c,d] = [u,v]$.

For each $t \in C \cap [u,v]$, we must have that $[a,b] \subset f_1(\pi_2(F'_m(t)))$, for otherwise $[a,b] \times \{t\} \not\subset G(f_{1,m+1}^{-1})|_{[u,v]}$ and thus, $[a,b] \times \{t\} \not\subset H|_{[u,v]}$, a contradiction. So $[a,b] \times G((F'_m)^{-1})|_{C\cap[u,v]}) \subset \ G_1^{m+1}|_{[u,v]} = H|_{[u,v]}$.

So we get that

$$\begin{aligned} [a,b] \times G_1|_{[u,v]} &= [a,b] \times \overline{G((F'_m)^{-1})|_{C\cap[u,v]})} \\ &\subset \overline{[a,b] \times G((F'_m)^{-1})|_{C\cap[u,v]})} \\ &\subset \overline{G_1^{m+1}} = \ G_1^{m+1}, \end{aligned}$$

which is the desired result. □

Corollary 23 is a necessary condition for the emergence of dimension two or greater in G_1^{m+1}.

Corollary 23. *Let $m \geq 2$ and let $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^m$ be an inverse sequence with surjective interval-valued functions. If $\dim(G_1^n) = 1$ for each $1 \leq n \leq m$, $\dim(G_2^{m+1}) = 1$, and $\dim(G_1^{m+1}) \geq 2$, then some flat spot of f_m composes to a nondegenerate value of f_1.*

Proof. Let $F_m \colon [0,1]_{m+1} \to G_1^m$ and $F'_m \colon [0,1]_{m+1} \to G_2^m$ be given. By Theorem 22, there exist a 1-dimensional continuum A in G_2^{m+1} and an interval $[a,b]$ such that $[a,b] \times A \subset G_1^{m+1}$. Again, by [18, Theorem 20.2 and p. 127], $\dim([a,b] \times A) = 2$. Suppose $\pi_{m+1}(A)$ is degenerate; let $\pi_{m+1}(A) = \{t\}$. Then $A \subset F'_m(t) \times \{t\}$. So $[a,b] \times A \subset [a,b] \times F'_m(t) \times \{t\}$, which is topologically a subset of $F_m(t) \times \{t\}$. But by Observation 12, $\dim(F_m(t) \times \{t\}) \leq 1$, a contradiction. So $\pi_{m+1}(A)$ is nondegenerate; let $\pi_{m+1}(A) = [u,v]$.

Suppose $\pi_i(A)$ is nondegenerate for some $2 \leq i \leq m$. Assume i is the largest such integer. Then $\pi_i(A)$ is an interval in $[0,1]_i$. Since $[a,b] \times A \subset G_1^{m+1}$, it follows that $F_{i-1}(t)$ contains $[a,b]$ for each $t \in \pi_i(A)$, where $F_{i-1} \colon [0,1]_i \to G_1^{i-1}$. So $[a,b] \times G_2^i|_{\pi_i(A)} \subset G_1^i|_{\pi_i(A)}$. By Corollary 6, $\dim(G_2^i|_{\pi_i(A)}) = 1$. So $\dim([a,b] \times G_2^i|_{\pi_i(A)}) = 2$. But by hypothesis, $\dim(G_1^i) = 1$, a contradiction.

Thus, $\pi_i(A)$ is degenerate for each $2 \leq i \leq m$. Let $A = \{(p_2, \ldots, p_m)\} \times [u,v]$. So f_m has a flat spot at $\pi_m(A) = \{p_m\}$, specifically $[u,v] \times \{p_m\} \subset G(f_m)$. Also, $f_1(p_2)$ is nondegenerate since $[a,b] \subset f_1(p_2)$. Hence, a flat spot of f_m composes to a nondegenerate value of f_1. □

Theorem 24 gives a sufficient condition for $\dim(G_1^{m+1})$ to be greater than one.

Theorem 24. *Let $m \geq 2$ and let $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^m$ be an inverse sequence with surjective interval-valued functions. If some flat spot $[s,t] \times \{x_m\}$ of f_m composes to a nondegenerate value of f_j for some $1 \leq j < m$,* *then* $G_j^{m+1}|_{[s,t]}$ *contains a two cell and* $\dim(G_1^{m+1}|_{[s,t]}) \geq 2$*. Hence,* G_j^{m+1} *contains a two cell and* $\dim(G_1^{m+1}) \geq 2$*.*

Proof. By hypothesis, there exists $x_{j+1} \in f_{j+1,m}(x_m)$ such that $f_j(x_{j+1})$ is a nondegenerate interval $[a, b]$. Also, there exists $x_{j+2}, \ldots, x_{m-1}$ such that $(x_{j+1}, \ldots, x_{m-1}, x_m)$ is in G_{j+1}^m. Hence, $[a, b] \times [s, t] \overset{T}{\approx} [a, b] \times \{x_{j+1}\} \times \ldots \times \{x_m\} \times [s, t]$ is a subset of $G_j^{m+1}|_{[s,t]}$. So, $G_j^{m+1}|_{[s,t]}$ contains a two cell. By $(*)$ (see page 4 at bottom) and Corollary 6, $\dim(G_1^{m+1}|_{[s,t]}) \geq 2$. It follows that G_j^{m+1} contains a two cell and $\dim(G_1^{m+1}) \geq 2$. $\square$

Corollary 25. *Let* $m \geq 2$ *and let* $\{f_i\colon [0, 1] \to [0, 1]\}_{i=1}^m$ *be an inverse sequence with surjective interval-valued functions. Suppose that* $\dim(G_1^n) = 1$ *for each* $1 \leq n \leq m$*, and that* $\dim(G_2^{m+1}) = 1$*. Then* $\dim(G_1^{m+1}) \geq 2$ *if and only if some flat spot of* f_m *composes to a nondegenerate value of* f_j *for some* $1 \leq j < m$*.*

Theorem 26 generalizes [10, Theorem 4.2] and gives sufficient conditions for $\varprojlim\{[0, 1], f_i\}$ to be tree-like.

Theorem 26. *Let* $\{f_i\colon [0, 1] \to [0, 1]\}_{i=1}^\infty$ *be an inverse sequence with surjective interval-valued bonding functions. If* $\dim(G(f_i)) = 1$ *for each* $i \geq 1$*, and for no* $j < m$ *does a flat spot of* f_m *compose to a nondegenerate value of* f_j*, then* $\varprojlim\{[0, 1], f_i\}$ *is a tree-like continuum.*

Proof. We only need to show that $\varprojlim\{[0, 1], f_i\}$ is 1-dimensional. By way of contradiction, suppose that $\dim(\varprojlim\{[0, 1], f_i\}) \geq 2$. By [10, Theorem 3.3], $\dim(G_1^{m+1}) \geq 2$ for some $m \geq 1$. Assume m is the least such integer. If $m = 1$, then $\dim(G(f_1)) = 2$, contradicting the hypothesis. So, $m \geq 2$.

Let j be the largest integer such that $\dim(G_j^{m+1}) \geq 2$. If $j = m$, then $\dim(G(f_m)) \geq 2$, contradicting the hypothesis. So, $j < m$. It follows from the hypothesis, our choice of m, and Corollary 6 that $\dim(G_j^n) = 1$ for $j \leq n \leq m$. Also, $\dim(G_{j+1}^{m+1}) = 1$ by choice of j. By Corollary 23, some flat spot of f_m composes to a nondegenerate value of f_j, contradicting the hypothesis. Therefore, $\dim(\varprojlim\{[0, 1], f_i\}) = 1$. By [3], $\varprojlim\{[0, 1], f_i\}$ has trivial shape and thus is tree-like. $\square$

Corollary 27 gives necessary conditions for $\varprojlim\{[0, 1], f_i\}$ to be 1-dimensional, and therefore to be tree-like.

Corollary 27. *Suppose* $\{f_i\colon [0, 1] \to [0, 1]\}_{i=1}^\infty$ *is an inverse sequence with surjective interval-valued bonding functions and* $\dim(G(f_i)) = 1$ *for each* $i \geq 1$*. If there exist integers* j *and* m *with* $j < m$ *such that a flat spot* $[s, t] \times \{x_m\}$ *of* f_m *composes to a nondegenerate value of* f_j*,* *and there exists an* $(m+1)$*-tail sequence of inverse mappings* $\{g_i\}_{i \geq m+1}$ *where the domain of* g_{m+1}^{-1} *is a nondegenerate subinterval of* $[s,t]$*, then* $\dim(\varprojlim\{[0,1], f_i\}) \geq 2$*.*

Proof. Suppose the domain of g_{m+1}^{-1} is $[u,v]$. Note that $[u,v] \times \{x_m\}$ is a flat spot of f_m that composes to a nondegenerate value of f_j. So, by Theorem 24, $\dim(G_1^{m+1}|_{[u,v]}) \geq 2$. By [16, Theorem 2.1], $\varprojlim\{[0,1], f_i\}$ contains a homeomorphic copy of $G_1^{m+1}|_{[u,v]}$. So $\dim(\varprojlim\{[0,1], f_i\}) \geq 2$. □

Corollary 28. *Suppose* $\{f_i \colon [0,1] \to [0,1]\}_{i=1}^{\infty}$ *is an inverse sequence with surjective interval-valued bonding functions and* $\dim(G(f_i)) = 1$ *for each* $i \geq 1$*. If there exist integers* j *and* m *with* $j < m$ *such that a flat spot* $[s,t] \times \{x_m\}$ *of* f_m *composes to a nondegenerate value of* f_j*, and there exists* $u \in [0,1]_{m+2}$ *such that* $f_{m+1}(u) \cap [s,t]$ *is nondegenerate, then* $\dim(\varprojlim\{[0,1], f_i\}) \geq 2$*.*

Proof. Let $x = (x_1, x_2, \ldots)$ be a point of $\varprojlim\{[0,1], f_i\}$ where $x_{m+2} = u$. Let $f_{m+1}(u) = [v,w] \subset [s,t]$. We define an $(m+1)$-tail sequence of inverse mappings $\{g_i\}_{i \geq m+1}$ as follows. Let $G(g_{m+1}) = \{u\} \times [v,w]$, and for each $i \geq m+2$, let $G(g_i) = \{(x_{i+1}, x_i)\}$. We see that $\{g_i\}_{i \geq m+1}$ is an $(m+1)$-tail sequence of inverse mappings satisfying the conditions of Corollary 27, and hence, $\dim(\varprojlim\{[0,1], f_i\}) \geq 2$. □

Corollary 29. *Suppose* $X = \varprojlim\{[0,1], f_i\}$*,* $\dim(G(f_i)) = 1$ *for each* $i \geq 1$*, and for each* $k \in \mathbb{N}$ *and each subinterval* $[s,t]$ *of* $[0,1]_k$*, there is a* k*-tail sequence* $\{g_i\}$ *of inverse mappings where the domain of* g_k^{-1} *is a nondegenerate subinterval of* $[s,t]$*. Then* X *is tree-like if and only if for no* $j < m$ *does a flat spot of* f_m *compose to a nondegenerate value of* f_j*.*

Corollary 30. *Suppose* $X = \varprojlim\{[0,1], f_i\}$*,* $\dim(G(f_i)) = 1$ *for each* $i \geq 1$*, and there exists a surjective* 3*-tail sequence* $\{g_i\}$ *of inverse mappings. Then* X *is tree-like if and only if for no* $j < m$ *does a flat spot of* f_m *compose to a nondegenerate value of* f_j*.*

Question 31. Can the assumption of the existence of k-tail sequences be omitted in corollaries 27, 28, 29, and 30?

As an application of Corollary 28, [10, examples 5.1, 5.2, and 5.6] are immediately seen to have dimension larger than one. By Theorem 26, [10, examples 5.3 and 5.5] and [11, Example 6.1] are seen to be tree-like.

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