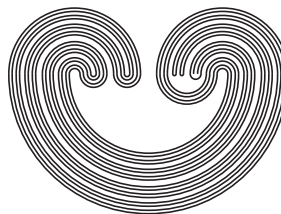


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A FINITE PRESENTATION FOR THE
AUTOMORPHISM GROUP OF THE FIRST
HOMOLOGY OF A NON-ORIENTABLE
SURFACE OVER \mathbb{Z}_2 PRESERVING THE MOD 2
INTERSECTION FORM

by

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A FINITE PRESENTATION FOR THE AUTOMORPHISM GROUP OF THE FIRST HOMOLOGY OF A NON-ORIENTABLE SURFACE OVER \mathbb{Z}_2 PRESERVING THE MOD 2 INTERSECTION FORM

RYOMA KOBAYASHI AND GENKI OMORI

ABSTRACT. Let $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ be the group of automorphisms on the first homology group with \mathbb{Z}_2 coefficients of a closed non-orientable surface N_g preserving the mod 2 intersection form. In this paper, we obtain a finite presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$. As an application we calculate the second homology group of $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$.

1. INTRODUCTION

For $g \geq 1$ and $n \geq 0$, let $N_{g,n}$ be a compact connected non-orientable surface of genus g with n boundary components (we denote $N_{g,0}$ by N_g) and a bilinear form $\cdot : H_1(N_g; \mathbb{Z}_2) \times H_1(N_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ the mod 2 intersection form on the first homology group $H_1(N_g; \mathbb{Z}_2)$ of N_g with \mathbb{Z}_2 coefficients. We represent N_g by a sphere with g crosscaps as in Figure 1; i.e., we regard N_g as a sphere with g boundary components and a Möbius band attached to each boundary component. We define $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ by the subgroup of the automorphism group $\text{Aut } H_1(N_g; \mathbb{Z}_2)$ of $H_1(N_g; \mathbb{Z}_2)$ preserving the mod 2 intersection form \cdot . Note that $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ is isomorphic to $O(g, \mathbb{Z}_2) = \{A \in GL(g, \mathbb{Z}_2) \mid {}^tAA = E\}$ by taking the basis $\{x_1, x_2, \dots, x_g\}$ for $H_1(N_g; \mathbb{Z}_2)$, where x_i is a homology class of a one-sided simple closed curve μ_i in Figure 1 and E is an identity matrix

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of $GL(g, \mathbb{Z}_2)$ (see [5]). By Mustafa Korkmaz [3] and Błażej Szepietowski [12] we have isomorphisms.

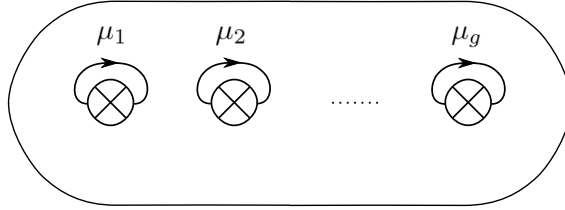


FIGURE 1. Simple closed curves $\mu_1, \mu_2, \dots, \mu_g$ in N_g representing the basis x_1, x_2, \dots, x_g for $H_1(N_g; \mathbb{Z}_2)$, respectively.

Let a_i ($i = 1, \dots, g-1$, for $g \geq 2$) and b (for $g \geq 4$) $\in \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ be the following elements:

$$a_i : \begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_i, \\ x_k & \mapsto x_k \quad (k \neq i, i+1), \end{cases} \quad b : \begin{cases} x_1 & \mapsto x_2 + x_3 + x_4, \\ x_2 & \mapsto x_1 + x_3 + x_4, \\ x_3 & \mapsto x_1 + x_2 + x_4, \\ x_4 & \mapsto x_1 + x_2 + x_3, \\ x_k & \mapsto x_k \quad (k \neq 1, 2, 3, 4). \end{cases}$$

In this paper, we give a finite presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$.

Theorem 1.1. *If $g = 1, 2, 3$, then $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ is the following group.*

- $\text{Aut}(H_1(N_1; \mathbb{Z}_2), \cdot) = 1$,
- $\text{Aut}(H_1(N_2; \mathbb{Z}_2), \cdot) = \langle a_1 \mid a_1^2 = 1 \rangle \cong \mathbb{Z}_2$,
- $\text{Aut}(H_1(N_3; \mathbb{Z}_2), \cdot) = \langle a_1, a_2 \mid a_1^2 = a_2^2 = (a_1 a_2)^3 = 1 \rangle$.

If $g \geq 4$ is odd, $g = 4$ or 6 , then $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ admits a presentation with generators a_1, \dots, a_{g-1}, b and relations

- (1) $a_i^2 = b^2 = 1$ for $i = 1, \dots, g-1$,
- (2) $(a_i a_j)^2 = 1$ for $g \geq 4$, $|i - j| > 1$,
- (3) $(a_i a_{i+1})^3 = 1$ for $g \geq 3$, $i = 1, \dots, g-2$,
- (4) $(a_i b)^2 = 1$ for $g \geq 4$, $i \neq 4$,
- (5) $(a_4 b)^3 = 1$ for $g \geq 5$,
- (6) $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$ for $g \geq 7$.

If $g \geq 8$ is even, then $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ admits a presentation with generators $a_1, \dots, a_{g-1}, b, b_0, \dots, b_{\frac{g-2}{2}}$ and relations (1)–(6) above and the following relations:

- (7) $b_0 = a_1$, $b_1 = b$, $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$,

- (8) $b_{i+1} = (b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$
 for $2 \leq i \leq \frac{g-4}{2}$,
 (9) $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$.

We read every word of every group in this paper from right to left. In §3, we will prove Theorem 1.1 for $g \geq 4$. Theorem 1.1 is clear for $g = 1, 2$. For $g = 3, 4$, Szepietowski [11, proof of Theorem 5.5] gave the presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$. Note that $\text{Aut}(H_1(N_3; \mathbb{Z}_2), \cdot)$ is isomorphic to the dihedral group D_6 and the symmetric group S_3 . By the result of Korkmaz [3, Corollary 4.1] and Theorem 1.1, the first homology group of $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ is as follows:

$$H_1(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z}) = \begin{cases} 0 & \text{for } g = 1, g \geq 7, \\ \langle [a_1] \rangle \cong \mathbb{Z}_2 & \text{for } g = 2, 3, 5, 6, \\ \langle [a_1], [b] \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } g = 4. \end{cases}$$

Note that the above equality is known for $g \geq 7$ odd (see, for instance, [13]).

In §4, by using the presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ obtained in Theorem 1.1, we calculate the second homology group of $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ for $g \geq 9$. We get the following theorem.

Theorem 1.2. *For $g \geq 9$ or $g = 7$, the second homology group of $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ is trivial.*

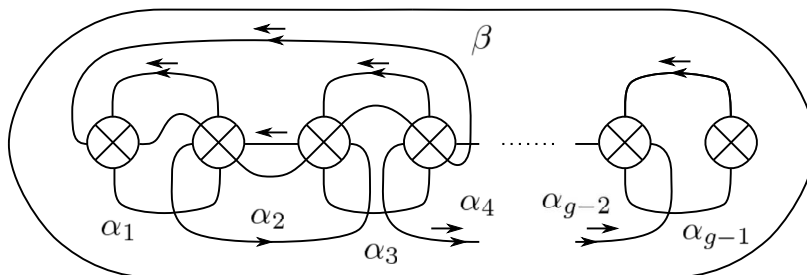
Theorem 1.2 was shown by Michael R. Stein [8] for odd g (see Theorem 2.13 and Proposition 3.3(a)). More precisely, he proved $H_2(\text{Sp}(2h, \mathbb{Z}_m); \mathbb{Z}) = 0$ when $h \geq 3$ and m is not divisible by 4 (see also [1]).

We give a generating set for $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ which consists of one element x_0 by an application of the discussion of Wolfgang Pitsch [7] to prove Theorem 1.2. By using the generator of $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ and Stein's result we show that x_0 is trivial for $g \geq 9$.

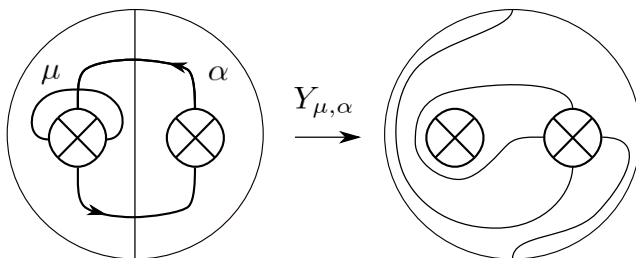
2. PRELIMINARIES

Let $\alpha_1, \dots, \alpha_{g-1}, \beta$ be two-sided simple closed curves on N_g as in Figure 2. Arrows on the side of simple closed curves in Figure 2 indicate directions of Dehn twists along their simple closed curves. Since the actions of the Dehn twists along $\alpha_1, \dots, \alpha_{g-1}, \beta$ induce a_1, \dots, a_{g-1}, b on $H_1(N_g; \mathbb{Z}_2)$, respectively, we denote Dehn twists along $\alpha_1, \dots, \alpha_{g-1}, \beta$ by a_1, \dots, a_{g-1}, b and abuse of notation.

Let μ be a one-sided simple closed curve and α a two-sided simple closed curve such that μ and α intersect transversely in one point. For the simple closed curves μ and α , we denote by $Y_{\mu, \alpha}$ a self-diffeomorphism on N_g which is described as the result of pushing the regular neighborhood of

FIGURE 2. Simple closed curves $\alpha_1, \dots, \alpha_{g-1}, \beta$ on N_g .

μ once along α (see Figure 3). We call $Y_{\mu, \alpha}$ a *Y-homeomorphism*. We set the direction of Y_{μ_i, α_j} ($1 \leq i \leq g$, $1 \leq j \leq g-1$) by the orientation of α_j in Figure 2 and $y := Y_{\mu_1, \alpha_1}$. Note that the action of the Y-homeomorphism on $H_1(N_g; \mathbb{Z}_2)$ is trivial.

FIGURE 3. The Y-homeomorphism on the regular neighborhood of $\mu \cup \alpha$.

The *mapping class group* $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on N_g fixing each boundary component pointwise. In [6], Luis Paris and Szepletowski give a finite presentation for $\mathcal{M}(N_g)$. The presentation has a generating set which consists of Dehn twists along two-sided simple closed curves and “crosscap transpositions.” In [10], Michał Stukow obtains a finite presentation for $\mathcal{M}(N_g)$ whose generators are Dehn twists and a Y-homeomorphism. Stukow’s presentation is the following.

Theorem 2.1. *If $g \geq 4$ is odd or $g = 4$, then $\mathcal{M}(N_g)$ admits a presentation with generators $a_1, \dots, a_{g-1}, b, y$, and ρ . The defining relations are*

- (A1) $[a_i, a_j] = 1$ for $|i - j| > 1$,
- (A2) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ for $i = 1, \dots, g - 2$,

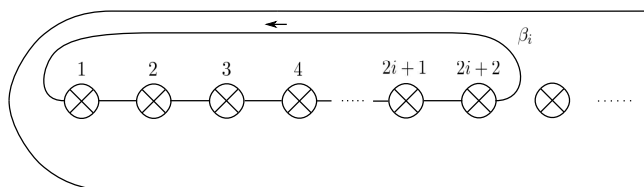
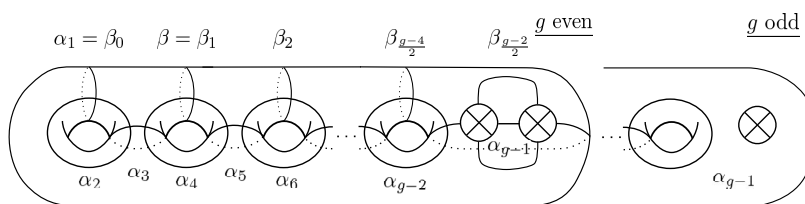
- (A3) $[a_i, b] = 1$ for $i \neq 4$,
 (A4) $a_4 b a_4 = b a_4 b$ for $g \geq 5$,
 (A5) $(a_2 a_3 a_4 b)^{10} = (a_1 a_2 a_3 a_4 b)^6$ for $g \geq 5$,
 (A6) $(a_2 a_3 a_4 a_5 a_6 b)^{12} = (a_1 a_2 a_3 a_4 a_5 a_6 b)^9$ for $g \geq 7$,
 (B1) $y(a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) = (a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1}) y$,
 (B2) $y(a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) y = a_1 (a_2 a_1 y^{-1} a_2^{-1} y a_1 a_2) a_1$,
 (B3) $[a_i, y] = 1$ for $i = 3, \dots, g-1$,
 (B4) $a_2 (y a_2 y^{-1}) = (y a_2 y^{-1}) a_2$,
 (B5) $y a_1 = a_1^{-1} y$,
 (B6) $b y b y^{-1} = \{a_1 a_2 a_3 (y^{-1} a_2 y) a_3^{-1} a_2^{-1} a_1^{-1}\} \{a_2^{-1} a_3^{-1} (y a_2 y^{-1}) a_3 a_2\}$,
 (B7) $[(a_4 a_5 a_3 a_4 a_2 a_3 a_1 a_2 y a_2^{-1} a_1^{-1} a_3^{-1} a_2^{-1} a_4^{-1} a_3^{-1} a_5^{-1} a_4^{-1}), b] = 1$
 for $g \geq 6$,
 (B8) $\{(y a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b (a_4 a_3 a_2 a_1 y^{-1})\} \{(a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1}) b^{-1} (a_4 a_3 a_2 a_1)\}$
 $= \{(a_4^{-1} a_3^{-1} a_2^{-1}) y (a_2 a_3 a_4)\} \{a_3^{-1} a_2^{-1} y^{-1} a_2 a_3\} \{a_2^{-1} y a_2\} y^{-1}$
 for $g \geq 5$,
 (C1a) $(a_1 a_2 \cdots a_{g-1})^g = \rho$ for g odd,
 (C1b) $(a_1 a_2 \cdots a_{g-1})^g = 1$ for g even,
 (C2) $[a_1, \rho] = 1$,
 (C3) $\rho^2 = 1$,
 (C4a) $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-1}{2}} = 1$ for g odd,
 (C4b) $(y^{-1} a_2 a_3 \cdots a_{g-1} y a_2 a_3 \cdots a_{g-1})^{\frac{g-2}{2}} y^{-1} a_2 a_3 \cdots a_{g-1} = \rho$ for g even,

where $[X, Y] = XYX^{-1}Y^{-1}$. If $g \geq 6$ is even, then $\mathcal{M}(N_g)$ admits a presentation with generators $a_1, \dots, a_{g-1}, y, b, \rho$, and $b_0, \dots, b_{\frac{g-2}{2}}$. The defining relations are (A1)–(A6), (B1)–(B8), and (C1a)–(C4b) and the following relations:

- (A7) $b_0 = a_1, \quad b_1 = b$,
 (A8) $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$
 for $1 \leq i \leq \frac{g-4}{2}$,
 (A9a) $[b_2, b] = 1$ for $g = 6$,
 (A9b) $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$ for $g \geq 8$.

Relations (A1) and (A3) are called *disjointness relations* and relations (A2) and (A4) are called *braid relations*. When we deform relations (or words) by disjointness relations and braid relations, we write “DI” and “BR” on the left-right arrow (or the equality sign), respectively.

b_i ($2 \leq i \leq \frac{g-2}{2}$) in the g even case of Theorem 2.1 is the Dehn twist along a simple closed curve β_i in Figure 4. The arrow on the side of the simple closed curve β_i in Figure 4 indicates the direction of the Dehn twist b_i . We note that N_g is diffeomorphic to a surface as in Figure 5 and we can choose the diffeomorphism such that simple closed curves

FIGURE 4. Simple closed curves β_i on N_g for $2 \leq i \leq \frac{g-2}{2}$.FIGURE 5. A different view of simple closed curves α_i ($1 \leq i \leq g-1$) and β_j ($0 \leq j \leq \frac{g-2}{2}$) on N_g .

α_i ($1 \leq i \leq g-1$) in Figure 2 and β_j ($0 \leq j \leq \frac{g-2}{2}$) in Figure 4 are sent to a position in Figure 5.

Since the action of $\mathcal{M}(N_g)$ on $H_1(N_g; \mathbb{Z}_2)$ preserves the mod 2 intersection form \cdot , we have a homomorphism $\rho_2 : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$. John D. McCarthy and Ulrich Pinkall [5] show that ρ_2 is surjective. $\Gamma_2(N_g) := \ker \rho_2$ is called the *level 2 mapping class group* of $\mathcal{M}(N_g)$. Szepietowski [11] proves that $\Gamma_2(N_g)$ is generated by Y-homeomorphisms for $g \geq 2$. More precisely, Szepietowski [11, Lemma 3.6 and Theorem 5.5] shows the following theorem.

Theorem 2.2. *For $g \geq 2$, $\Gamma_2(N_g)$ is normally generated by y in $\mathcal{M}(N_g)$.*

We note that squares of Dehn twists along non-separating two-sided simple closed curves are elements of $\Gamma_2(N_g)$. Hence, $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$ is a normal generating set for $\Gamma_2(N_g)$ in $\mathcal{M}(N_g)$.

We now explain about the Tietze transformations. Let G be a group with presentation $G = \langle X | R \rangle$, where X is a subset of G and R is a set consisting of words of elements of X . Then G is isomorphic to the quotient group F/K , where F is the free group which is generated by X and K is the normal subgroup of F which is normally generated by R . Then the following transformations among presentations do not change the isomorphism class of G .

$$\begin{aligned} \langle X|R \rangle &\longleftrightarrow \langle X|R \cup \{k\} \rangle && \text{for } k \in K - R, \\ &\longleftrightarrow \langle X \cup \{v\}|R \cup \{vw^{-1}\} \rangle && \text{for } w \in F - X. \end{aligned}$$

These transformations are called the *Tietze transformations*. In this paper, we use these transformations without any comment when we deform presentations (or relations).

3. PROOF OF THEOREM 1.1 FOR $g \geq 4$

By surjectivity of $\rho_2 : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ and the definition of $\Gamma_2(N_g)$, we have the following short exact sequence.

$$(3.1) \quad 1 \longrightarrow \Gamma_2(N_g) \longrightarrow \mathcal{M}(N_g) \xrightarrow{\rho_2} \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \longrightarrow 1.$$

We have the finite presentation for $\mathcal{M}(N_g)$ (Theorem 2.1) and the normal generating set $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$ for $\Gamma_2(N_g)$ (Theorem 2.2). We can get a presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ by adding $\{a_1^2, \dots, a_{g-1}^2, b^2, y\}$ to the relations of the presentation for $\mathcal{M}(N_g)$ in Theorem 2.1.

The relations $a_1^2 = \dots = a_{g-1}^2 = b^2 = 1$ are clearly nothing but relation (1) in Theorem 1.1. By Claim 3.2 and relation (1), we have

$$\begin{aligned} (a_1 a_2 a_3 a_4 a_5)^6 &= (a_1^2 a_2 a_3 a_4 a_5)^5 \\ &= (a_2 a_3 a_4 a_5)^5 \\ &= (a_2^2 a_3 a_4 a_5)^4 \\ &\vdots \\ &= a_5^2 \\ &= 1. \end{aligned}$$

Hence, we obtain the relation $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$ in relation (7) from relation (A8) for $i = 1$. Relations (2), (3), (4), (5), and (9) in Theorem 1.1 are obtained from relations (A1), (A2), (A3), (A4), and (A9b) in Theorem 2.1 and relation (1).

Since $y = 1$ in $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$, relations (B1), (B3), (B4), (B7), and (B8) are unnecessary. By using relation (1) and braid relations (Theorem 2.1(A2) and (A4)), relations (B2), (B5), and (B6) are deformed as follows.

$$\begin{aligned}
\text{(B2)} \quad & \xleftrightarrow{y=1\&(1)} a_2 a_1 a_2 a_1 a_2 = \underline{a_1 a_2 a_1 a_2 a_1 a_2 a_1} \\
& \xleftrightarrow{\text{BR}} a_2 a_1 a_2 a_1 a_2 = a_2 a_1 \underline{a_2 a_2} a_2 a_1 a_2 \\
& \xleftrightarrow{(1)} a_2 a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1 a_2. \\
\text{(B5)} \quad & \xleftrightarrow{y=1} a_1 = a_1^{-1} \iff (1). \\
\text{(B6)} \quad & \xleftrightarrow{y=1\&(1)} 1 = a_1 \underline{a_2 a_3 a_2 a_3} a_2 a_1 a_2 a_3 \underline{a_2 a_3 a_2} \\
& \xleftrightarrow{\text{BR}} 1 = a_1 a_3 \underline{a_2 a_3 a_3 a_2} a_1 \underline{a_2 a_3 a_3 a_2} a_3 \\
& \iff 1 = a_1 a_3 a_1 a_3 \xleftrightarrow{(A1)\&(1)} 1 = a_1^2.
\end{aligned}$$

Therefore, relations (B1), (B2), \dots , and (B8) drop out.

It is sufficient for proof of this theorem to show the following three claims.

Claim 3.1. *Relations (C1a) and (C4b) are equivalent to $\rho = 1$ under $y = 1$, relations (1), (BR), and (DI). It allows us to rule out generator ρ and relations (C1a), (C2), (C3), and (C4b) from the presentation.*

Claim 3.2. *Let G be a group and assume that $g_1, g_2, \dots, g_n \in G$ satisfy relations*

$$\begin{aligned}
\text{(BR)} \quad & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for } i = 1, \dots, n-1, \\
\text{(DI)} \quad & [g_i, g_j] = 1 \quad \text{for } |i - j| > 1.
\end{aligned}$$

Then we have a relation $(g_1 g_2 \cdots g_n)^{n+1} = (g_1^2 g_2 \cdots g_n)^n$ on G .

“BR” and “DI” means braid relations and disjointness relations, respectively.

Claim 3.3. *Relation (A9a) follows from relations (1), (2), (3), (4), and (5).*

We suppose that Claim 3.2 and Claim 3.3 are true.

Proof of Claim 3.1. Since $y = 1$ in $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ and $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ has relations (C1a) and (C4b), ρ is represented by the form

$$\rho = \begin{cases} (a_1 a_2 \cdots a_{g-1})^g & \text{for } g \text{ odd,} \\ (a_2 \cdots a_{g-1})^{g-1} & \text{for } g \text{ even.} \end{cases}$$

We get $\rho = 1$ by repeatedly applying Claim 3.2 and relation (1) to the right-hand side of the above equation. For example, in g odd case,

$$\begin{aligned}
\rho &= (a_1 a_2 \cdots a_{g-1})^g \stackrel{\text{Claim 3.2}}{=} (a_2 \cdots a_{g-1})^{g-1} \stackrel{\text{Claim 3.2}}{=} (a_3 \cdots a_{g-2})^{g-2} \\
&\stackrel{\text{Claim 3.2}}{=} \cdots \stackrel{\text{Claim 3.2}}{=} a_{g-1}^2 \stackrel{(1)}{=} 1.
\end{aligned}$$

Thus, we obtain the claim. \square

By Claim 3.1, relations (C2) and (C3) are unnecessary. By a discussion similar to the proof of Claim 3.1, relations (C1b) and (C4a) are unnecessary, too. Therefore, relations (C1a), (C1b), (C2), (C3), (C4a), and (C4b) drop out. For relation (A5), we apply Claim 3.2 as follows.

$$\begin{aligned} (A5) &\iff (a_2 a_3 a_4 b)^{10} = \underline{(a_1 a_2 a_3 a_4 b)^6} \stackrel{\text{Claim 3.2}}{\iff} (a_2 a_3 a_4 b)^{10} = (a_2 a_3 a_4 b)^5 \\ &\iff (a_2 a_3 a_4 b)^5 = 1 \stackrel{\text{Claim 3.2}}{\iff} \dots \stackrel{\text{Claim 3.2}}{\iff} b^2 = 1 \iff (1). \end{aligned}$$

We have completed the proof of Theorem 1.1 without proofs of Claim 3.2 and Claim 3.3.

Proof of Claim 3.2.

$$(g_1 g_2 \cdots g_n)^{n+1} = (g_1 g_2 \cdots g_n)(g_1 g_2 \cdots g_n) \cdots (g_1 g_2 \cdots g_n).$$

Let A_i ($i = n+1, n, \dots, 1$) be the i -th sequence $(g_1 g_2 \cdots g_n)$ from the right in the right-hand side. By using disjointness relations and braid relations, the above equation is deformed as follows.

$$\begin{aligned} (g_1 g_2 \cdots g_n)^{n+1} &= A_{n+1} A_n \cdots A_1 \\ &= (g_1 g_2 \cdots g_{n-1} \underline{g_n}) A_n A_{n-1} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1})(g_1 g_2 \cdots g_{n-2} \underline{g_n g_{n-1} g_n}) A_{n-1} \cdots A_1 \\ &\stackrel{\text{BR}}{=} (g_1 g_2 \cdots g_{n-1})(g_1 g_2 \cdots g_n) g_{n-1} A_{n-1} \cdots A_1. \end{aligned}$$

We replace the first sequence $(g_1 g_2 \cdots g_n)$ from the left in the bottom with A_n . Then we have

$$\begin{aligned} (g_1 g_2 \cdots g_n)^{n+1} &= (g_1 g_2 \cdots g_{n-1}) A_n \underline{g_{n-1}} A_{n-1} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_{n-3} \underline{g_{n-1} g_{n-2} g_{n-1} g_n}) A_{n-2} \cdots A_1 \\ &\stackrel{\text{BR}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_{n-3} g_{n-2} g_{n-1} \underline{g_{n-2} g_n}) A_{n-2} \cdots A_1 \\ &\stackrel{\text{DI}}{=} (g_1 g_2 \cdots g_{n-1}) A_n (g_1 g_2 \cdots g_n) g_{n-2} A_{n-2} \cdots A_1. \end{aligned}$$

We replace the second sequence $(g_1 g_2 \cdots g_n)$ from the left in the bottom with A_{n-1} and repeat it. Then we have

$$\begin{aligned}
(g_1 g_2 \cdots g_n)^{n+1} &= (g_1 g_2 \cdots g_{n-1}) A_n A_{n-1} g_{n-2} A_{n-2} \cdots A_1 \\
&= (g_1 g_2 \cdots g_{n-1}) A_n A_{n-1} A_{n-2} g_{n-3} A_{n-3} \cdots A_1 \\
&\vdots \\
&= (g_1 g_2 \cdots g_{n-1}) A_n \cdots A_2 g_1 A_1 \\
&= (g_1 g_2 \cdots g_{n-2}) A_n g_{n-2} A_{n-1} \cdots A_2 g_1 A_1 \\
&\vdots \\
&= (g_1 g_2 \cdots g_{n-2}) A_n \cdots A_3 g_1 A_2 g_1 A_1 \\
&\vdots \\
&= g_1 A_n \cdots g_1 A_3 g_1 A_2 g_1 A_1 \\
&= (g_1^2 g_2 \cdots g_n)^n.
\end{aligned}$$

Thus, we obtain the claim. \square

Proof of Claim 3.3. Note that $b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$. We first show the following.

- (a) $a_i(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_{i-1}$ for $i = 2, 3, 4$.
- (b) $b(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_4 a_5 a_4$.
- (c) $a_5(a_1 a_2 a_3 a_4 a_5 b) = (a_1 a_2 a_3 a_4 a_5 b) a_4 b a_4$.

Relation (a) is obtained by an argument similar to the proof of Claim 3.2. The other relations are obtained by the following deformations.

$$\begin{aligned}
\text{(b)} \quad b(a_1 a_2 a_3 a_4 a_5 b) &\stackrel{\text{DI}}{=} a_1 a_2 a_3 \underline{b a_4 b} a_5 \stackrel{\text{BR}}{=} a_1 a_2 a_3 a_4 b a_4 a_5 \\
&\stackrel{(1)}{=} a_1 a_2 a_3 a_4 b (a_5 a_5) a_4 a_5 \stackrel{\text{BR}}{=} (a_1 a_2 a_3 a_4 a_5 b) a_4 a_5 a_4. \\
\text{(c)} \quad a_5(a_1 a_2 a_3 a_4 a_5 b) &\stackrel{\text{DI}}{=} a_1 a_2 a_3 \underline{a_5 a_4 a_5} b \stackrel{\text{BR}}{=} a_1 a_2 a_3 a_4 a_5 a_4 b \\
&\stackrel{(1)}{=} a_1 a_2 a_3 a_4 a_5 (b b) a_4 b \stackrel{\text{BR}}{=} (a_1 a_2 a_3 a_4 a_5 b) a_4 b a_4.
\end{aligned}$$

We now prove $bb_2 = b_2 b$ by using only relations (a), (b), (c), (1), and disjointness relations. It means the relation $bb_2 = b_2 b$ is unnecessary.

$$\begin{aligned}
bb_2 &= b(a_1a_2a_3a_4a_5b)^5 \\
&\stackrel{(b)}{=} (a_1a_2a_3a_4a_5b)a_4a_5a_4(a_1a_2a_3a_4a_5b)^4 \\
&\stackrel{(a),(c)}{=} (a_1a_2a_3a_4a_5b)^2a_3a_4ba_4a_3(a_1a_2a_3a_4a_5b)^3 \\
&\stackrel{(a),(b)}{=} (a_1a_2a_3a_4a_5b)^3a_2a_3a_4a_5a_4a_3a_2(a_1a_2a_3a_4a_5b)^2 \\
&\stackrel{(a),(c)}{=} (a_1a_2a_3a_4a_5b)^4a_1a_2a_3a_4\underline{ba_4a_3a_2a_1}(a_1a_2a_3a_4a_5b) \\
&\stackrel{(1)}{=} (a_1a_2a_3a_4a_5b)^4a_1a_2a_3a_4\underline{ba_5b} \\
&\stackrel{\text{DI}}{=} (a_1a_2a_3a_4a_5b)^5b \\
&= b_2b.
\end{aligned}$$

Thus, we obtain the claim. \square

4. THE SECOND HOMOLOGY GROUP OF $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$

In this section, we prove Theorem 1.2. First, we obtain a generating set for $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ when $g \geq 9$ by using the Hopf formula and applying the discussion of Pitsch [7]. More precisely, we obtain the following proposition.

Proposition 4.1. *For $g \geq 9$, $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is generated by one element x_0 which is represented by the following element:*

$$A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2,$$

where A , B_i ($i = 1, \dots, 5$), B , and C are the following.

$$\begin{aligned}
A &:= b^2, \\
B_i &:= a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}, \\
B &:= ba_4 ba_4^{-1} b^{-1} a_4^{-1}, \\
C &:= (a_2 a_3 a_4 a_5 a_6 b)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} b^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 b)^{-4} \\
&\quad \cdot (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} b^{-1})^{-5}.
\end{aligned}$$

Now we recall the classical Hopf formula. Let G be a group with finite presentation $G = \langle X | R \rangle$, where X is a finite subset of G and R is a finite set consisting of words of the elements of X . Then G is isomorphic to the quotient group F/K , where F is the free group which is generated by X and K is the normal subgroup of F which is normally generated by R . The classical *Hopf formula* states that

$$H_2(G; \mathbb{Z}) \cong \frac{K \cap [F, F]}{[K, F]}.$$

We remark that $(K \cap [F, F])/[K, F]$ is an abelian group and any element of $(K \cap [F, F])/[K, F]$ is represented by a product of commutators of elements of F and by a product of conjugations of elements of R on F . Since $f k f^{-1} \equiv k$ in $K/[K, F]$ for any $f \in F$ and $k \in K$, every element of $H_2(G; \mathbb{Z})$ is represented by $\prod r_i^{n_i}$, where $R = \{r_1, \dots, r_N\}$ and $n_i \in \mathbb{Z}$.

We modify the presentation for $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ for $g \geq 9$ in Theorem 1.1 to easily apply the Hopf formula to $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$.

At first we know that $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ admits a presentation with generators a_0, a_1, \dots, a_{g-1} and relators

- (1) a_i^2 for $i = 0, \dots, g-1$,
- (2) $[a_i, a_j]$ for “ $j - i > 1$ and $i \neq 0$ ” or “ $i = 0$ and $j \neq 4$,”
- (3) $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$ for $i = 1, \dots, g-2$ $a_0 a_4 a_0 a_4^{-1} a_0^{-1} a_4^{-1}$,
- (4) $(a_2 a_3 a_4 a_5 a_6 a_0)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 a_0)^{-4} (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^{-5}$,
- (5) $[a_{g-5}, b_{\frac{g-2}{2}}]$ for $g \geq 8$ even,

where $a_0 = b$ and $b_{\frac{g-2}{2}}$ is inductively defined as $b_1 = a_0, b_2 = (a_1 a_2 a_3 a_4 a_5 b)^5$, and $b_{i+1} = (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3} b_i)^5 (b_{i-1} a_{2i} a_{2i+1} a_{2i+2} a_{2i+3})^{-6}$ for $2 \leq i \leq \frac{g-4}{2}$.

Lemma 4.2. *In the above presentation, relators a_1^2, \dots, a_{g-1}^2 in (1) are unnecessary.*

Proof. By relator (3), we can write a_1, \dots, a_{g-1} as conjugations of a_0 in $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ inductively, as follows.

$$\begin{aligned}
 a_4 &= a_0 a_4 a_0 a_4^{-1} a_0^{-1}, \\
 a_5 &= a_4 a_5 a_4 a_5^{-1} a_4^{-1}, & a_3 &= a_4 a_3 a_4 a_3^{-1} a_4^{-1}, \\
 a_6 &= a_5 a_6 a_5 a_6^{-1} a_5^{-1}, & a_2 &= a_3 a_2 a_3 a_2^{-1} a_3^{-1}, \\
 &\vdots & a_1 &= a_2 a_1 a_2 a_1^{-1} a_2^{-1}. \\
 a_{g-1} &= a_{g-2} a_{g-1} a_{g-2} a_{g-1}^{-1} a_{g-2}^{-1},
 \end{aligned}$$

Thus, a_1^2, \dots, a_{g-1}^2 are conjugations of a_0^2 in $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$. \square

We set

$$\begin{aligned}
A &:= a_0^2, \\
D_{i,j} &:= [a_i, a_j], \\
B_i &:= a_i a_{i+1} a_i^{-1} a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}, \\
B &:= a_0 a_4 a_0 a_4^{-1} a_0^{-1} a_4^{-1}, \\
C &:= (a_2 a_3 a_4 a_5 a_6 a_0)^6 (a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^6 (a_1 a_2 a_3 a_4 a_5 a_6 a_0)^{-4} \\
&\quad (a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1} a_6^{-1} a_0^{-1})^{-5}, \\
D &:= [a_{g-5}, b_{\frac{g-2}{2}}].
\end{aligned}$$

Then any element x of $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is represented by

$$x = A^n \left(\prod_{(*)} D_{i,j}^{n_{i,j}} \right) \left(\prod_{i=1}^{g-2} B_i^{m_i} \right) B^m C^l D^{l'},$$

where $n, n_{i,j}, m_i, m, l, l' \in \mathbb{Z}$, and $(*)$ means the condition “ $j - i > 1$ and $i \neq 0$ ” or “ $i = 0$ and $j \neq 4$.”

Definition 4.3. Let G and F be groups which are given in the Hopf formula. For $g, h \in F$ such that $[g, h] = 1$ in G , we denote by $\{g, h\}$ the equivalence class of the commutator $[g, h] \in [F, F]$ in $H_2(G; \mathbb{Z})$.

Korkmaz and András Stipsicz [4, Lemma 3.3] give the following relations in $H_2(G; \mathbb{Z})$. For $g, h, k \in G$ such that g commutes with h and k ,

$$(I) \quad \{g, hk\} = \{g, h\} + \{g, k\},$$

$$(II) \quad \{g, h^{-1}\} = -\{g, h\}.$$

Note that relation (I) is obtained from relation (II).

Let $\mathcal{T}(N_{g,n})$ be the subgroup of $\mathcal{M}(N_{g,n})$ generated by all Dehn twists and let $\mathcal{M}(\Sigma_{g,n})$ be the mapping class group of a compact connected orientable surface $\Sigma_{g,n}$ of genus g with n boundary components (i.e., $\mathcal{M}(\Sigma_{g,n})$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ which fix each boundary component pointwise).

Lemma 4.4. *Let $g \geq 9$. If α and β are disjoint non-separating two-sided simple closed curves on N_g , then $\{t_\alpha, t_\beta\} = 0$ in $H_2(\mathcal{T}(N_g); \mathbb{Z})$, where t_α and t_β are Dehn twists along simple closed curves α and β , respectively.*

Proof. Let S be the surface obtained by cutting N_g along the simple closed curve α and let g' be the genus of S . Note that if g is even and S is orientable, then $g' = \frac{g-2}{2} \geq \frac{10-2}{2} = 4$, and if g is odd or S is non-orientable, then $g' = g-2 \geq 7$ since $g \geq 9$. We regard t_β as an element of $\mathcal{M}(\Sigma_{g',2})$ when g is even and S is orientable or $\mathcal{T}(N_{g',2})$ when g is odd or S

is non-orientable. John Harer [2] proved that $H_1(\mathcal{M}(\Sigma_{h,n}); \mathbb{Z}) = 1$ for $h \geq 3$ and Stukow [9] proved that $H_1(\mathcal{T}(N_{h,n}); \mathbb{Z}) = 1$ for $h \geq 7$. Thus, there exist either $X_i, Y_i \in \mathcal{M}(S)$ or $X_i, Y_i \in \mathcal{T}(S)$ such that $t_\beta = \prod_i [X_i, Y_i]$. Note that X_i and Y_i commute with t_α . Therefore, in $H_2(\mathcal{T}(N_g); \mathbb{Z})$, we have

$$\begin{aligned} \{t_\alpha, t_\beta\} &= \left\{t_\alpha, \prod_i [X_i, Y_i]\right\} \stackrel{(I)}{=} \sum_i \{t_\alpha, [X_i, Y_i]\} \\ &\stackrel{(I) \& (II)}{=} \sum_i \left[\{t_\alpha, X_i\} + \{t_\alpha, Y_i\} - \{t_\alpha, X_i\} - \{t_\alpha, Y_i\} \right] \\ &= 0. \end{aligned}$$

Thus, we obtain the claim. \square

The homomorphism $\rho_2|_{\mathcal{T}(N_g)} : \mathcal{T}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ induces a homomorphism $H_2(\mathcal{T}(N_g); \mathbb{Z}) \rightarrow H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ on their homology groups. Hence, the equivalence classes of $D_{i,j}$ and D in $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ are trivial by Lemma 4.4, and any element of $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is represented by

$$x = A^n \left(\prod_{i=1}^{g-2} B_i^{m_i} \right) B^m C^l.$$

Proof of Proposition 4.1. By the Hopf formula, any element of $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is a product of commutators of the free group generated by $\{a_0, a_1, \dots, a_{g-1}\}$. Hence, the exponent sum of each a_i in x is zero. The exponent sum of each a_i in x is the following.

$$\begin{aligned} (\text{the exponent sum of } a_0) &= 2n + m + l, \\ (\text{the exponent sum of } a_1) &= m_1 + l, \\ (\text{the exponent sum of } a_2) &= -m_1 + m_2 + l, \\ (\text{the exponent sum of } a_3) &= -m_2 + m_3 + l, \\ (\text{the exponent sum of } a_4) &= -m_3 + m_4 - m + l, \\ (\text{the exponent sum of } a_5) &= -m_4 + m_5 + l, \\ (\text{the exponent sum of } a_6) &= -m_5 + m_6 + l, \\ (\text{the exponent sum of } a_7) &= -m_6 + m_7, \\ &\vdots \\ (\text{the exponent sum of } a_{g-2}) &= -m_{g-3} + m_{g-2}, \\ (\text{the exponent sum of } a_{g-1}) &= -m_{g-2}. \end{aligned}$$

The above equations give $m_{g-2} = m_{g-3} = \cdots = m_7 = m_6 = 0$ and the following system of equations.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By an elementary calculation, this matrix has rank 7 and so the linear map $\mathbb{Z}^8 \rightarrow \mathbb{Z}^7$ has a 1-dimensional kernel. We can check that the kernel is generated by the vector $(-7, -2, -4, -6, 4, 2, 12, 2)$. Therefore, $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is generated by x_0 which is represented by an element

$$A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2.$$

Thus, we finish the proof. \square

When $g \geq 7$ is odd, Theorem 1.2 is proved by Stein [8]. It is sufficient for a proof of Theorem 1.2 to show that $x_0 = 0$ when $g \geq 10$ is even.

Proof of Theorem 1.2. Recall that $\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ is isomorphic to $O(g, \mathbb{Z}_2) = \{A \in GL(g, \mathbb{Z}_2) \mid {}^tAA = E\}$. Under this identification, we define the injective homomorphism

$$\begin{array}{ccc} \iota_g : \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot) & \hookrightarrow & \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \\ \cup & & \cup \\ A & \mapsto & \left(\begin{array}{c|c} A & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right). \end{array}$$

Note that $\iota_g(a_i) = a_i$ for $i = 1, \dots, g-2$ and $\iota_g(b) = b$. Let F and F' be free groups generated by $\{a_1, \dots, a_{g-1}, b\}$ and $\{a_1, \dots, a_{g-2}, b\}$, respectively, and let $\nu : F \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot)$ and $\nu' : F' \rightarrow \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot)$ be natural projections. Then the following diagram is commutative.

$$\begin{array}{ccc} F' & \xrightarrow{\tilde{\iota}_g} & F \\ \nu' \downarrow & \circlearrowleft & \downarrow \nu \\ \text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot) & \xrightarrow{\iota_g} & \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot) \end{array}$$

The homomorphism $\tilde{\iota}_g : F' \rightarrow F$ is defined by $\tilde{\iota}_g(a_i) = a_i$ for $i = 1, \dots, g-2$ and $\tilde{\iota}_g(b) = b$. We denote the kernels of ν and ν' by K and K' , respectively. By the Hopf formula, the restriction $\tilde{\iota}_g : K' \cap [F', F'] \rightarrow K \cap [F, F]$ of $\tilde{\iota}_g$ induces the homomorphism $\tilde{\iota}_{g*} : H_2(\text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot); \mathbb{Z}) \rightarrow H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$. Since $H_2(\text{Aut}(H_1(N_{g-1}; \mathbb{Z}_2), \cdot); \mathbb{Z}) = 0$ for $g \geq 10$ even ([8]), it is enough for the proof of Theorem 1.2 to show that $\tilde{\iota}_{g*}$ is surjective for $g \geq 10$. By Proposition 4.1, $H_2(\text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot); \mathbb{Z})$ is generated by x_0 for $g \geq 9$ such that x_0 is represented by $A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2$. Thus, we can check $\tilde{\iota}_{g*}(x'_0) = x_0$ by the definition of $\tilde{\iota}_g$, where x'_0 is represented by an element $A^{-7}B_1^{-2}B_2^{-4}B_3^{-6}B_4^4B_5^2B^{12}C^2 \in K' \cap [F', F']$. Therefore, x_0 is trivial and we complete the proof. \square

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