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José G. Anaya, David Maya, and Fernando Orozco-Zitli

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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MAKING HOLES IN THE SECOND SYMMETRIC PRODUCT OF UNICOHERENT LOCALLY CONNECTED CONTINUA

JOSÉ G. ANAYA, DAVID MAYA, AND FERNANDO OROZCO-ZITLI

Abstract. A *continuum* is a compact connected metric space. The *second symmetric product* of a continuum X, $\mathcal{F}_2(X)$ is the hyperspace of all nonempty subsets of X having at most two points. Let X be a continuum such that $\mathcal{F}_2(X)$ is unicoherent. Then an element $A \in \mathcal{F}_2(X)$ *makes a hole* in $\mathcal{F}_2(X)$ if $\mathcal{F}_2(X) - \{A\}$ is not unicoherent. In this paper, we characterize the elements $A \in \mathcal{F}_2(X)$ satisfying A makes a hole in $\mathcal{F}_2(X)$ when X is a unicoherent locally connected continuum.

1. Introduction

A *continuum* is a connected compact metric space. Let X be a continuum. For each positive integer n, let $\mathcal{F}_n(X) = \{A \subseteq X : A \text{ has at most } n \text{ points and } A \neq \emptyset\}$. The hyperspace $\mathcal{F}_n(X)$ is called the n^{th} *symmetric product of* X. It is known that each $\mathcal{F}_n(X)$ is a continuum (see [6, pp. 876, 877] and [11, Theorem 4.10]).

A connected topological space Z is *unicoherent* provided that $A \cap B$ is connected whenever A and B are connected closed subsets of Z such that $Z = A \cup B$. A point z in a unicoherent topological space Z *makes a hole in* Z if $Z - \{z\}$ is not unicoherent.

In this paper, we are interested in the following problem which arises in [1, p. 2000].

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Problem. Let $\mathcal{H}(X)$ be a unicoherent hyperspace of a continuum X. Which elements A in $\mathcal{H}(X)$ make a hole in $\mathcal{H}(X)$?

A proof of the unicoherence of each n^{th} symmetric product of a unicoherent locally connected continuum can be found in [8, Theorem 1]. In the current paper, we present the solution to this problem when X is a unicoherent locally connected continuum and $\mathcal{H}(X) = \mathcal{F}_2(X)$.

Readers especially interested in this problem are referred to [1], [2], [3] and [4].

2. Preliminaries and Auxiliary Results

A point z of a connected topological space Z is called *cut point* (*non-cut point*, respectively) if $Z - \{z\}$ is disconnected (connected, respectively). The set of all cut points (non-cut points, respectively) of Z is denoted by $\text{Cut}(Z)$ ($\text{NCut}(Z)$, respectively).

We use the symbols $\mathbb{R}$ and S^1 to denote the set of real numbers and the unit circle in the Euclidean plane $\mathbb{R}^2$, respectively. An *arc* is any space homeomorphic to $[0, 1]$.

For a metric space (X, d_X), a subset F of X, a point $x \in X$, and $\varepsilon > 0$, let $\text{diam}_X(F) = \sup\{d_X(w, y) : w, y \in F\}$ and $B_X(x, \varepsilon) = \{y \in X : d_X(x, y) < \varepsilon\}$.

In this paper the word "map" stands for a continuous function. By exp, we denote the exponential map from $\mathbb{R}$ to S^1 defined by $\exp(t) = (\cos t, \sin t)$ for each $t \in \mathbb{R}$.

Let Z be a topological space. A map $f : Z \to S^1$ has a *lifting* if there exists a map $h : Z \to \mathbb{R}$ satisfying $f = \exp \circ h$. The space Z has *property* (b) if each map from Z to S^1 has a lifting. It is known that in connected locally connected metric spaces, the unicoherence and to have property (b) are equivalent (see [7, Theorem 3]).

Given subsets K and L of a continuum X, the symbol $\langle K, L\rangle$ denotes the set $\{\{x, y\} \in \mathcal{F}_2(X) : (x, y) \in K \times L\}$.

Proposition 2.1. *Let X be a continuum and let $p, q \in X$. If $p, q \in \text{Cut}(X)$ and $p \neq q$, then there exist nondegenerate subcontinua K, L, and M of X such that $X = K \cup L \cup M$, $K \cap L = \{p\}$, $K \cap M = \{q\}$, and $L \cap M = \emptyset$.*

Proof. Since $X - \{p\}$ is disconnected, there exist nondegenerate subcontinua Y and W of X such that $X = Y \cup W$ and $Y \cap W = \{p\}$. Then either $q \in Y$ or $q \in W$. Without loss of generality, we may assume that $q \in Y$. It is easy to see that $q \in \text{Cut}(Y)$. Hence, there exist two nondegenerate subcontinua A and B of Y such that $Y = A \cup B$ and $A \cap B = \{q\}$. So either $p \in A$ or $p \in B$. Without loss of generality, we may assume that

$p \in A$. Then the sets $K = A$, $L = W$, and $M = B$ satisfy the required properties. □

Proposition 2.2. *Let X be a continuum and let K and L be connected subsets (subcontinua) of X. Then $\langle K, L\rangle$ is a connected subset (subcontinuum) of $\mathcal{F}_2(X)$ and it does not have cut points when K and L are nondegenerate sets.*

Proof. The connectedness of $\langle K, L\rangle$ follows from [10, Lemma 1].

In order to prove the second part of this proposition, let $\{p, q\} \in \langle K, L\rangle$. Using the fact that K and L are nondegenerate sets and the arguments in [9, p. 137, Theorem 11], it can be shown that $(K \times L) - \{(p, q), (q, p)\}$ is connected. So since $\langle K, L\rangle - \{\{p, q\}\}$ is a continuous image of $(K \times L) - \{(p, q), (q, p)\}$, $\langle K, L\rangle - \{\{p, q\}\}$ is connected (see [11, 2.4.3 of Proposition 2.4]). □

Lemma 2.3. *Let Y and Z be unicoherent locally connected continua and let $(y, z) \in Y \times Z$. Then $(Y \times \{z\}) \cup (\{y\} \times Z)$ is unicoherent.*

Proof. The proof follows from the fact that the unicoherence and the local connectedness are topological properties and [5, Theorem 40] □

Lemma 2.4. *Let X be a unicoherent locally connected continuum and let $p, q \in X$. If $p \in \mathrm{NCut}(X)$ and $p \neq q$, then $(X \times X) - \{(p, q), (q, p)\}$ has property* (b).

Proof. Put $W = (X \times X) - \{(p, q), (q, p)\}$. Let $f : W \to S$ be a map. We shall prove that f has a lifting.

First, we define $D = \{(x, x) : x \in X\}$ and, for each $x \in X - \{p, q\}$, let $G(x) = (X \times \{x\}) \cup (\{x\} \times X)$. Since X is a unicoherent locally connected continuum, D and each $G(x)$ are unicoherent (see Lemma 2.3). Then, by [7, Theorem 3]), D and each $G(x)$ have property (b). So there exists a map $\psi : D \to \mathbb{R}$ such that $f|_D = \exp \circ \psi$, and for each $x \in X - \{p, q\}$, by [9, p. 407, Theorem 4], there exists a map $\varphi_x : G(x) \to \mathbb{R}$ satisfying $f|_{G(x)} = \exp \circ \varphi_x$ and $\varphi_x(x, x) = \psi(x, x)$.

Now, we are going to verify that $\varphi_x(x, y) = \varphi_y(x, y)$ for every $x, y \in X - \{p, q\}$. Take $x, y \in X - \{p, q\}$. Since $X - \{p\}$ is a connected open subset of X, by [12, p. 132, Theorem 8.26], there exists an arc L from x to y contained in $X - \{p\}$. Now, by [7, Theorem 3 and Corollary 6], $L \times L$ has property (b). Put $N = L \times L$. Then there exists a map $\lambda : N \to \mathbb{R}$ satisfying $f|_N = \exp \circ \lambda$ and $\lambda(x, x) = \psi(x, x)$ (see [9, p. 407, Theorem 4]). From the facts that $N \cap D = \{(x, x) : x \in L\}$ and $N \cap G(x) = (\{x\} \times L) \cup (L \times \{x\})$ are connected, $(x, x) \in N \cap D \cap G(x)$, $\lambda(x, x) = \psi(x, x) = \varphi_x(x, x)$, $\exp \circ \lambda|_{N \cap D} = f|_{N \cap D} = \exp \circ \psi|_{N \cap D}$, and $\exp \circ \lambda|_{N \cap G(x)} = f|_{N \cap G(x)} = \exp \circ \varphi_x|_{N \cap G(x)}$, by [7, p. 64, (3)], we have

$\lambda|_{N\cap D} = \psi|_{N\cap D}$ and $\lambda|_{N\cap G(x)} = \varphi_x|_{N\cap G(x)}$. So, using $(y,y) \in N \cap D$, it follows $\lambda(y,y) = \psi(y,y)$. Hence, since $\exp\circ\lambda|_{N\cap G(y)} = f|_{N\cap G(y)} = \exp\circ\varphi_y|_{N\cap G(y)}$, and $N \cap G(y)$ is connected, by [7, p. 64, (3)], $\lambda|_{N\cap G(y)} = \varphi_y|_{N\cap G(y)}$. Thus, $\varphi_x(x,y) = \lambda(x,y) = \varphi_y(x,y)$ since $(x,y) \in N \cap G(x) \cap G(y)$.

Define $h : W \to \mathbb{R}$ as follows: for each $(x,y) \in W$, let

$$h(x,y) = \begin{cases} \psi(x,x), & \text{if } x = y, \\ \varphi_x(x,y), & \text{if } x \notin \{p,q\}, \\ \varphi_y(x,y), & \text{if } y \notin \{p,q\}. \end{cases}$$

Since $\varphi_x(x,y) = \varphi_y(x,y)$ for every $x, y \in X - \{p,q\}$ and $\varphi_x(x,x) = \psi(x,x)$ for every $x \in X$, h is well defined. It is easy to see that $f = \exp\circ h$. Finally, we shall prove that h is continuous.

To check the continuity of h, we will prove that h is continuous at each point of W. Take $(x_0,y_0) \in W$. Let $\varepsilon > 0$. From the continuity of f and the local connectedness of X, it follows the existence of connected open subsets U and V of X satisfying $(x_0,y_0) \in U \times V \subseteq W$ and $\operatorname{diam}_{S^1}(f(U \times V)) < \frac{\varepsilon}{\pi}$. Now, to show that $h(U \times V) \subseteq B_{\mathbb{R}}(h(x_0,y_0),\varepsilon)$, we are going to prove the existence of an arc $J(w,z)$ in W such that $(x_0,y_0),(w,z) \in J(w,z) \subseteq U \times V$ and $h|_{J(w,z)}$ is a map for each $(w,z) \in U \times V$. Let $(w,z) \in U \times V$. By [12, p. 132, Theorem 8.26], there exist arcs L_1 and L_2 from x_0 to w contained in U and from y_0 to z contained in V. We consider the following cases.

Case I. $x_0 = y_0$.

Without loss of generality, we suppose that $U = V$, $w \neq x_0$, and $w \in X - \{p,q\}$. Let L_3 be an arc from w to z contained in U (see [12, p. 132, Theorem 8.26]). Put $J = J(w,z) = (D \cap (L_1 \times L_1)) \cup (\{w\} \times L_3)$. Clearly, $J \subseteq U \times V$, $D \cap (L_1 \times L_1) \subseteq D$, $\{w\} \times L_3 \subseteq G(w)$, and $(D \cap (L_1 \times L_1)) \cap (\{w\} \times L_3) = \{(w,w)\}$ is closed in W. Thus, since $h|_{D\cap(L_1\times L_1)} = \psi|_{D\cap(L_1\times L_1)}$ and $h|_{\{w\}\times L_3} = \varphi_w|_{\{w\}\times L_3}$ are maps, $h|_J$ is a map.

Case II. $x_0 \neq y_0$.

We have that either $x_0 \in X - \{p,q\}$ or $y_0 \in X - \{p,q\}$. Without loss of generality, suppose that $x_0 \in X - \{p,q\}$. Also, we may assume that $z \neq y_0$. So $z \in X - \{p,q\}$. Put $J = J(w,z) = (\{x_0\} \times L_2) \cup (L_1 \times \{z\})$. It is easy to see that $J \subseteq U \times V$, $\{x_0\} \times L_2 \subseteq G(x_0)$, $L_1 \times \{z\} \subseteq G(z)$, and $(\{x_0\} \times L_2) \cap (L_1 \times \{z\}) = \{(x_0,z)\}$ is closed. So, from the fact that $h|_{\{x_0\}\times L_2} = \varphi_{x_0}|_{\{x_0\}\times L_2}$ and $h|_{L_1\times\{z\}} = \varphi_z|_{L_1\times\{z\}}$ are maps, we have that $h|_J$ is a map.

So there exists an arc $J(w,z)$ from (x_0,y_0) to (w,z) contained in $U \times V$ such that $h|_{J(w,z)}$ is a map and $f|_{J(w,z)} = \exp\circ h|_{J(w,z)}$. Then, since $\operatorname{diam}_{S^1}(f|_{J(w,z)}(J(w,z))) < \frac{\varepsilon}{\pi}$, by [7, p. 64, (4)], we obtain that

$\operatorname{diam}_{\mathbb{R}}(h|_{J(w,z)}(J(w,z))) < \varepsilon$. Thus, from the fact that $(x_0,y_0),(w,z) \in J(w,z)$, it follows that $|h(w,z) - h(x_0,y_0)| < \varepsilon$. This finishes the proof of the continuity of h.

Then each map from W to S^1 has a lifting. So W has property (b). □

Lemma 2.5. *Let Y and Z be unicoherent locally connected subcontinua of a locally connected continuum X and let $p \in X$. If $p \in \operatorname{NCut}(Y) \cap \operatorname{NCut}(Z)$, then $(Y \times Z) - \{(p,p)\}$ has property* (b).

Proof. Put $W = (Y \times Z) - \{(p,p)\}$. Let $f : W \to S$ be a map. We shall show that f has a lifting.

Fix $b \in Y - \{p\}$ and set $P = (Y - \{p\}) \times (Z - \{p\})$. Define $D = \{b\} \times Z$ and, for each $(y,z) \in P$, let $G(y,z) = (\{y\} \times Z) \cup (Y \times \{z\})$. Since Y and Z are unicoherent locally connected continua, D and each $G(y,z)$ are unicoherent (see Lemma 2.3). Hence, by [7, Theorem 3], D and each $G(y,z)$ have property (b). Then there exists a map $\psi : D \to \mathbb{R}$ such that $f|_D = \exp \circ \psi$ and, for each $(y,z) \in P$, by [9, p. 407, Theorem 4], there exists a map $\varphi_{(y,z)} : G(y,z) \to \mathbb{R}$ such that $f|_{G(y,z)} = \exp \circ \varphi_{(y,z)}$ and $\varphi_{(y,z)}(b,z) = \psi(b,z)$.

Let $(y,z),(u,v) \in P$. We are going to verify that $\varphi_{(y,z)}(y,v) = \varphi_{(u,v)}(y,v)$. By the local connectedness of X and [12, p. 132, Theorem 8.26], since $Y - \{p\}$ and $Z - \{p\}$ are connected open subsets of X, there exist arcs I and L from y to u and from z to v contained in $Y - \{p\}$ and $Z - \{p\}$, respectively. Put $N = (I \times Z) \cup (Y \times L)$. Clearly, $N \subseteq W$ and $(I \times Z) \cap (Y \times L) = I \times L$ is connected. So, by [7, Corollary 6] and [5, Theorem 40 and Corollary 48], N is unicoherent. Then, by [7, Theorem 3], N has property (b). Hence, there exists a map $\lambda : N \to \mathbb{R}$ satisfying $f|_N = \exp \circ \lambda$ and $\lambda(b,v) = \psi(b,v)$ (see [9, p. 407, Theorem 4]). Notice $\{b\} \times L \subseteq N$. Then, since $\exp \circ \lambda|_{\{b\}\times L} = f|_{\{b\}\times L} = \exp \circ \psi|_{\{b\}\times L}$ and $\lambda(b,v) = \psi(b,v)$, by [7, p. 64, (3)], $\lambda|_{\{b\}\times L} = \psi|_{\{b\}\times L}$. Thus, $\lambda(b,z) = \psi(b,z)$. Now, from the fact that $G(y,z), G(u,v) \subseteq N$, $\lambda(b,z) = \psi(b,z) = \varphi_{(y,z)}(b,z)$, $\lambda(b,v) = \psi(b,v) = \varphi_{(u,v)}(b,v)$, $\exp \circ \lambda|_{G(y,z)} = f|_{G(y,z)} = \exp \circ \varphi_{(y,z)}$, and $\exp \circ \lambda|_{G(u,v)} = f|_{G(u,v)} = \exp \circ \varphi_{(u,v)}$, it follows that $\lambda|_{G(y,z)} = \varphi_{(y,z)}$ and $\lambda|_{G(u,v)} = \varphi_{(u,v)}$ (see [7, p. 64, (3)]). Thus, $\varphi_{(y,z)}(y,v) = \lambda(y,v) = \varphi_{(u,v)}(y,v)$.

Define $h : W \to \mathbb{R}$ as follows: for each $(y,z) \in W$, if $(y,z) \in G(u,v)$ for some $(u,v) \in P$, let

$$h(y,z) = \varphi_{(u,v)}(y,z).$$

From the fact that $\varphi_{(y,z)}(y,z) = \varphi_{(u,v)}(y,z)$ for each $(u,v) \in P$ such that $(y,z) \in G(u,v)$, we have that h is well defined. Clearly, $f = \exp \circ h$. We shall prove that h is continuous.

To check the continuity of h, we are going to show that h is continuous at each point of W. Take $(y_0, z_0) \in W$. Let $\varepsilon > 0$. By the continuity of f and the local connectedness of Y and Z, there exist connected open subsets U of Y and V of Z, such that $(y_0, z_0) \subseteq U \times V \subseteq W$ and $\operatorname{diam}_{S^1}(f(U \times V)) < \frac{\varepsilon}{\pi}$. Now, to prove that $h(U \times V) \subseteq B_{\mathbb{R}}(h(y_0, z_0), \varepsilon)$, we will verify that there exists an arc $J(w, x)$ in W such that $(y_0, z_0), (w, x) \in J(w, x) \subseteq U \times V$ and $h|_{J(w,x)}$ is a map for each $(w, x) \in U \times V$. Let $(w, x) \in U \times V$. Since $(y_0, z_0) \in W$, either $y_0 \in Y - \{p\}$ or $z_0 \in Z - \{p\}$. Without loss of generality, we may assume that $y_0 \neq p$. Also, we suppose that $w \neq p$. By the local connectedness of Y and Z and [12, p. 132, Theorem 8.26], since U and V are connected open subsets of Y of Z, respectively, there exist arcs L_1 and L_2 such that $L_1 \times L_2 \subseteq U \times V$, $\operatorname{NCut}(L_1) = \{y_0, w\}$, $\{z_0, x\} \subseteq \operatorname{NCut}(L_2)$, and $\operatorname{NCut}(L_2) - \{z_0\} \subseteq Z - \{p\}$. Take $a \in \operatorname{NCut}(L_2) - \{z_0\}$. Put $J = J(w, x) = (\{y_0\} \times L_2) \cup (L_1 \times \{a\}) \cup (\{w\} \times L_2)$. Notice that $(w, x) \in J \subseteq U \times V$, $h|_{(\{y_0\} \times L_2) \cup (L_1 \times \{a\})} = \varphi_{(y_0,a)}|_{(\{y_0\} \times L_2) \cup (L_1 \times \{a\})}$, and $h|_{\{w\} \times L_2} = \varphi_{(w,x)}|_{\{w\} \times L_2}$ are maps. So, since $((\{y_0\} \times L_2) \cup (L_1 \times \{a\})) \cap (\{w\} \times L_2)$ is closed, $h|_J$ is a map. Then, from the fact that $\operatorname{diam}(f|_J(J)) < \frac{\varepsilon}{\pi}$, by [7, p. 64, (4)], it follows that $\operatorname{diam}(h|_J(J)) < \varepsilon$. Thus, $|h(w, x) - h(y_0, z_0)| < \varepsilon$. This proves that h is continuous.

Therefore, each map from W to S^1 has a lifting. So W has property (b). □

3. Main Results

Theorem 3.1. *Let X be a continuum such that $\mathcal{F}_2(X)$ is unicoherent and let $p \in X$. If $X - \{p\}$ has at least three components, then $\{p\}$ makes a hole in $\mathcal{F}_2(X)$.*

Proof. Since $X - \{p\}$ has at least three components, there exist subcontinua Y and Z of X such that $X = Y \cup Z$, $Y \cap Z = \{p\}$, and $p \in \operatorname{Cut}(Y) \cup \operatorname{Cut}(Z)$. Assume that $p \in \operatorname{Cut}(Y)$. Define $\mathcal{A} = \mathcal{F}_2(Y) - \{\{p\}\}$ and $\mathcal{B} = \langle X, Z\rangle - \{\{p\}\}$. Then $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $\mathcal{F}_2(X) - \{\{p\}\}$ and, by Proposition 2.2, $\mathcal{A}$ and $\mathcal{B}$ are connected. Notice that $\mathcal{A} \cap \mathcal{B} = \langle\{p\}, Y\rangle - \{\{p\}\}$ and $\{p\} \in \operatorname{Cut}(\langle\{p\}, Y\rangle)$. So, since $\mathcal{F}_2(X) - \{\{p\}\} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$ is disconnected, $\mathcal{F}_2(X) - \{\{p\}\}$ is not unicoherent. □

Theorem 3.2. *Let X be a continuum such that $\mathcal{F}_2(X)$ is unicoherent and let $p, q \in X$. If $p, q \in \operatorname{Cut}(X)$ and $p \neq q$, then $\{p, q\}$ makes a hole in $\mathcal{F}_2(X)$.*

Proof. By Proposition 2.1, there exist nondegenerate subcontinua K, L, and M of X such that $X = K \cup L \cup M$, $K \cap L = \{p\}$, $K \cap M = \{q\}$, and

$L \cap M = \emptyset$. Define $\mathcal{A} = \mathcal{F}_2(K) - \{\{p,q\}\}$ and $\mathcal{B} = (\langle X, L\rangle \cup \langle X, M\rangle) - \{\{p,q\}\}$. Thus, $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $\mathcal{F}_2(X) - \{\{p,q\}\}$. Now, since $\langle L, M\rangle - \{\{p,q\}\} \subseteq (\langle X, L\rangle - \{\{p,q\}\}) \cap (\langle X, M\rangle - \{\{p,q\}\})$, using Proposition 2.2, it can be proved that $\mathcal{A}$ and $\mathcal{B}$ are connected. Finally, from the facts that $\mathcal{F}_2(X) - \{\{p,q\}\} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = (\langle \{p\}, K\rangle \cup \langle \{q\}, K\rangle) - \{\{p,q\}\}$ is disconnected, we have that $\mathcal{F}_2(X) - \{\{p,q\}\}$ is not unicoherent. □

Theorem 3.3. *Let X be a unicoherent locally connected continuum and let $p, q \in X$. If $p \in \mathrm{NCut}(X)$, then $\{p,q\}$ does not make a hole in $\mathcal{F}_2(X)$.*

Proof. By [8, Theorem 1], $\mathcal{F}_2(X)$ is unicoherent. In order to prove the unicoherence of $\mathcal{F}_2(X) - \{\{p,q\}\}$, by [7, Theorem 2], it suffices to show that $\mathcal{F}_2(X) - \{\{p,q\}\}$ has property (b).

Put $\mathcal{W} = \mathcal{F}_2(X) - \{\{p,q\}\}$ and let $f : \mathcal{W} \to S^1$ be a map. Define $Z = (X \times X) - \{(p,q),(q,p)\}$ and consider the surjective map $\pi : Z \to \mathcal{W}$ defined by $\pi(x,y) = \{x,y\}$. Notice that Z has property (b) (see lemmas 2.4 and 2.5). Since $f \circ \pi$ is a map from Z to S^1, there exists a map $\varphi : Z \to \mathbb{R}$ such that $f \circ \pi = \exp \circ \varphi$. Now, we are going to prove that $\varphi(x,y) = \varphi(y,x)$ for every $(x,y) \in Z$.

Define $\sigma : Z \to Z$ by $\sigma(x,y) = (y,x)$ for each $(x,y) \in Z$. The surjective map σ satisfies $\exp \circ \varphi \circ \sigma(x,y) = f(\{x,y\}) = \exp \circ \varphi(x,y)$ for every $(x,y) \in Z$ and $\varphi(u,u) = \varphi \circ \sigma(u,u)$ for every $u \in X - \{p,q\}$. Then, by Proposition 2.2 and [7, p. 64, (3)], $\varphi = \varphi \circ \sigma$. Thus, $\varphi(x,y) = \varphi \circ \sigma(x,y) = \varphi(y,x)$ for every $(x,y) \in Z$.

Finally, define $h : \mathcal{W} \to \mathbb{R}$ by $h(\{x,y\}) = \varphi(x,y)$ for each $\{x,y\} \in \mathcal{W}$. From the fact that $\varphi(x,y) = \varphi(y,x)$ for every $(x,y) \in Z$, we have that h is well defined. Notice that $f = \exp \circ h$. To prove the continuity of h, let $\{w,z\} \in \mathcal{W}$ and let V be an open subset of S^1 such that $h(\{w,z\}) \in V$. Then $(w,z) \in Z$ and $\varphi(w,z) \in V$. Since φ is continuous, there exist open subsets U_1 and U_2 of X such that $(w,z) \in U_1 \times U_2 \subseteq Z$ and $\varphi(U_1 \times U_2) \subseteq V$. So $\{w,z\} \in \langle U_1, U_2\rangle$ and $h(\langle U_1, U_2\rangle) \subseteq V$. This proves that h is continuous.

Thus, $\mathcal{F}_2(X) - \{\{p,q\}\}$ has property (b). □

Theorem 3.4. *Let X be a unicoherent locally connected continuum and let $p \in X$. If $X - \{p\}$ has exactly two components, then $\{p\}$ does not make a hole in $\mathcal{F}_2(X)$.*

Proof. The unicoherence of $\mathcal{F}_2(X)$ follows from [8, Theorem 1]. Now, using [7, Theorem 4] to prove that $\mathcal{F}_2(X) - \{\{p\}\}$ is unicoherent, it suffices to prove that there exist unicoherent locally connected closed subspaces $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ of $\mathcal{F}_2(X) - \{\{p\}\}$ such that $\mathcal{F}_2(X) - \{\{p\}\} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $\mathcal{A} \cap \mathcal{B}$ and $(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C}$ are connected.

Since $X - \{p\}$ has exactly two components, then there exist subcontinua K and L of X such that $X = K \cup L$, $K \cap L = \{p\}$, and $p \in \mathrm{NCut}(K) \cap \mathrm{NCut}(L)$. By [12, p. 134, 8.37], we have that K and L are locally connected.

Now, define $\mathcal{A} = \mathcal{F}_2(K) - \{\{p\}\}$, $\mathcal{B} = \langle K, L\rangle - \{\{p\}\}$, and $\mathcal{C} = \mathcal{F}_2(L) - \{\{p\}\}$. It is easy to see that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are closed subsets $\mathcal{F}_2(X) - \{\{p\}\}$. By Theorem 3.3, $\mathcal{A}$ and $\mathcal{C}$ are unicoherent. The local connectedness of $\mathcal{A}$ and $\mathcal{C}$ follows from the fact that K and L are locally connected and [8, Theorem 1]. On the other hand, since $K \times L - \{(p, p)\}$ is homeomorphic to $\mathcal{B}$, $\mathcal{B}$ is locally connected and, by Lemma 2.5 and [7, Theorem 2], $\mathcal{B}$ is unicoherent. Finally, since $p \in \mathrm{NCut}(K) \cap \mathrm{NCut}(L)$, $\mathcal{A} \cap \mathcal{B} = \langle \{p\} K\rangle - \{\{p\}\}$ and $(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = \langle \{p\}, L\rangle - \{\{p\}\}$ are connected. Thus, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy the required properties.

Therefore, $\mathcal{F}_2(X) - \{\{p\}\}$ is unicoherent. □

Classification

Theorem 3.5. *Let X be a unicoherent locally connected continuum and let $p, q \in X$. Then $\{p, q\}$ makes a hole in $\mathcal{F}_2(X)$ if and only if either $p = q$ and $X - \{p\}$ has at least three components or $p \neq q$ and p and q are cut points of X.*

Proof. If $\{p, q\}$ makes a hole in $\mathcal{F}_2(X)$, by Theorem 3.3, $p, q \notin \mathrm{NCut}(X)$. So $p, q \in \mathrm{Cut}(X)$. Now, if $p = q$, by Theorem 3.4, then $X - \{p\}$ has at least three components. This proves the first part.

The second part follows from theorems 3.1 and 3.2 since $\mathcal{F}_2(X)$ is unicoherent (see [8, Theorem 1]). □

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(Anaya, Maya, Orozco-Zitli) Universidad Autónoma del Estado de México; Facultad de Ciencias; Instituto Literario 100; Col. Centro, C.P. 50000, Toluca, Estado de México, México.

E-mail address, Anaya: jgao@uaemex.mx

E-mail address, Maya: dmayae@outlook.com

E-mail address, Orozco-Zitli: forozcozitli@gmail.com