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by

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ITERATED FUNCTION SYSTEMS WITH THE AVERAGE SHADOWING PROPERTY

MEHDI FATEHI NIA

Abstract. The average shadowing property is considered for set-valued dynamical systems, generated by parameterized iterated function systems (IFSs), which are uniformly contracting, or conjugacy, or products of such ones. We also prove that if a continuous surjective IFS $\mathcal{F}$ on a compact metric space X has the average shadowing property, then every point x is chain recurrent. Moreover, we introduce some examples and investigate the relationship between the original shadowing property and the shadowing property for an IFS. For example, we prove that the Sierpinski IFS has the average shadowing property. Then we show that there is an IFS $\mathcal{F}$ on S^1 such that $\mathcal{F}$ does not satisfy the average shadowing property, but every point x in S^1 is chain recurrent.

1. Introduction

The shadowing property of a dynamical system is one of the most important notions in dynamical systems (see [12], [15]). The notion of the average shadowing property was introduced by Michael Blank [6] in order to study chaotic dynamical systems, which is a good tool to characterize Anosov diffeomorphisms. The average shadowing property was further studied by several authors, with particular emphasis on connections with other notions from topological dynamics, or more narrowly, shadowing theory (e.g., see [10], [11], [13]).

Iterated function systems (IFSs) are used for the construction of deterministic fractals and have found numerous applications, in particular,

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in image compression and image processing [2]. Important notions in dynamics, like attractors, minimality, transitivity, and shadowing, can be extended to an IFS (see [4], [3], [7], [8]). The authors define the shadowing property for a parameterized IFS and prove that if it is uniformly contracting or expanding, then it has the shadowing property [9]. They also generalize the shadowing property on an affine parameterized IFS and prove the following theorem.

Theorem 1.1. *Consider a closed nonempty subset $A \subset \mathbb{C}$, situated strictly inside or strictly outside the unit circle. For any closed disc centered at 0 with radius $r > 1$ in $\mathbb{C}$ and any subset $B \subset \mathbb{C}$, the parameterized IFS $F = \mathbb{C}; f_{a,b} | a \in A, b \in B$, with $f_{a,b}(z) = az + b$, has the shadowing property on Z_+.*

The present paper concerns the average shadowing property for parameterized IFSs, and some important results about this notion are extended to iterated function systems. Specifically, we prove that each uniformly contracting parameterized IFS has the average shadowing property on $\mathbb{Z}_+$.

We will also give some examples illustrating our results. Example 3.3 shows that Sierpinski's IFS is an important IFS and has the average shadowing property. Minimality plays an important role in dynamical systems; in [1], the authors have shown that a minimal homeomorphism on a compact metric space does not have the shadowing property. In Example 3.10, we show that there is a minimal IFS which has the average shadowing property.

In §4, we investigate the average shadowing property and chain recurrent sets on IFSs for which the stated space is the unite circle. Theorem 4.1 shows that if an IFS on the unit circle has more than one fixed point, then it does not have the average shadowing property. Example 4.4 shows that there is an IFS $\mathcal{F}$ on S^1 such that $\mathcal{F}$ does not satisfy the average shadowing property, but every point x in S^1 is chain recurrent.

2. Preliminaries

In [10], [11], and [13], the average shadowing property is defined and discussed for continuous maps. Let $f : X \longrightarrow X$ be a continuous map. For $\delta > 0$, a sequence $\{x_i\}_{i \geq 0}$ of points in X is called a *δ-average-pseudo-orbit* of f if there is a number $N = N(\delta)$ such that for all $n \geq N$ and $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average shadowing property* if, for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ϵ-shadowed in average by some point $y \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.$$

We use the notion of chain recurrent points to study chaotic dynamical systems.

The set $CR(f)$ consisting of all chain recurrent points, i.e., such points $x \in X$ that for any $\delta > 0$, there exists a periodic δ-pseudo-orbit through x, is called the *chain recurrent set* of the discrete dynamical system (X, f).

Let (X, d) be a complete metric space. Let us recall that a *parameterized iterated function system (IFS)* $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \to X$ where $\lambda \in \Lambda$ and Λ is a finite nonempty set (see [9]).

Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ and Λ^T denote the set of all infinite sequences $\{\lambda_i\}_{i\in T}$ of symbols belonging to Λ. A typical element of $\Lambda^{\mathbb{Z}_+}$ can be denoted as $\sigma = \{\lambda_0, \lambda_1, ...\}$, and we use the shortened notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_{n-1}} o ... o f_{\lambda_1} o f_{\lambda_0}.$$

A sequence $\{x_n\}_{n\in T}$ in X is called an *orbit* of the IFS $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for each $\lambda_n \in \sigma$.

Given $\delta > 0$, a sequence $\{x_n\}_{n\in T}$ in X is called a *δ-pseudo-orbit* of $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that for every $\lambda_n \in \sigma$, we have $d(f_{\lambda_n}(x_n), x_{n+1}) < \delta$.

One says that the IFS $\mathcal{F}$ has the *shadowing property* (on T) if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_n\}_{n\in T}$ there exists an orbit $\{y_n\}_{n\in T}$, satisfying the inequality $d(x_n, y_n) \leq \epsilon$ for all $n \in T$. In this case, one says that the $\{y_n\}_{n\in T}$ ϵ-shadows the δ-pseudo-orbit $\{x_n\}_{n\in T}$.

For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ of points in X is called a *δ-average-pseudo-orbit* of $\mathcal{F}$, if there exists a natural number $N = N(\delta) > 0$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\}$ in $\Lambda^{\mathbb{Z}_+}$, such that for all $n \geq N$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f_{\lambda_i}(x_i), x_{i+1}) < \delta.$$

A sequence $\{x_i\}_{i\geq 0}$ in X is said to be *ϵ-shadowed in average* by a point z in X, if there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(z), x_i) < \epsilon.$$

A continuous IFS $\mathcal{F}$ is said to have the average shadowing property if, for every $\epsilon >$, there is $\delta > 0$ such that every δ-average-pseudo-orbit of f is ϵ-shadowed in average by some point in X.

Please note that if Λ is a set with one member, then the parameterized IFS $\mathcal{F}$ is an ordinary discrete dynamical system. In this case, the average shadowing property for $\mathcal{F}$ is an ordinary average shadowing property for a discrete dynamical system.

The parameterized IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is *uniformly contracting* if there exists

$$\beta = sup_{\lambda\in\Lambda} sup_{x\neq y} \frac{d(f_\lambda(x), f_\lambda(y))}{d(x, y)}$$

and this number, also called the *contracting ratio*, is less than 1 [9].

Definition 2.1. Suppose (X, d) and (Y, d') are compact metric spaces and Λ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two IFSs where $f_\lambda : X \to X$ and $g_\lambda : Y \to Y$ are continuous maps for all $\lambda \in \Lambda$. We say that $\mathcal{F}$ is topologically conjugate to $\mathcal{G}$ if there is a homeomorphism $h : X \to Y$ such that $g_\lambda = hof_\lambda oh^{-1}$ for all $\lambda \in \Lambda$. In this case, h is called a *topological conjugacy.*

3. Average Shadowing Property for Iterated Function Systems

In this section we investigate the structure of a parameterized IFS with the average shadowing property. It is well known that if $f : X \to X$ and $g : Y \to Y$ are conjugated, then f has the shadowing property if and only if so does g. In the next theorem we extend this property for iterated function systems.

Theorem 3.1. *Suppose (X, d) and (Y, d') are compact metric spaces and Λ is a finite set. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two conjugated IFSs with topological conjugacy h. If there exist two positive real numbers, K and L, such that $L < \frac{d'(h(p),h(q))}{d(p,q)} < K$ for all $p, q \in X$, then $\mathcal{F}$ has the average shadowing property if and only if so does $\mathcal{G}$.*

Proof. Given $\epsilon > 0$, let $\delta > 0$ be an ϵ/K-modulus of the average shadowing property $\mathcal{F}$. Suppose $\{y_i\}_{i\geq 0}$ is an $L\delta$-average-pseudo-orbit of $\mathcal{G}$. This means that there exists a natural number $N = N(\delta) > 0$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\}$ in $\Lambda^{\mathbb{Z}_+}$, such that for all $n \geq N$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d'(g_{\lambda_i}(y_i), y_{i+1}) < L\delta.$$

Put $x_i = h^{-1}(y_i)$ for all $i \geq 0$. Then

$$d(f_{\lambda_i}(x_i), x_{i+1}) = d(h^{-1}og_{\lambda_i}(y_i), h^{-1}(y_{i+1})) < \frac{d'(g_{\lambda_i}(y_i), y_{i+1})}{L}$$

for all $i \geq 0$. Thus,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f_{\lambda_i}(x_i), x_{i+1})) < \frac{L\delta}{L} = \delta$$

for all $n \geq N$. Hence, $\{x_i\}_{i\geq 0}$ is a δ-average-pseudo-orbit for $\mathcal{F}$, and so there is an orbit $\{z_i\}_{i\geq 0}$ of $\mathcal{F}$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(z_i, x_i) < \frac{\epsilon}{K}.$$

Note that for each $i \geq 0$, there is $\mu_i \in \Lambda$ such that $z_{i+1} = f_{\mu_i}(z_i)$. Let $w_i = h(z_i)$; it is clear that $w_{i+1} = h(z_{i+1}) = h(f_{\mu_i}(z_i)) = g_{\mu_i}(h(z_i)) = g_{\mu_i}(w_i)$ for all $i \geq 0$. Therefore, $\{w_i\}_{i\geq 0}$ is an orbit of $\mathcal{G}$ and $d'(w_i, y_i) = d'(h(z_i), h(x_i)) < Kd(z_i, x_i)$ for all $i \geq 0$. Then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d'(w_i, y_i) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d'(h(z_i), h(x_i)) < K\frac{\epsilon}{K} = \epsilon. \quad \square$$

Theorem 3.2. *If a parameterized IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is uniformly contracting, then it has the average shadowing property on $\mathbb{Z}_+$.*

Proof. Assume that $\beta < 1$ is the contracting ratio of $\mathcal{F}$. Given $\epsilon > 0$, take $\delta = (1-\beta)\epsilon/2 \leq \epsilon/2$ and suppose $\{x_i\}_{i\geq 0}$ is a δ-pseudo-orbit for $\mathcal{F}$. Therefore, there exists a natural number $N = N(\delta)$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\} \in \Lambda^{\mathbb{Z}_+}$ such that $\frac{1}{n}\sum_{i=0}^{n-1} d(f_{\lambda_i}(x_i), x_{i+1}) < \delta$ for all $n \geq N(\delta)$. Put $\alpha_i = d(f_{\lambda_i}(x_i), x_{i+1})$ for all $i \geq 0$. Consider an orbit $\{y_i\}_{i\geq 0}$ such that $d(x_0, y_0) < \delta \leq \frac{\epsilon}{2}$ and $y_{i+1} = f_{\lambda_i}(y_i)$ for all $i \geq 0$.

Now we show that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(y_i, x_i) < \epsilon$. Take $M = d(x_0, y_0)$. Obviously,

$$d(x_1, y_1) \leq d(x_1, f_{\lambda_0}(x_0)) + d(f_{\lambda_0}(x_0), f_{\lambda_0}(y_0)) \leq \alpha_0 + \beta M.$$

Similarly,

$$\begin{aligned} d(x_2, y_2) &\leq d(x_2, f_{\lambda_1}(x_1)) + d(f_{\lambda_1}(x_1), f_{\lambda_1}(y_1)) \\ &\leq \alpha_1 + \beta d(x_1, y_1) \\ &\leq \alpha_1 + \beta(\alpha_0 + \beta M) \end{aligned}$$

and

$$\begin{aligned} d(x_3, y_3) &\leq d(x_3, f_{\lambda_2}(x_2)) + d(f_{\lambda_2}(x_2), f_{\lambda_2}(y_2)) \\ &\leq \alpha_2 + \beta d(x_2, y_2) \\ &\leq \alpha_2 + \beta(\alpha_1 + \beta d(x_1, y_1)) \\ &\leq \alpha_2 + \beta(\alpha_1 + \beta(\alpha_0 + \beta M) \\ &= \alpha_2 + \beta\alpha_1 + \beta^2\alpha_0 + \beta^3 M. \end{aligned}$$

By induction, one can prove that for each $i > 2$,

$$d(x_i, y_i) \leq \alpha_{i-1} + \beta\alpha_{i-2} + ... + \beta^{i-1}\alpha_0 + \beta^i M.$$

This implies that

$$\begin{aligned} \sum_{i=0}^{n-1} d(y_i, x_i) &= M(1 + \beta + .. + \beta^{n-1}) \\ &+ \alpha_0(1 + \beta + .. + \beta^{n-2}) \\ &+ \alpha_1(1 + \beta + .. + \beta^{n-3}) \\ &\vdots \\ &+ \alpha_{n-2} \\ &\leq \frac{1}{1-\beta}(M + \sum_{i=0}^{n-2} \alpha_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(y_i, x_i) &\leq \frac{1}{1-\beta}(M + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-2} \alpha_i) \\ &< \frac{1}{1-\beta}(M + \delta) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So the proof is complete. □

The next example shows that the Sierpinski IFS has the average shadowing property.

Example 3.3 (Sierpinski Gasket)**.** Consider the solid (filled) equilateral triangle with vertices at $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Now we define the following iterated function system on X, the so-called Sierpinski IFS (see Figure 1) [2, pp. 50, 62, 84, etc.].

$$f_1(\mathtt{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathtt{x}.$$

$$f_2(\mathtt{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathtt{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$f_3(\mathtt{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathtt{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$$

It is clear that $\mathcal{F} = \{X; f_1, f_2, f_3\}$ is uniformly contracting and, by Theorem 3.2, has the average shadowing property.

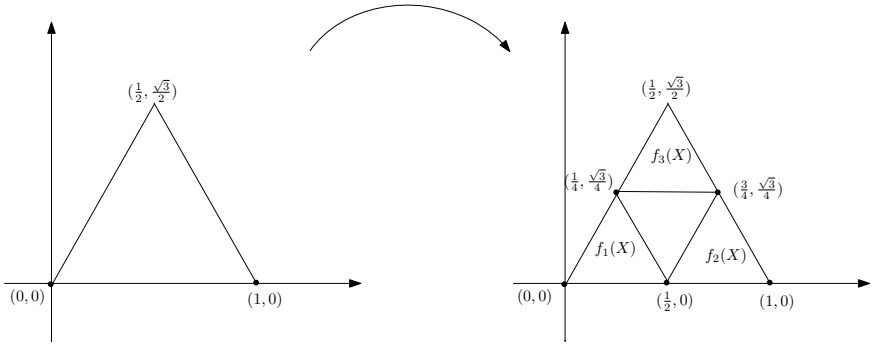

FIGURE 1. Sierpinski Gasket.

Let S denote the Sierpinski Gasket generated by $\mathcal{F}$ (see [2, pp. 50, 62, 84, etc.] for more details). A similar argument shows that the IFS $\mathcal{G} = \{S; f_1, f_2, f_3\}$ has the average shadowing property and every point $x \in S$ is a chain recurrent point for $\mathcal{G}$.

Let us recall some notions related to symbolic dynamics.

Let $\Sigma_2\{(s_0 s_1 s_2 ...) | s_i = 0 \text{ or } 1\}$. We will refer to elements of Σ_2 as points in Σ_2. Let $s = s_0 s_1 s_2 ...$ and $t = t_0 t_1 t_2 ...$ be points in Σ_2. We denote the distance between s and t as $d(s,t)$ and define it by

$$d(s,t) = \begin{cases} 0, & s = t \\ \frac{1}{2^{k-1}}, & k = min\{i; s_i \neq t_i\} \end{cases}.$$

Example 3.4. Let $f_0,\ f_1 : \Sigma_2 \to \Sigma_2$ be two maps defined as $f_0(s_0 s_1 s_2 ...) = 0 s_0 s_1 s_2 ...$ and $f_1(s_0 s_1 s_2 ...) = 1 s_0 s_1 s_2 ...$ for each $s = s_0 s_1 s_2 ... \in \Sigma_2$. It is clear that $\mathcal{F} = \{\Sigma_2; f_0, f_1\}$ is uniformly contracting and, by Theorem 3.2, has the average shadowing property.

Yingxuan Niu [11] shows that, for a dynamical system (X, f), if f has the average-shadowing property, then so does f^k for every $k \in \mathbb{N}$. The following theorem generalizes a similar result for an IFS.

Theorem 3.5. *Let Λ be a finite set, let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an IFS, and let $k \geq 2$ be an integer. Set*

$$\mathcal{F}^k = \{X; g_\mu | \mu \in \Pi\} = \{X; f_{\lambda_{k-1}} o ... o f_{\lambda_0} | \lambda_0, ..., \lambda_{k-1} \in \Lambda\}.$$

If $\mathcal{F}$ has the average shadowing property (on $\mathbb{Z}_+$), then so does $\mathcal{F}^k$.

Proof. Suppose $\mathcal{F}$ has the average shadowing property and ϵ is an arbitrary positive number. There exists $\delta > 0$ such that every δ-average-pseudo-orbit is ϵ/k-shadowed in average by some point in X. Suppose $\{x_i\}_{i\geq 0}$ is a δ-average-pseudo-orbit for $\mathcal{F}^k$. So there exists a natural number $N = N(\delta) > 0$ and $\gamma = \{\mu_0, \mu_1, \mu_2, \mu_3, ...\}$ in $\Pi^{\mathbb{Z}_+}$, such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(g_{\mu_i}(x_i), x_{i+1}) < \delta$$

for all $n \geq N$. Now we put $y_{nk+j} = f_{\lambda_j^n} o ... o f_{\lambda_0^n}(x_n)$ and $y_{nk} = x_n$. Then $g_{\mu_n} = f_{\lambda_{k-1}^n} o ... o f_{\lambda_0^n}$, for $0 < j < k-1$, $n \in \mathbb{Z}_+$, that is

$$\{y_i\}_{i\geq 0} =$$

$$\{x_0, f_{\lambda_0^0}(x_0), ..., f_{\lambda_{k-2}^0} o ... o f_{\lambda_0^0}(x_0), x_1, f_{\lambda_0^1}(x_1), ..., f_{\lambda_{k-2}^1} o ... o f_{\lambda_0^1}(x_1), ...\}.$$

It is clear that $\{y_i\}_{i\geq 0}$ is a δ-average-pseudo-orbit for $\mathcal{F}$. So there exists $x \in X$, and $\sigma \in \Lambda^{\mathbb{Z}_+}$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(x), y_i) < \frac{\epsilon}{k}.$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{kn} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_{ki}}(x), y_{ki}) \leq \limsup_{n\to\infty} \frac{1}{kn} \sum_{j=0}^{kn-1} d(\mathcal{F}_{\sigma_j}(x), y_j) < \frac{\epsilon}{k}.$$

So

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_{ki}}(x), y_{ki}) < \epsilon.$$

Since $y_{ik} = x_i$, the theorem is proved. □

Let (X, d) and (Y, d') be two complete metric spaces. Consider the product set $X \times Y$ endowed with the metric

$$D((x_1, y_1), (x_2, y_2)) = max\{d(x_1, x_2), d'(y_1, y_2)\}.$$

Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\gamma | \gamma \in \Gamma\}$ be two parameterized IFSs. The IFS $\mathcal{H} = \mathcal{F} \times \mathcal{G} = \{X \times Y; f_\lambda \times g_\gamma : \lambda \in \Lambda, \gamma \in \Gamma\}$, defined by

$(f_\lambda \times g_\gamma)(x,y) = (f_\lambda(x), g_\gamma(y))$, is called the *product* of the IFSs $\mathcal{F}$ and $\mathcal{G}$.

Theorem 3.6. *The product of two parameterized IFSs has the average shadowing property if and only if each projection has.*

Proof. Given $\epsilon > 0$, let M be the diameter of $X \times Y$ and $\alpha = (\epsilon/2(M+1))^2$. Since $\mathcal{F}$ and $\mathcal{G}$ have the average shadowing property, there exists $\delta > 0$ such that every δ-average-pseudo-orbit for $\mathcal{F}(\mathcal{G})$ is α-shadowed by some point $u \in X (v \in Y)$.

Suppose $\{(x_i, y_i)\}_{i\geq 0}$ is a δ-average-pseudo-orbit for $\mathcal{F} \times \mathcal{G}$. By definition of the δ-average-pseudo-orbit, $\{x_i\}_{i\geq 0}$ and $\{y_i\}_{i\geq 0}$ are δ-average-pseudo-orbits for $\mathcal{F}$ and $\mathcal{G}$, respectively. Then there exist the sequences $\sigma \in \Lambda^{\mathbb{Z}_+}$ and $\gamma \in \Gamma^{\mathbb{Z}_+}$ and points $u \in X$ and $v \in Y$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(u), x_i) < \alpha$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d^{'}(\mathcal{G}_{\gamma_i}(v), y_i) < \alpha.$$

Now, [11, Lemma 3.4] implies that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} D(\mathcal{F}_{\sigma_i} \times \mathcal{G}_{\gamma_i}(u,v), (x_i, y_i))$$

$$= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} max(d(\mathcal{F}_{\sigma_i}(u), x_i), d^{'}(\mathcal{G}_{\gamma_i}(v), y_i)) < \epsilon.$$

So $\mathcal{F} \times \mathcal{G}$ has the average shadowing property.

Conversely, suppose that $\mathcal{F} \times \mathcal{G}$ has the average shadowing property and ϵ is an arbitrary positive number. There exists $\delta > 0$ such that every δ-average-pseudo-orbit of $\mathcal{F} \times \mathcal{G}$ can be ϵ-shadowed by some points in X. Let $\{x_i\}_{i\geq 0}$ be a δ-average-pseudo-orbit for $\mathcal{F}$. Take an orbit $\{y_i\}_{i\geq 0}$ in $\mathcal{G}$. So $\{(x_i, y_i)\}_{i\geq 0}$ is a δ-average-pseudo-orbit for $\mathcal{F} \times \mathcal{G}$. Then there exists the sequences $\sigma \in \Lambda^{\mathbb{Z}_+}$ and $\gamma \in \Gamma^{\mathbb{Z}_+}$ and point $(u,v) \in X \times Y$ such that $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} D(\mathcal{F}_{\sigma_i} \times \mathcal{G}_{\gamma_i}(u,v), (x_i, y_i)) < \epsilon$. Therefore, $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(u), x_i) < \epsilon$. □

The following theorem gives another characterization of the average shadowing property on an IFS.

Theorem 3.7. *If a continuous surjective* IFS $\mathcal{F}$ *on a compact metric space X has the average shadowing property, then every point x is a chain*

recurrent point for $\mathcal{F}$. Moreover, $\mathcal{F}$ has only one chain component which is the whole space.

Proof. Suppose ϵ_0 is an arbitrary positive number and

$$D = max_{(x,y)\in X\times X} d(x,y).$$

Take $0 < \epsilon < \epsilon_0/2$ such that if $d(x,y) < 2\epsilon$, then $d(f_\lambda(x), f_\lambda(y)) < \epsilon_0$ for all $\lambda \in \Lambda$. Let $\delta > 0$ be an ϵ modulus of the average shadowing property $\mathcal{F}$ and let N_0 be a sufficiently large natural number for which $\frac{3D}{N_0} < \delta$. Suppose x and y are two different points of X. We assume that y is not in the positive orbit of x; otherwise, it is clear.

Fix $\lambda' \in \Lambda$. Put $T_1 = \{t \in X : f_{\lambda'}(t) = y\}$. Take a point $y_1 \in T_1$. Again, we consider a subset T_i of X and take a point $y_i \in T_i$ satisfying

$$f_{\lambda'}(T_i) = y_{i-1}, 1 < i \leq N_0 - 2.$$

Set a sequence $\{x_i\}_{i\geq 0}$ of points by

$$x_i = \begin{cases} f_{\lambda'}^{[i \ mod \ 2N_0]}(x) & \text{if } [i \ mod \ 2N_0] \in [0, N_0] \\ y_{2N_0-([i \ mod \ 2N_0]+1)} & \text{if } [i \ mod \ 2N_0] \in [N_0+1, 2N_0-2] \\ y & \text{if } [i \ mod \ 2N_0] = 2N_0 - 1. \end{cases}$$

Then, for $n \geq N_0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f_{\lambda'}(x_i), x_{i+1}) < \frac{1}{n}.\frac{n}{N_0}3D \leq \frac{3D}{N_0} < \delta.$$

Therefore, $\{x_i\}_{i\geq 0}$ is a cyclic δ-average-pseudo-orbit of $f_{\lambda'}$ and, consequently, is a cyclic δ-average-pseudo-orbit of $\mathcal{F}$. Hence, there exist $z \in X$ and $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(z), x_i) < \epsilon.$$

Put $S_1 = \{x, f_{\lambda'}(x), ..., f_{\lambda'}^{N_0}(x)\}$ and $S_2 = \{y_{N_0-2}, ..., y_1, y\}$. Since $\{x_i\}_{i\geq 0}$ is a periodic sequence of period $2N_0$ constructed from $\{x, f_{\lambda'}(x), ..., f_{\lambda'}^{N_0}(x), y_{N_0-2}, ..., y_1, y\}$, there exist infinite increasing sequences $\{i_1, i_2, ...\}$ and $\{l_1, l_2, ...\}$ such that $x_{i_j} \in S_1$ and $d(\mathcal{F}_{\sigma_{i_j}}(z), x_{i_j}) < 2\epsilon$ for all $i_j \in \{i_1, i_2, ...\}$. Similarly, $x_{l_j} \in S_2$ and $d(\mathcal{F}_{\sigma_{l_j}}(z), x_{l_j}) < 2\epsilon$ for all $l_j \in \{l_1, l_2, ...\}$. Now we can find $i_0 \in \{i_1, i_2, ...\}$ and $l_0 \in \{l_1, l_2, ...\}$ with $i_0 < l_0$ such that $x_{i_0} \in S_1$ and $x_{l_0} \in S_2$. This implies that

$$d(\mathcal{F}_{\sigma_{i_0}}(z), x_{i_0}) < 2\epsilon \text{ and } d(\mathcal{F}_{\sigma_{l_0}}(z), x_{l_0}) < 2\epsilon.$$

Let $x_{i_0} = f_{\lambda'}^{j_1}(x)$ and $x_{l_0} = y_{j_2}$ for some $j_1 > 0$ and $j_2 > 0$. So it is clear that

$$x, f_{\lambda'}(x), ..., f_{\lambda'}^{j_1}(x) = x_{i_0}$$
$$\mathcal{F}_{\sigma_{i_0+1}}(z), \mathcal{F}_{\sigma_{i_0+1}}(z), ..., \mathcal{F}_{\sigma_{l_0-1}}(z)$$
$$x_{l_0} = y_{j_2}, y_{j_2-1}, ..., y$$

is an ϵ_0-pseudo-orbit from x to y. □

Remark 3.8. Example 4.4 shows that the converse of Theorem 3.7 is not true.

Definition 3.9. The iterated function system $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is *minimal* if each closed subset $A \subset X$ such that $f_\lambda(A) \subset A$ for all $\lambda \in \Lambda$, is empty or coincides with X.

Equivalently, for a minimal iterated function system $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$, for any $x \in X$, the collection of iterates, $f_{\lambda_k} o ... o f_{\lambda_1} o f_{\lambda_0}(x)$, $k > 0$, and $\lambda_1, ..., \lambda_k \in \Lambda$, is dense in X [16].

In the next example we introduce a minimal IFS that has the average shadowing property.

Example 3.10. Take the following maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = (\frac{1}{2} + 2\alpha)x - \alpha; \ f_2(x) = (\frac{1}{2} + 2\alpha)x + \frac{1}{2} - \alpha$$

where $0 < \alpha < \frac{1}{4}$. It is clear that $\mathcal{G} = \{\mathbb{R}; f_1, f_2\}$ is a uniformly contracting IFS. By [2, Theorem 7.1] and [17, Lemma 4.1], there exists a compact subset A of $\mathbb{R}$ with nonempty interior such that $f_1(A) \subset A$, $f_2(A) \subset A$, and $\mathcal{F} = \{A; f_1, f_2\}$ is minimal. But $\mathcal{F}$ is a uniformly contracting IFS and, by Theorem 3.2, has the average shadowing property.

4. Average Shadowing Property and Iterated Function Systems on the Unit Circle

The average shadowing property is an important concept in dynamical systems, which is closely related to stability and chaos of systems (see [11], [14]). So it is interesting to find some IFSs which do not have this property.

Consider a circle S^1 with coordinate $x \in [0; 1)$ and we denote by d the metric on S^1 induced by the usual distance on the real line. Let $\pi(x) : \mathbb{R} \to S^1$ be the covering projection defined by the relations

$$\pi(x) \in [0; 1) \text{ and } \pi(x) = x(x \ mod \ 1)$$

with respect to the considered coordinates on S^1.

Theorem 4.1. *Suppose F_1 and F_2 are two homeomorphisms on $[0, 1]$ and f_1 and f_2 are their induced homeomorphisms on S^1. Assume that 0, a, and 1 are fixed points of F_1 and F_2 for some $0 < a < 1$. Also $F_1(t) > t$ and $F_2(t) > t$ for all $t \in [0, 1] - \{0, a, 1\}$. Then the* IFS,

$\mathcal{F} = \{[0,1], f_\lambda | \lambda \in \{0,1\}\}$ *where* $\Lambda = \{1,2\}$*, does not satisfy the average shadowing property.*

Proof. Assume that $\pi(0) = \pi(1) = b$ and $\pi(a) = c$. So b and c are two fixed points of f_1 and f_2. Take $\epsilon > 0$ such that $min\{d(b,c), d(c,b)\} > 3\epsilon$ and $D = max_{(x,y)\in S^1\times S^1} d(x,y)$. Given $\delta > 0$, choose a natural number K such that $\frac{3D}{K} < \delta$.

We define $\{x_i\}_{i\geq 0}$ by setting

$$x_i = \begin{cases} b & \text{if } 0 \leq i \leq K \\ c & \text{if } K+1 \leq i \leq 3K \\ b & \text{if } 3.2^j.K+1 \leq i \leq 3.2^{j+1}.K,\ j = 0,2,4,... \\ c & \text{if } 3.2^j.K+1 \leq i \leq 3.2^{j+1}.K,\ j = 1,3,5,... \end{cases}$$

Obviously, for $n > K$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f_1(x_i), x_{i+1}) < \frac{1}{n}.\frac{n}{K}.3D < \delta.$$

Then $\{x_i\}_{i\geq 0}$ is a δ-average-pseudo orbit of f_1 and consequently a δ-average-pseudo orbit of $\mathcal{F}$. We assume z to be a point in S^1 such that $\{x_i\}_{i\geq 0}$ is ϵ-shadowed in average, respect to $\mathcal{F}$, by z. That is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(z), x_i) < \epsilon$$

for some $\sigma \in \Lambda^{\mathbb{Z}_+}$.

Claim. There is a natural number M and a point $e \in \{b,c\}$ such that $d(\mathcal{F}_{\sigma_i}(z), e) \geq \epsilon$ for all $i > M$. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\mathcal{F}_{\sigma_i}(z), x_i) \geq \epsilon,$$

which is a contradiction. So for any $\delta > 0$ we can find a δ-average-pseudo orbit of $\mathcal{F}$ that can not be ϵ-shadowed in average, with respect to $\mathcal{F}$, by some point in S^1.

Proof of Claim. The case $z = b$ or $z = c$ is clear. Let $w = \pi^{-1}(z)$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\}$. Suppose $0 < w < a$. Since $F_1(t) > t$ and $F_2(t) > t$ for all $t \in [0,1] - \{0, a, 1\}$, then $lim_{n\to\infty} F_{\lambda_n} o F_{\lambda_{n-1}} o ... o F_{\lambda_0}(w) = a$. Thus, $lim_{n\to\infty} \mathcal{F}_{\sigma_i}(z) = c$.

Similarly, if $a < w < 1$, then $lim_{n\to\infty} \mathcal{F}_{\sigma_i}(z) = b$. We recall that a point $x \in X$ is a *chain recurrent* for $\mathcal{F}$ if, for every $\epsilon > 0$, there exist a finite sequence of points $\{p_i \in X : i = 0, 1, ..., n\}$ with $p_0 = p_n = x$ and a corresponding sequence of indices $\{\lambda_i \in \Lambda : i = 1, 2, ..., n\}$ satisfying $d(f_{\lambda_i}(p_i), p_{i+1}) \leq \epsilon$ for $i = 1, 2, ..., n-1$. Such a sequence of points is

called an ϵ-chain from x to x; similarly, we can define an ϵ-chain from x to y [5, Theorem 3.1.]. □

Remark 4.2. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\lambda \in \Lambda$. Every chain recurrent of f_λ is a chain recurrent of $\mathcal{F}$, but by Example 4.3, the converse is not true.

Example 4.3. Let $\pi : [0,1] \to S^1$ be a map defined by $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$. Let $F_1 : [0,1] \to [0,1]$ be a homeomorphism defined by

$$F_1(t) = \begin{cases} t + (\frac{1}{2} - t)t & \text{if } 0 \le t \le \frac{1}{2} \\ t - (t - \frac{1}{2})(1-t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

and let $F_2 : [0,1] \to [0,1]$ be a homeomorphism defined by

$$F_2(t) = \begin{cases} t + (\frac{1}{2} - t)t & \text{if } 0 \le t \le \frac{1}{2} \\ t + (1-t)(t - \frac{1}{2}) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

Assume that f_i is a homeomorphism on S^1 defined by $f_i(\cos(2\pi t), \sin(2\pi t)) = (\cos(2\pi F_i(t)), \sin(2\pi F_i(t)))$, for $i \in \{0,1\}$.

Consider $\mathcal{F} = \{S^1, f_\lambda | \lambda \in \{0,1\}\}$; it is clear that $x = \pi(\frac{1}{2})$ is not a chain recurrent point for f_1. We show that x is a chain recurrent point for $\mathcal{F}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|s - t| < \delta$ implies $d(\pi(s), \pi(t)) < \epsilon$. By definition of F_1, it is clear that if $0 < t < a$, then $\{F_1^n(t)\}_{n \ge 0}$ is an increasing sequence that converges to $\frac{1}{2}$. Similarly, if $\frac{1}{2} < t < 1$, then $\{F_2^n(t)\}_{n \ge 0}$ is an increasing sequence that converges to 1. There is a δ-chain, with respect to F_2, $\frac{1}{2} = y_0, ...y_N = 1$ from $\frac{1}{2}$ to 1 and a δ-chain, with respect to F_1, $0 = y_{N+1}, ..., y_{N+K} = \frac{1}{2}$ from 0 to $\frac{1}{2}$. Hence, $x_0, ..., x_{N+K}$ is an ϵ-chain from x to x. Up to this point, x is chain recurrent for $\mathcal{F}$.

The following example shows that there is an IFS $\mathcal{F}$ on S^1 satisfying the following:

1. $\mathcal{F}$ does not satisfy the average shadowing property;
2. every point x in S^1 is chain recurrent.

Example 4.4. Let $\pi : [0,1] \to S^1$ be a map defined by $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$. Fix $0 < a < 1$. Let $F_1 : [0,1] \to [0,1]$ be a homeomorphism defined by

$$F_1(t) = \begin{cases} t + (a - t)t & \text{if } 0 \le t \le a \\ t + (1-t)(t - a) & \text{if } a \le t \le 1 \end{cases}$$

and let $F_2 : [0,1] \to [0,1]$ be a homeomorphism defined by

$$F_2(t) = \begin{cases} t + (a - t)t^2 & \text{if } 0 \le t \le a \\ t + (1-t^2)(t - a) & \text{if } a \le t \le 1 \end{cases}.$$

Assume that f_i is a homeomorphism on S^1 defined by $f_i(\cos(2\pi t), \sin(2\pi t)) = (\cos(2\pi F_i(t)), \sin(2\pi F_i(t)))$, for $i \in \{0, 1\}$.

Consider $\mathcal{F} = \{S^1, f_\lambda | \lambda \in \{0, 1\}\}$; by Theorem 4.1, $\mathcal{F}$ does not have the average shadowing property.

Now we show that every point x in S^1 is a chain recurrent point for $\mathcal{F}$. Suppose x is a non-fixed point in S^1 and $y = \pi^{-1}(x)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|s-t| < \delta$ implies $d(\pi(s), \pi(t)) < \epsilon$. By definition of F_1, it is clear that if $0 < t < a$, then $\{F_1^n(t)\}_{n\geq 0}$ is an increasing sequence that converges to a. Similarly, if $a < t < 1$, then $\{F_1^n(t)\}_{n\geq 0}$ is an increasing sequence that converges to 1. Suppose $a < y < 1$; there is a δ-chain, $y = y_0, ...y_N = 1$ from y to 1 and a δ-chain, $0 = y_{N+1}, ..., y_{N+K} = y$ from 0 to y which contains a. Hence, $x_0, ..., x_{N+K}$ is an ϵ-chain from x to x. Then x is a chain recurrent point for f_1 and, consequently, is a chain recurrent point for $\mathcal{F}$. The case of $0 < y < a$ is similar.

References

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems: Recent Advances*. North-Holland Mathematical Library, 52. Amsterdam: North-Holland Publishing Co., 1994.

[2] Michael Barnsley, *Fractals Everywhere*. Boston, MA: Academic Press, Inc., 1988.

[3] Michael F. Barnsley and Andrew Vince, *Fractal continuation of analytic (fractal) functions*. Available at arXiv:1209.6100v1 [math.DS].

[4] _____, *The Conley attractors of an iterated function system*, Bull. Aust. Math. Soc. **88** (2013), no. 2, 267–279.

[5] Michael F. Barnsley, David C. Wilson, and Krzysztof Leśniak, *Some recent progress concerning topology of fractals* in Recent Progress in General Topology. III. Ed. K. P. Hart, J. van Mill, and P. Simon. Paris: Atlantis Press, 2014. 69–92.

[6] M. L. Blank, *Metric properties of ϵ-trajectories of dynamical systems with stochastic behaviour*, Ergodic Theory Dynam. Systems **8** (1988), no. 3, 365–378.

[7] Mehdi Fatehi Nia, *Iterated function systems with the shadowing property*, J. Adv. Res. Pure Math. **7** (2015), no. 1, 83–91.

[8] Vasile Glăvan and Valeriu Guţu, *Attractors and fixed points of weakly contracting relations*, Fixed Point Theory **5** (2004), no. 2, 265–284.

[9] _____, *Shadowing in parameterized IFS*, Fixed Point Theory **7** (2006), no. 2, 263–274.

[10] Rongbao Gu, *The average-shadowing property and topological ergodicity*, J. Comput. Appl. Math. **206** (2007), no. 2, 796–800.

[11] Yingxuan Niu, *The average-shadowing property and strong ergodicity*, J. Math. Anal. Appl. **376** (2011), no. 2, 528–534.

[12] Ken Palmer, *Shadowing in Dynamical Systems: Theory and Applications*. Mathematics and its Applications, 501. Dordrecht: Kluwer Academic Publishers, 2000.

[13] Jong-Jin Park and Yong Zhang, *Average shadowing properties on compact metric spaces*, Commun. Korean Math. Soc. **21** (2006), no. 2, 355–361.

[14] Kazuhiro Sakai, *Diffeomorphisms with the average-shadowing property on two-dimensional closed manifolds*, Rocky Mountain J. Math. **30** (2000), no. 3, 1129–1137.

[15] ———, *Various shadowing properties for positively expansive maps*, Topology Appl. **131** (2003), no. 1, 15–31.

[16] Artem Raibekas, *C^2-Iterated function systems on the circle and symbolic blender-like*. PhD Thesis. IMPA, Rio de Janeiro, Brazil, 2011. Available at http://www.impa.br/opencms/pt/ensino/downloads/teses_de_doutorado/teses_2011/Artem_Raibekas.pdf

[17] Daniël Faruk Younis, *Robustly minimal iterated function systems on compact manifolds*. Master's Thesis. University of Amsterdam, 2011. Available at https://esc.fnwi.uva.nl/thesis/centraal/files/f752978221.pdf

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