http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 48, 2016

Pages 277–287

http://topology.nipissingu.ca/tp/

CAT(0) Extensions of Right-Angled Coxeter Groups

by

Charles Cunningham, Andy Eisenberg, Adam Piggott, and Kim Ruane

Electronically published on December 3, 2015

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 48 (2016)
Pages 277-287

E-Published on December 3, 2015

CAT(0) EXTENSIONS OF RIGHT-ANGLED COXETER GROUPS

CHARLES CUNNINGHAM, ANDY EISENBERG, ADAM PIGGOTT, AND KIM RUANE

Abstract. We show that any split extension of a right-angled Coxeter group W_Γ by a generating automorphism of finite order acts faithfully and geometrically on a CAT(0) metric space.

1. Introduction

An isometric group action is *faithful* if its kernel is trivial, and it is *geometric* if it is cocompact and properly discontinuous. A finitely generated group G is a *CAT(0) group* if there exists a CAT(0) metric space X equipped with a faithful geometric G-action. The CAT(0) property is not an invariant of the quasi-isometry class of a group (see, for example, [1], [6], and [3, p. 258]). Whether or not it is an invariant of the abstract commensurability class of a group is as yet unknown. Attention was brought to this matter in [8]. In this article we illustrate that answering this question for any family of CAT(0) groups may require a variety of techniques.

It is well known that an arbitrary right-angled Coxeter group W is a CAT(0) group because it acts faithfully and geometrically on a CAT(0) cube complex X. It is also well known that the automorphism group $\mathrm{Aut}(W)$ is generated by three types of finite-order automorphisms. As a natural source of examples we consider split extensions of right-angled Coxeter groups by finite cyclic groups, where, in each case, the cyclic

2010 *Mathematics Subject Classification.* Primary 20F65, 20F55.

Key words and phrases. automorphisms, CAT(0) extensions, CAT(0) groups, Coxeter groups.

This work was partially supported by a grant from the Simons Foundation (#317466 to Adam Piggott).

group acts on W as the group generated by one of these various generating automorphisms. Our theorem is the following.

Theorem 1.1. *Suppose W is a right-angled Coxeter group and $\phi \in \mathrm{Aut}(W)$ is an automorphism induced by a graph automorphism, a partial conjugation, or a transvection. Let m denote the order of ϕ. Then the group $G = W \rtimes_\phi \mathbb{Z}/m\mathbb{Z}$ is a CAT(0) group.*

What is most interesting is that G is a CAT(0) group for different reasons in each of the three cases. When ϕ is an automorphism induced by a graph automorphism, the left multiplication action $W \circlearrowright X$ extends to an action $G \circlearrowright X$; when ϕ is a partial conjugation, G is itself a right-angled Coxeter group; when ϕ is a transvection, G is not a right-angled Coxeter group and the action $W \circlearrowright X$ cannot extend to all of G, but we can explicitly construct a new CAT(0) space Y and describe a faithful geometric action $G \circlearrowright Y$.

After necessary background material is described in §2, the three cases of the theorem are treated, in turn, in sections 3, 4, and 5.

We also note that, in each case of the theorem, we take an extension $W_\Gamma \rtimes H$ where $H \leqslant \mathrm{Aut}(W_\Gamma)$ is finite. In [4], we give an example in which H is infinite and $W_\Gamma \rtimes H$ is not a right-angled Coxeter group. We currently do not know whether such extensions with infinite H are CAT(0) or not. Since this question does not address the abstract commensurability of the CAT(0) property, we will not address it further in this paper.

2. Right-Angled Coxeter Groups and Their Automorphisms

In this section we briefly recall a very small part of the rich combinatorial and geometric theory of right-angled Coxeter groups. The interested reader may consult [5] for a thorough account of the more general subject of Coxeter groups from the geometric group theory point of view.

Fix an arbitrary finite simple graph Γ with vertex set S and edge set E. The *right-angled Coxeter group defined by* Γ is the group $W = W_\Gamma$ generated by S, with relations declaring that the generators all have order 2, and adjacent vertices commute with each other. The pair (W, S) is called a right-angled Coxeter system. As described in [5, Proposition 7.3.4], we construct a cube complex $X = X(W, S)$ inductively as follows:

- The set of vertices is indexed by W, say $X^0 = \{v_w \mid w \in W\}$.
- To complete the construction of the one-skeleton X^1, we add edges of unit length so that vertices v_u and v_w are adjacent if and only if $u^{-1}w \in S$.

- For each $k \geqslant 2$, we construct the k-skeleton by gluing in Euclidean unit cubes of dimension k whenever X^{k-1} contains the $(k-1)$-skeleton of such a cube.

Remark 2.1. We note the following about this construction:

- The dimension of X equals the number of vertices in the largest clique in Γ.
- The barycentric subdivision of X is the well-known Davis complex $\Sigma = \Sigma(W, S)$. By a result of Gromov, Σ, hence also X, is a CAT(0) metric space (see [5, Theorem 12.3.3] for a generalization due to Moussong).

By construction, the geometry of X is determined entirely by its 1-skeleton X^1. It follows that a permutation σ of the vertex set X^0 determines an isometry of X if it respects the adjacency relation. In particular, for all $w \in W$, the map $v_u \mapsto v_{wu}$ extends to an isometry $\Phi_w \in \mathrm{Isom}(X)$. The map $w \mapsto \Phi_w$ is a faithful geometric action $W \circlearrowright X$ known as the *left multiplication action*.

From the graph Γ, we may infer the existence of certain finite-order automorphisms of W. For each vertex $a \in S$, we write $\mathrm{Lk}(a)$ for the set of vertices adjacent to a and $\mathrm{St}(a)$ for $\mathrm{Lk}(a) \cup \{a\}$.

- Each graph automorphism $f \in \mathrm{Aut}(\Gamma)$ restricts to a permutation of S which determines an automorphism $\phi_f \in \mathrm{Aut}(W)$.
- For each union of non-empty connected components D of $\Gamma \setminus \mathrm{St}(a)$, the map

$$s \mapsto \begin{cases} asa & s \in D, \\ s & s \in S \setminus D \end{cases}$$

  determines an automorphism of W called the *partial conjugation with acting letter a and domain D*.
- If $a, d \in S$ are such that $\mathrm{St}(d) \subseteq \mathrm{St}(a)$, then the rule

$$s \mapsto s \text{ for all } s \in S \setminus \{d\} \text{ and } d \mapsto da$$

  determines an automorphism of W called the *transvection with acting letter a and domain d*.

Together, the automorphisms induced by graph automorphisms, the partial conjugations, and the transvections comprise a generating set for $\mathrm{Aut}(W)$ [7]. We note that partial conjugations and transvections are involutions and graph automorphisms have finite order.

In what follows, $\phi \in \mathrm{Aut}(W)$ shall always denote a non-trivial automorphism of finite order m, and G shall denote the semi-direct product

$G = W \rtimes_\phi \mathbb{Z}/m\mathbb{Z}$. So G is presented by

$$P_1 = \langle S \cup \{z\} \mid s^2 = 1 \text{ for all } s \in S, [s,t] = 1 \text{ for all } \{s,t\} \in E,$$
$$z^m = 1, zsz^{-1} = \phi(s) \text{ for all } s \in S\rangle.$$

3. When ϕ Is Induced by a Graph Automorphism

Suppose ϕ is induced by a graph automorphism $f \in \mathrm{Aut}(\Gamma)$. Then the map $v_w \mapsto v_{\phi(w)}$ preserves the adjacency relation in X^1 and hence determines an isometry $\Phi \in \mathrm{Isom}(X)$. By simple computation, the reader may confirm that the relations in the presentation P_1 are satisfied when each $s \in S$ is replaced by Φ_s and z is replaced by Φ. Hence, the rule

$$s \mapsto \Phi_s \text{ for all } s \in S \text{ and } z \mapsto \Phi$$

determines an action $G \circlearrowright X$. We leave the reader to confirm that the action is faithful and geometric, and hence Theorem 1.1 holds in the first of the three cases.

In fact, a stronger result holds for similar reasons.

Lemma 3.1. *If $\mathcal{H} \leqslant \mathrm{Aut}(\Gamma)$ is the group of graph automorphisms and H is the corresponding subgroup of $\mathrm{Aut}(W)$, then the natural action $W \circlearrowright X$ extends to a faithful geometric action $W \rtimes H \circlearrowright X$.*

4. When ϕ Is a Partial Conjugation

Now suppose that ϕ is the partial conjugation with acting letter a and domain D. Recall that v_w denotes the vertex of X indexed by the group element $w \in W$. For any $d \in D$, v_1 and v_d are adjacent in X^1, but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in X^1, the left multiplication action $W \circlearrowright X$ does not naturally extend to an action $G \circlearrowright X$. However, G is itself a right-angled Coxeter group and hence also a CAT(0) group.

Lemma 4.1. *If ϕ is a partial conjugation with acting letter a and domain D, then G is itself a right-angled Coxeter group.*

We will omit the details of the proof, which may be found in [4]. In that paper, we engage more broadly with the problem of identifying a right-angled Coxeter presentation in a given group (or proving that no such presentation exists). We find various families of extensions of right-angled Coxeter groups which are again right-angled Coxeter, and these include Lemma 4.1 as a special case.

Here we will give a description of how to construct the defining graph Λ for G based on the original graph Γ. The procedure is as follows:

(1) Add a new vertex labeled x, which we connect to everything in $\Gamma \backslash D$.
(2) Replace the label of vertex a with the label ax and add edges connecting ax to each vertex in D.

An example is shown in Figure 1.

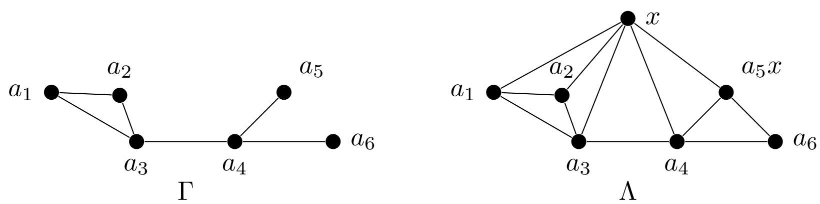

FIGURE 1. Λ is the defining graph of $W_\Gamma \rtimes \langle x \rangle$, where x has acting letter a_5 and domain $\{a_6\}$.

5. When ϕ Is a Transvection

Finally, we suppose that ϕ is the transvection with acting letter a and domain d. Recall that this means that $\mathrm{St}(d) \subseteq \mathrm{St}(a)$ and ϕ is determined by the rule

$$d \mapsto da \text{ and } s \mapsto s \text{ for all } s \in S \backslash \{d\}.$$

We note that v_1 and v_d are adjacent in X^1, but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in X^1, the left multiplication action $W \circlearrowright X$ does not naturally extend to an action $G \circlearrowright X$. In fact, a stronger statement is true. It follows from [5, §13.2] that $\mathrm{Fix}(d)$ is a codimension 1 subspace of Σ and $\mathrm{Fix}(da)$ is codimension 2. Hence, there is no isometry of X which can conjugate the isometry representing d to give the isometry representing da, so the left multiplication action $W \circlearrowright X$ cannot be extended in any way to an action $G \circlearrowright X$.

We also note that G does not embed in a right-angled Coxeter group since G contains an element of order 4. Since $xdx = ad$, we have that $(xd)^2 = a$ and xd has order 4. In a right-angled Coxeter group, any non-trivial element of finite order is an involution.

It seems that to show that G is a CAT(0) group, we must identify a new CAT(0) space Y and describe a faithful geometric action $G \circlearrowright Y$. The key to our success in doing exactly this is the existence of a certain finite index subgroup of W which is itself a right-angled Coxeter group. Although the existence of such a subgroup is well known (see [2, Example

1.4], for example, where the analogous subgroup is used in the context of right-angled Artin groups), we provide the details here for completeness.

Let $h_a : W \to \mathbb{Z}/2\mathbb{Z}$ denote the homomorphism determined by the rule

$$a \mapsto 1 \text{ and } s \mapsto 0 \text{ for all } s \in S\backslash\{a\}.$$

Let U denote the kernel of h_a and let

$$S' = (S\backslash\{a\}) \cup \{asa \mid s \in S\backslash \operatorname{St}(a)\}.$$

Lemma 5.1. *The pair (U, S') is a right-angled Coxeter system, and hence U is a right-angled Coxeter group. Further, conjugation by a in W restricts to an automorphism $\theta \in \operatorname{Aut}(U)$ induced by a permutation of S'; this automorphism is trivial if and only if a is central in W.*

Proof. If a is central in W, then $S' = S\backslash\{a\}$, and the result is evident. In this case, conjugation by a restricts to the trivial automorphism of U and hence is the automorphism of U induced by the trivial permutation of S'.

Suppose a is not central in W. An alternative presentation for W may be constructed from the standard Coxeter presentation for W by the following Tietze transformations:

- For each vertex $s \in S\backslash \operatorname{St}(a)$, introduce a new generator $\widehat{s}$, the defining relation $asa = \widehat{s}$, and redundant relations $a\widehat{s}a = s$ and $\widehat{s}^2 = 1$.
- For each pair of adjacent vertices $s, t \in S\backslash \operatorname{St}(a)$, introduce the redundant relation $\widehat{s}\widehat{t} = \widehat{t}\widehat{s}$.
- For each pair of adjacent vertices $s \in S\backslash \operatorname{St}(a)$ and $t \in \operatorname{Lk}(a)$, introduce the redundant relation $\widehat{s}t = t\widehat{s}$.
- For each vertex $x \in \operatorname{Lk}(a)$, we rewrite the relation $xa = ax$ as $axa = x$.

The resulting presentation of W is

$$\begin{aligned} P_2 = \langle S' \cup \{a\} \mid\ & x^2 = 1 \text{ for all } x \in S', \\ & [s,t] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s,t \neq a, \\ & [\widehat{s},\widehat{t}] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s,t \in S\backslash \operatorname{St}(a), \\ & [\widehat{s},t] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s \in S\backslash \operatorname{St}(a) \text{ and} \\ & \qquad t \in \operatorname{Lk}(a), \\ & a^2 = 1 \text{ and } asa = s \text{ for all } s \in \operatorname{Lk}(a), \\ & asa = \widehat{s} \text{ and } a\widehat{s}a = s \text{ for all } s \in S\backslash \operatorname{St}(a)\rangle. \end{aligned}$$

Evidently, this is the presentation of a semidirect product in which the non-normal factor is $\langle a\rangle$, the normal factor is a right-angled Coxeter group with generating set

$$S' = (S\backslash\{a\}) \cup \{\widehat{x} \mid x \in S\backslash \operatorname{St}(a)\},$$

and a acts on the normal factor as the automorphism θ induced by permuting the generators according to the rule

$$x \mapsto \widehat{x},\ \widehat{x} \mapsto x \text{ for all } x \in S \setminus \operatorname{St}(a), \text{ and } y \mapsto y \text{ for all } y \in \operatorname{Lk}(a).$$

The action of a on U is non-trivial because $S \neq \operatorname{St}(a)$. □

We now have the following refined decomposition of G:

$$G = (U \rtimes_\theta \langle a \rangle) \rtimes_\phi \langle z \rangle.$$

A presentation P_3 for G is obtained from the presentation P_2 for W by appending the generator z and relations

$$z^2 = 1,\ zsz = s \text{ for all } s \in S' \setminus \{d\},\ zdz = da, \text{ and } zaz = a.$$

It follows that for each $g \in G$, there exist unique choices $u_g \in U$ and $\epsilon_g, \delta_g \in \{0, 1\}$, such that $g = u_g a^{\epsilon_g} z^{\delta_g}$. We shall write Y for the CAT(0) cube complex on which U acts geometrically and faithfully as defined in §2, and we write $p : G \to U$ for the projection map $g \mapsto u_g$. The projection map is not a homomorphism because, for $s \in S \setminus \operatorname{St}(a)$, we have $p(a)p(s)p(a) = s \neq s' = p(s')$. Even so, it allows us to parlay the left multiplication action of G on itself into an action of $G \circlearrowright Y$.

Lemma 5.2. *For all $g \in S' \cup \{a, z\}$, the rule*

$$v_u \mapsto v_{p(gu)} \text{ for all } u \in U$$

respects adjacency in Y^1 and hence determines an isometry $\Phi_g \in \operatorname{Isom}(Y)$.

Proof. Let $u \in U$, $s \in S'$, and $g \in S' \cup \{a, z\}$. To prove the result we must establish that $v_{p(gu)}$ and $v_{p(gus)}$ are adjacent. For this it suffices to show that $(p(gu))^{-1} p(gus) \in S'$.

If $g \in S'$, then

$$(p(gu))^{-1} p(gus) = (gu)^{-1} gus = s \in S'.$$

If $g = a$, then

$$\begin{aligned}(p(au))^{-1} p(aus) &= (p(\theta(u)a))^{-1} p(\theta(us)a) \\ &= (\theta(u))^{-1} \theta(us) \\ &= \theta(s) \in S'.\end{aligned}$$

Finally, we consider the case $g = z$. We note that if d occurs an even number of times in any word for u, then a occurs an even number of times in any word for $\phi(u)$, and $p(zu) = \phi(u)$. If, on the other hand, d occurs an odd number of times in any word for u, then a occurs an odd number of times in any word for $\phi(u)$, and $p(zu) = \phi(u)a$. The parity of d in a group element $u \in U$ is identified by the homomorphism $h_d : U \to \mathbb{Z}/2\mathbb{Z}$ determined by the rule

$$d \mapsto 1 \text{ and } s \mapsto 0 \text{ for all } s \in S' \backslash \{d\}.$$

Therefore, we consider cases based on the value of $h_d(u)$ and whether or not $s = d$.

If $h_d(u) = 0$ and $s \neq d$, then

$$(p(zu))^{-1} p(zus) = (\phi(u))^{-1} \phi(us) = s \in S'.$$

If $h_d(u) = 0$ and $s = d$, then

$$(p(zu))^{-1} p(zud) = (\phi(u))^{-1} \phi(ud) a = \phi(d) a = d \in S'.$$

If $h_d(u) = 1$ and $s \neq d$, then

$$(p(zu))^{-1} p(zus) = (\phi(u)a)^{-1} \phi(us) a = a \phi(u)^{-1} \phi(u) sa = asa = \theta(s) \in S'.$$

If $h_d(u) = 1$ and $s = d$, then

$$(p(zu))^{-1} p(zud) = (\phi(u)a)^{-1} \phi(ud) = a \phi(u)^{-1} \phi(u) da = ada = d \in S'.$$

Adjacency is respected in all cases, so the result holds in the case that $g = z$ and thus, Φ_g is an isometry of Y as required. $\square$

In summary, we have that G is presented by

$$\begin{aligned} P_3 = \langle S' \cup \{a, z\} \mid\ & x^2 = 1 \text{ for all } x \in S', \\ & [s,t] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s, t \neq a, \\ & [\hat{s}, \hat{t}] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s, t \in S \backslash \operatorname{St}(a), \\ & [\hat{s}, t] = 1 \text{ for all } \{s,t\} \in E \text{ such that } s \in S \backslash \operatorname{St}(a) \text{ and} \\ & \qquad t \in \operatorname{Lk}(a), \\ & a^2 = 1 \text{ and } asa = s \text{ for all } s \in \operatorname{Lk}(a), \\ & asa = \hat{s} \text{ and } a\hat{s}a = s \text{ for all } s \in S \backslash \operatorname{St}(a), \\ & z^2 = 1 \text{ and } zsz = s \text{ for all } s \in S' \backslash \{d\}, zdz = da, zaz = a \rangle; \end{aligned}$$

and

$$\begin{aligned} \Phi_s(v_u) &= v_{su} \text{ for all } s \in S', \\ \Phi_a(v_u) &= v_{\theta(u)}, \\ \Phi_z(v_u) &= v_{\phi(u)} \text{ if } h_d(u) = 0, \\ \Phi_z(v_u) &= v_{\phi(u)a} \text{ if } h_d(u) = 1. \end{aligned}$$

Lemma 5.3. *The map*

$$g \mapsto \Phi_g \text{ for all } g \in S' \cup \{a, z\}$$

determines a geometric action $G \circlearrowright Y$ which extends the left multiplication action $U \circlearrowright Y$. If a is not central in W, the action is faithful. If a is central in W, the kernel is the subgroup generated by $\{a, z\}$.

Proof. To prove that the map determines an isometric group action, we must prove that the relations in the presentation P_3 for G hold when each $g \in S' \cup \{a, z\}$ is replaced by Φ_g. It is clear that those relations not involving either a or z remain true when each $g \in S'$ is replaced by Φ_g. We leave the reader to verify that the following relations hold (using the rules listed immediately before the statement of the lemma).

$$\begin{aligned}
&\Phi_a^2 = 1,\\
&\Phi_a\Phi_s\Phi_a = \Phi_s \text{ for all } s \in \mathrm{Lk}(a),\\
&\Phi_a\Phi_s\Phi_a = \Phi_{\hat{s}} \text{ for all } s \in S \setminus \mathrm{St}(a),\\
&\Phi_a\Phi_{\hat{s}}\Phi_a = \Phi_s \text{ for all } s \in S \setminus \mathrm{St}(a),\\
&\Phi_z^2 = 1,\\
&\Phi_z\Phi_s\Phi_z = \Phi_s \text{ for all } s \in S' \setminus \{d\},\\
&\Phi_z\Phi_d\Phi_z = \Phi_d\Phi_a,\\
&\Phi_z\Phi_a\Phi_z = \Phi_a.
\end{aligned}$$

We note that, because $v_1 \mapsto v_{p(g)}$, the stabilizer of v_1 is a subgroup of the finite abelian group $\langle a, z\rangle$. If a is not central in W, there exists $s \in S \setminus \mathrm{St}(a)$. Computation shows that Φ_a and Φ_{az} do not fix v_s, and Φ_z does not fix v_{ds}. Our claims about the kernel of the action follow immediately. □

If a is central in W, then there is no obvious way in which a should act non-trivially on Y. We can, however, extend Y to a new space Y^+ by appending two unit length edges in a V shape at each vertex, thereby providing pieces on which a and ϕ can act non-trivially. More formally, to construct Y^+ from Y, we write v_u^0 for v_u, and we append new vertices

$$\left\{v_u^i \mid \text{for all } u \in U \text{ and } i \in \{-1, 1\}\right\}$$

and new unit length edges

$$\left\{\left\{v_u^0, v_u^{-1}\right\}, \left\{v_u^0, v_u^1\right\} \mid \text{for all } u \in U\right\}.$$

It is evident that appending such V shapes at each vertex does not cause the CAT(0) property to fail; hence, Y^+ is a CAT(0) cube complex.

Proposition 5.4. *If a is central in W, then G acts faithfully and geometrically on Y^+.*

Proof. Suppose that a is central in W, i.e., that $\mathrm{St}(a) = \Gamma$. Then (U, S') is a right-angled Coxeter system, and $W = U \times \langle a \rangle$.

We now define a homomorphism $\Phi \colon G \to \mathrm{Isom}(Y^+)$. For each $s \in S'$, we declare $\Phi(s)$ to be the isometry determined by the rule

$$v_u^i \mapsto v_{su}^i \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.$$

We declare $\Phi(a)$ to be the isometry determined by the rule

$$v_u^i \mapsto v_u^{-i} \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.$$

We declare $\Phi(z)$ to be the isometry determined by the rule

$$v_u^i \mapsto \begin{cases} v_u^i & \text{if } h_d(u) = 0, \\ v_u^{-i} & \text{if } h_d(u) = 1, \end{cases}$$

for all $u \in U$ and $i \in \{-1, 0, 1\}$. The maps can be described informally: Each $s \in S'$ acts on Y^+ in the way which most naturally extends the left multiplication action $U \circlearrowright Y$; a flips the V attached to every vertex, while z flips only half the V shapes, because it flips the V attached to a vertex v_u if and only if d has an odd parity in u.

It is evident that the maps described above preserve adjacency in the one-skeleton of Y^+ and hence determine isometries of Y^+. Simple computations confirm that these definitions respect the relations in the presentation P_3 of G (some of the relations listed are vacuous). Therefore, these definitions do indeed determine an isometric action $G \circlearrowright Y^+$. That the action is geometric follows easily from the fact that the left multiplication action $U \circlearrowright Y$ is geometric. □

Acknowledgment. We thank the anonymous referee for his/her constructive suggestions which helped improve the clarity of our exposition.

References

[1] Juan M. Alonso and Martin R. Bridson *Semihyperbolic groups*, Proc. London Math. Soc. (3) **70** (1995), no. 1, 56–114.

[2] Mladen Bestvina, Bruce Kleiner, and Michah Sageev, *The asymptotic geometry of right-angled Artin groups. I*, Geom. Topol. **12** (2008), no. 3, 1653–1699.

[3] Martin R. Bridson and André Haefliger, *Metric Spaces of Non-Positive Curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Berlin: Springer-Verlag, 1999.

[4] Charles Cunningham, Andy Eisenberg, Adam Piggott, and Kim Ruane, *Recognizing right-angled Coxeter groups using involutions*. Preprint. Available at arXiv:1410.4589v1 [math.GR].

[5] Michael W. Davis, *The Geometry and Topology of Coxeter Groups*. London Mathematical Society Monographs Series, 32. Princeton, NJ: Princeton University Press, 2008.

[6] M. Kapovich and B. Leeb, *3-manifold groups and nonpositive curvature*, Geom. Funct. Anal. **8** (1998), no. 5, 841–852.

[7] Michael R. Laurence, *Automorphisms of graph products of groups*. Ph.D. Thesis, University of London, 1992.

[8] Adam Piggott, Kim Ruane, and Genevieve S. Walsh, *The automorphism group of the free group of rank 2 is a CAT(0) group*, Michigan Math. J. **59** (2010), no. 2, 297–302.

(Cunningham) Department of Mathematics; Bowdoin College; Brunswick, Maine 04011
E-mail address: charles.cunningham@tufts.edu

(Eisenberg) Department of Mathematics; Oklahoma State University; Stillwater, OK 74078
E-mail address: andy.eisenberg@okstate.com

(Piggott) Department of Mathematics; Bucknell University; Lewisburg, Pennsylvania 17837
E-mail address: adam.piggott@bucknell.edu

(Ruane) Department of Mathematics; Tufts University; Medford, Massachusetts 02155
E-mail address: kim.ruane@tufts.edu