http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 48, 2016

Pages 299–308

http://topology.nipissingu.ca/tp/

Uniformly Path Connected Homogeneous Continua

by

Janusz R. Prajs

Electronically published on January 18, 2016

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 48 (2016)
Pages 299-308

E-Published on January 18, 2016

UNIFORMLY PATH CONNECTED HOMOGENEOUS CONTINUA

JANUSZ R. PRAJS

Abstract. It is proven that every path connected homogeneous continuum is uniformly path connected, which answers a question of David P. Bellamy. From this result it follows that every path connected homogeneous continuum is continuously equivalent either to an arc, or to the Cantor fan.

Homogeneous path connected continua have been attracting attention for several decades [2] [3] [10] [13] [14] [16]. The main focus has been on finding additional structural properties of these continua, which would define some "small" classes of spaces containing all such continua. In [10] Krystyna Kuperberg asked whether every homogeneous path connected continuum is locally connected. This question was answered in the negative by the author [13]. Interestingly, if a continuum X is path connected and admits a topological group structure, then X is locally connected [8, Theorem 9.68]. Also, if X is path connected and isometrically homogeneous, then it is locally connected [15]. In [14] the author has shown that path connected homogeneous continua are weakly chainable, that is, they are continuous images of the pseudo-arc.

In [2], David P. Bellamy asked whether every homogeneous path connected continuum is uniformly path connected, and presented a substantial partial result related to this question. In this paper we answer Bellamy's question in the affirmative.

2010 *Mathematics Subject Classification.* Primary: 54F15, Secondary: 54E52.

Key words and phrases. continuum, homogeneous, quasi-interior, uniformly path connected.

©2016 Topology Proceedings.

The *uniform path connectedness* of metric spaces is a geometric property introduced by Włodzimierz Kuperberg in [11]. Among compact metric spaces it becomes a topological property, which is invariant with respect to continuous maps. Indeed, if X is compact, the uniform path connectedness of X is equivalent to the existence of a compact collection of paths in X (in the *sup* metric) connecting each two points of X. Showing that every homogeneous path connected continuum is uniformly path connected is the main result of this paper.

Two spaces X and Y are *continuously equivalent* if there exist continuous surjections $f : X \to Y$ and $g : Y \to X$. If an equivalence class of this relation is composed of compact spaces, then either all of its members are uniformly path connected or none are. In general, there are uncountably many equivalence classes of path connected continua because path connected continua have no common model [12, p. 51]. Nevertheless, from the main result of this paper it follows that path connected homogeneous continua can only be found in two of these classes: the one containing an arc, and the one containing the Cantor fan. This is a major reduction, which brings a new focus to the systematic study of homogeneous path connected continua.

1. Preliminaries

In this paper, all spaces are metric and mappings are continuous. A *continuum* is a non-empty, compact, connected metric space. The *Cantor fan* is the cone over the Cantor set. A collection of maps $f_\alpha : X \to Y$, $\alpha \in I$, between metric spaces (X, d_X) and (Y, d_Y) is *uniformly equicontinuous* provided that for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\alpha \in I$ and $x_1, x_2 \in X$ if $d_X(x_1, x_2) < \delta$, then $d_Y(f_\alpha(x_1), f_\alpha(x_2)) < \varepsilon$. A space X is *uniformly path connected* provided there exists a uniformly equicontinuous family of paths in X joining each point of X with each other point. It is known that a continuum is uniformly path connected if and only if it is a continuous image of the Cantor fan [11].

A space X is called *homogeneous* if, for all $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$. If X is a homogeneous compact space, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$, there is some homeomorphism $h : X \to X$ satisfying $h(x) = y$ and $d(z, h(z)) < \varepsilon$ for each $z \in X$. This is called the Effros theorem. It follows from the more general statement that for each $x \in X$, the evaluation map, $h \mapsto h(x)$, from the self-homeomorphism group of X onto X is open. The latter follows from [7, Theorem 2].

In our main argument we use the concept of quasi-interior defined below. This concept was also used in [2] in a similar context. A subset A of a topologically complete metric space X is said to have the *Baire property*, if there is an open set U such that $(A-U)\cup(U-A)$ is first category. A set in a topological space is said to be *analytic* if it is a continuous image of a Borel subset of a separable, complete metric space. It is known that analytic sets have the Baire property. For a set A in a space X, we define the *quasi-interior* of A as $A^* = \bigcup\{U \mid U$ is open in X and $U - A$ is of the first category$\}$. If $A \subset B$, then $A^* \subset B^*$ (see [6, (3.3)]).

Let $f : X \to Y$ be a map between metric spaces and let $x \in X$. We say that f is *quasi-interior at* x, if, for every open set U in X containing x, the point $f(x)$ is in the quasi-interior $(f(U))^*$ of $f(U)$. The first part of the following theorem has been shown in [14]. Here we prove the second part.

Theorem 1.1. *Let $f : X \to Y$ be a continuous surjection between topologically complete separable metric spaces X and Y. Then the set $X_Q = \{x \in X \mid f$ is quasi-interior at $x\}$ is a G_δ subset of X, and the quasi-interior $(f(X_Q))^*$ of the set $f(X_Q)$ equals Y.*

Proof. In [14, Theorem 1.9] it has been shown that X_Q is non-empty. Since X is separable, for every natural number k there is a collection $\{B_1^k, B_2^k, \cdots\}$ of open balls in X of radius $1/k$ that covers X. For all n and k, the quasi-interior $(f(B_n^k))^*$ is an open set in Y, and thus the set $V_n^k = B_n^k \cap f^{-1}((f(B_n^k))^*)$ is open in X. Hence, $\bigcup_n V_n^k$ is open for each k, and $X_Q = \bigcap_k(\bigcup_n V_n^k)$ is a G_δ subset of X.

Observe that for each analytic set $A \subset Y$, if A^* is dense in Y, then $Y - A^*$ is first category because Y is topologically complete. Consequently, $A^* = Y$. As a continuous image of a G_δ subset X_Q of a topologically complete space X, the set $f(X_Q)$ is analytic. Thus, to prove $(f(X_Q))^* = Y$, it suffices to show that $(f(X_Q))^*$ is dense in Y.

Suppose there is a non-empty open set $U \subset Y$ such that $U \cap (f(X_Q))^* = \emptyset$. It follows $U \cap f(X_Q)$ is first category, and thus $U \cap f(X_Q) \subset F = \bigcup_{n=1}^\infty F_n$ for some closed nowhere dense subsets F_n of U. By the Baire category theorem, $W = U - F$ is dense in U. The set W is topologically complete as a G_δ subset of a topologically complete set U. Clearly, W is also a G_δ subset of Y, and thus $Z = f^{-1}(W)$ is a G_δ subset of X. Consequently, Z is topologically complete. The map $f|Z : Z \to W$ is a continuous surjection between topologically complete spaces Z and W. Thus, there exists a $z \in Z$ such that $f|Z$ is quasi-interior at z. Let V be an open neighborhood of z in X. Then $V \cap Z$ is an open neighborhood of z in Z, and thus $f(z)$ is in the quasi-interior of $(f|Z)(V \cap Z) = f(V \cap Z)$ relative to W. Since $F = U - W$ is first category, $f(z)$ is also in the quasi-interior of $f(V \cap Z)$ relative to U. Further, since U is open in Y, $f(z)$ is in the quasi-interior of $f(V \cap Z) \subset f(V)$ in Y. Hence, $f(z)$ is in the quasi-interior of $f(V)$ in Y; that is, $z \in X_Q$. On the other hand,

$z \in f^{-1}(W) = f^{-1}(U - F) \subset f^{-1}(U - f(X_Q))$, and thus $z \notin X_Q$. This contradiction completes the proof. $\square$

Theorem 1.2. *Let X, Y, and Z be topologically complete separable metric spaces and let $f : X \to Z$ and $g : Y \to Z$ be continuous maps. Suppose U is open in X and V is open in Y with $(f(U))^* \cap (g(V))^* \neq \emptyset$. Then for some $x_0 \in U$ and $y_0 \in V$ with $f(x_0) = g(y_0)$, the map f is quasi-interior at x_0 and the map g is quasi-interior at y_0.*

Proof. The set $W = (f(U))^* \cap (g(V))^*$ is non-empty and open in Z, and by assumption there are two sequences, F_n and G_n of subsets of W, which are nowhere dense and closed relative to W, such that $W - \bigcup_n F_n \subset f(U)$ and $W - \bigcup_n G_n \subset g(V)$. Let $W_0 = W - \bigcup_n (F_n \cup G_n)$. Note that $W_0 \subset f(U) \cap g(V)$ and $W - W_0$ is first category. Since W_0 is a G_δ subset of W, and thus also of Z, it follows W_0 is topologically complete. Let $U_0 = U \cap f^{-1}(W_0)$ and $V_0 = V \cap g^{-1}(W_0)$. Then U_0 and V_0 are G_δ subsets of X and Y, respectively. Thus, U_0 and V_0 are topologically complete. Let $f_0 : U_0 \to W_0$ and $g_0 : V_0 \to W_0$ be the corresponding restrictions of f and g. Note that f_0 and g_0 are surjective. Consequently, f_0 and g_0 satisfy the conditions of Theorem 1.1. Let $U_{0,Q} = \{x \in U_0 \mid f_0 \text{ in quasi-interior at } x\}$ and $V_{0,Q} = \{y \in V_0 \mid g_0 \text{ in quasi-interior at } y\}$. By Theorem 1.1, the quasi-interiors of $f_0(U_{0,Q})$ and $g_0(V_{0,Q})$ relative to W_0 both equal W_0. Therefore, $f_0(U_{0,Q}) \cap g_0(V_{0,Q}) \neq \emptyset$. Let $z_0 \in f_0(U_{0,Q}) \cap g_0(V_{0,Q})$, $x_0 \in U_{0,Q} \subset U$, and $y_0 \in V_{0,Q} \subset V$ be such that $f_0(x_0) = z_0 = g_0(y_0)$.

Let N be an open neighborhood of x_0 in X. Then $N \cap U_0$ is an open neighborhood of x_0 in U_0, and thus $f_0(x_0) = f(x_0)$ is in the quasi-interior of $f_0(N \cap U_0) = f(N \cap U_0)$ relative to W_0. Since $W - W_0$ is first category, $f(x_0)$ is also in the quasi-interior of $f(N \cap U_0)$ relative to W. The point $f(x_0)$ is also in the quasi-interior of $f(N \cap U_0)$ relative to Y because W is open in Y. Hence, $f(x_0)$ is in the quasi-interior of $f(N)$ in Y. Therefore, f is quasi-interior at x_0. Similarly, we argue that g is quasi-interior at y_0. The proof is complete. $\square$

In our argument we use the following classic theorem [4, p. 252].

Theorem 1.3. *Let $f : X \to Y$ be an open surjective map such that X is a topologically complete metric space and Y is compact. Then there is a compact set Z in X such that $f|Z : Z \to Y$ is surjective.*

2. The Main Result

In this section, X is a metric space. The following definitions apply to any such X under consideration. Define the space $\mathcal{F}$ of all paths $p : [0,1] \to X$ in X with the *sup* metric. For $p \in \mathcal{F}$ the points $p(0)$ and $p(1)$ are called the *beginning* and *destination* of p, respectively. If $p \in \mathcal{F}$, the *inverse path* of $p(t)$ is defined by $p^{-1}(t) = p(1-t)$. The notation for inverse paths should not be confused with the notation for inverse functions and pre-images of sets. For paths $p, q \in \mathcal{F}$ with $p(1) = q(0)$, we define the *path product* $p * q : [0,1] \to X$ of p and q in the usual way; that is, $(p * q)(t) = p(2t)$ for $t \in [0, 1/2]$ and $(p * q)(t) = q(2t-1)$ for $t \in [1/2, 1]$.

If $x_0 \in X$, the set of all paths $p \in \mathcal{F}$ such that $p(0) = x_0$ is denoted by $\mathcal{F}_{x_0}$. The map $F_{x_0} : \mathcal{F}_{x_0} \to X$ defined by $F_{x_0}(p) = p(1)$ is called the *destination map* (for the paths that begin at x_0). Let $\mathcal{F}^Q_{x_0}$ be the collection of the members p of $\mathcal{F}_{x_0}$ such that the destination map F_{x_0} is quasi-interior at p, and define $\mathcal{F}^Q = \bigcup\{\mathcal{F}^Q_x \mid x \in X\}$. If X is compact, the space $\mathcal{F}$ of all paths in X is topologically complete and so are its closed subspaces $\mathcal{F}_x$. If, additionally, X is path connected, the destination map is surjective. By Theorem 1.1 we have the following proposition, which has also been used in [2].

Proposition 2.1. *If X is compact and path connected and $x_0 \in X$, then for some path $p \in \mathcal{F}_{x_0}$ the destination map $F_{x_0} : \mathcal{F}_{x_0} \to X$ is quasi-interior at p. Moreover, the set $\mathcal{F}^Q_{x_0}$ of all such paths p is a G_δ subset of $\mathcal{F}_{x_0}$.*

Next we show that if X is compact and homogeneous, then $\mathcal{F}^Q$ is a G_δ subset of $\mathcal{F}$ (Theorem 2.3 below). For the proof we use some additional notation and a lemma. For a set $U \subset \mathcal{F}$ define

$$W(U) = \{p \in U \mid p(1) = F_{p(0)}(p) \in (F_{p(0)}(U \cap \mathcal{F}_{p(0)}))^*\}.$$

Note that $W(U) \subset W(V)$ whenever $U \subset V$. With every open set U in $\mathcal{F}$ we associate a sequence $\widetilde{U}^k$ of the sets $\widetilde{U}^k = \{x \in U \mid B(x, \varepsilon + 1/k) \subset U$ for some $\varepsilon > 0\}$. Note that each $\widetilde{U}^k$ is open and $\bigcup_k \widetilde{U}^k = U$. Moreover, for $\delta < \frac{1}{k} - \frac{1}{k+1}$, the δ-neighborhood of $\widetilde{U}^k$ is contained in $\widetilde{U}^{k+1}$. For each open $U \subset \mathcal{F}$ let

$$W^0(U) = \bigcup_{k=1}^{\infty} W(\widetilde{U}^k).$$

The sets $W(\widetilde{U}^k)$ form an increasing nested sequence and all are contained in $W(U)$. Therefore, $W^0(U) \subset W(U)$. It is not clear whether $W^0(U)$ always equals $W(U)$.

Lemma 2.2. *If X is compact and homogeneous, then $W^0(U)$ is open in $\mathcal{F}$ for each open set $U \subset \mathcal{F}$.*

Proof. Let $U \subset \mathcal{F}$ be open and $p \in W^0(U)$. Fix a k such that $p \in W(\widetilde{U}^k)$. To prove that p is in the interior of $W^0(U)$, we show that for each sequence $\{p_n\} \subset \mathcal{F}$ with $\lim p_n = p$, some p_n's are in $W^0(U)$. Let

$\{p_n\}$ be such a sequence. Since $p \in \widetilde{U}^k \subset U$ and $\widetilde{U}^k$ is open, without loss of generality assume $p_n \in \widetilde{U}^k$ for each n. To simplify notation, let $a = p(0)$ and $a_n = p_n(0)$ for each n. Also, given $x \in X$, let $U_x = U \cap \mathcal{F}_x$ and $W_x(U) = \{p \in U_x \mid F_x(p) \in (F_x(U_x))^*\}$. Note that each $W_x(U)$ is open in $\mathcal{F}_x$ by the openness of U_x in $\mathcal{F}_x$, the openness of $(F_x(U_x))^*$ in X, and the continuity of F_x.

By the Effros theorem there are homeomorphisms $h_n : X \to X$ converging to the identity such that $h_n(a_n) = a$. Let $q_n = h_n \circ p_n$ and note that $\lim q_n = p$ and $q_n \in U_a$ for each n. Since $p \in \widetilde{U}^k$ and $\widetilde{U}^k$ is open, $q_n \in \widetilde{U}^k$ for sufficiently large n. Moreover, letting $h_n^{-1}\widetilde{U}^k = \{h_n^{-1} \circ r \mid r \in \widetilde{U}^k\}$, for sufficiently large n, $h_n^{-1}\widetilde{U}^k \subset \widetilde{U}^{k+1}$ because $\widetilde{U}^{k+1}$ contains some δ-neighborhood of $\widetilde{U}^k$ and $\lim h_n^{-1} = \mathrm{Id}_X$. Also, p is in $W_a(\widetilde{U}^k)$ which is open in $\mathcal{F}_a$. Thus, for sufficiently large n, $q_n \in W_a(\widetilde{U}^k)$ because $\{q_n\} \subset \mathcal{F}_a$. Fix an n with $h_n^{-1}\widetilde{U}^k \subset \widetilde{U}^{k+1}$ and $q_n \in W_a(\widetilde{U}^k)$.

We have $F_a(q_n) \in (F_a(\widetilde{U}^k \cap \mathcal{F}_a))^*$. Therefore, $F_{a_n}(p_n) \in (F_{a_n}(h_n^{-1}\widetilde{U}^k \cap \mathcal{F}_{a_n}))^*$ because $p_n = h_n^{-1} \circ h_n \circ p_n = h_n^{-1} \circ q_n$, $h_n^{-1}(a) = a_n$, and h_n^{-1} is a self-homeomorphism of X. Since $h_n^{-1}\widetilde{U}^k \subset \widetilde{U}^{k+1}$, it follows that $F_{a_n}(p_n) \in (F_{a_n}(\widetilde{U}^{k+1} \cap \mathcal{F}_{a_n}))^*$, and thus $p_n \in W(\widetilde{U}^{k+1}) \subset W^0(U)$, which completes the proof. □

Theorem 2.3. *If X is compact and homogeneous, then $\mathcal{F}^Q$ is a G_δ subspace of $\mathcal{F}$.*

Proof. Since $\mathcal{F}$ is separable, for every natural number k there is a collection $\{B_1^k, B_2^k, \cdots\}$ of open balls in $\mathcal{F}$ of radius $1/k$ that covers $\mathcal{F}$. Observe that a path $p \in \mathcal{F}$ is in $\mathcal{F}^Q$ if and only if, for each k, $p \in W^0(B_{n_k}^k)$ for some n_k. By Lemma 2.2 the sets $W^0(B_n^k)$ are open in $\mathcal{F}$ for all k and n and so are the unions $V^k = \bigcup_{n=1}^\infty W^0(B_n^k)$. Hence, $\mathcal{F}^Q = \bigcap_{k=1}^\infty V^k$ is a G_δ subspace of $\mathcal{F}$. □

Proposition 2.4. *Let X be compact; $a, b, c \in X$; $p \in \mathcal{F}_a$; $q \in \mathcal{F}_b$; and $p(1) = b$. If the destination map $F_b : \mathcal{F}_b \to X$ is quasi-interior at q, then the destination map $F_a : \mathcal{F}_a \to X$ is quasi-interior at the path product $p * q$.*

Proof. Let U be a neighborhood of $p*q$ in F_a and let V be a neighborhood of q in F_b such that $\{p * r \mid r \in V\} \subset U$. Then $F_b(V) \subset F_a(U)$. Hence, $F_a(p * q) = F_b(q) \in (F_b(V))^* \subset (F_a(U))^*$, as needed. □

From now on let X be a homogeneous path connected continuum. We fix $x_0 \in X$. Let $\mathcal{F}_{x_0}^\Lambda$ be the set of all path products $p * q^{-1}$ such that

$p(0) = x_0$, $p(1) = q(1)$, F_{x_0} is quasi-interior at p, and $F_{q(0)}$ is quasi-interior at q. Let $F^{\Lambda}_{x_0} : \mathcal{F}^{\Lambda}_{x_0} \to X$ be the destination map F_{x_0} restricted to $\mathcal{F}^{\Lambda}_{x_0}$.

Proposition 2.5. *For a path connected homogeneous continuum X with $x_0 \in X$, the set $\mathcal{F}^{\Lambda}_{x_0}$ is topologically complete and the destination map for this set, $F^{\Lambda}_{x_0} : \mathcal{F}^{\Lambda}_{x_0} \to X$, is surjective.*

Proof. The set $\mathcal{F}^{Q}_{x_0} = \{p \in \mathcal{F}_{x_0} \,|\, F_{x_0}$ is quasi-interior at $p\}$ is a G_δ subset of a topologically complete set $\mathcal{F}_{x_0}$ by Proposition 2.1. Thus, $\mathcal{F}^{Q}_{x_0}$ is topologically complete. Similarly, $\mathcal{F}^{Q} = \bigcup\{\mathcal{F}^{Q}_{x} \,|\, x \in X\}$ is a G_δ subset of the topologically complete space $\mathcal{F}$ by Theorem 2.3, and thus $\mathcal{F}^{Q}$ is topologically complete. Consequently, the product $\mathcal{F}^{Q}_{x_0} \times \mathcal{F}^{Q}$ is topologically complete. Let $\mathcal{H} = \{(p,q) \in \mathcal{F}^{Q}_{x_0} \times \mathcal{F}^{Q} \,|\, p(1) = q(1)\}$ and note that $\mathcal{H}$ is closed in $\mathcal{F}^{Q}_{x_0} \times \mathcal{F}^{Q}$. Therefore, $\mathcal{H}$ is topologically complete. For $(p,q) \in \mathcal{H}$, let $H(p,q) = p * q^{-1}$ and note that $H : \mathcal{H} \to \mathcal{F}^{\Lambda}_{x_0}$ is a homeomorphism. Hence, $\mathcal{F}^{\Lambda}_{x_0}$ is topologically complete.

Let $x \in X$. By Proposition 2.1 there is a path $p \in \mathcal{F}_{x_0}$ such that $F_{x_0} : \mathcal{F}_{x_0} \to X$ is quasi-interior at p. Let r be a path from x to x_0. By Proposition 2.4 the destination map $F_x : \mathcal{F}_x \to X$ is quasi-interior at $q = r * p$, and $p(1) = q(1)$. Thus, $p * q^{-1}$ is a member of $\mathcal{F}^{\Lambda}_{x_0}$ and $F^{\Lambda}_{x_0}(p * q^{-1}) = x$. Hence, $F^{\Lambda}_{x_0} : \mathcal{F}^{\Lambda}_{x_0} \to X$ is surjective. □

Proposition 2.6. *For a path connected homogeneous continuum X with $x_0 \in X$, the destination map $F^{\Lambda}_{x_0} : \mathcal{F}^{\Lambda}_{x_0} \to X$ is open.*

Proof. Suppose the map $F^{\Lambda}_{x_0} : \mathcal{F}^{\Lambda}_{x_0} \to X$ is not open. Then for some path $s = p * q^{-1} \in \mathcal{F}^{\Lambda}_{x_0}$ with $p \in \mathcal{F}^{Q}_{x_0}$, $q \in \mathcal{F}^{Q}_{q(0)}$, and $p(1) = q(1)$, and for some $\varepsilon > 0$, there exists a sequence $\{c_n\}$ in X converging to $c = s(1) = q(0)$ such that for every member s' of $\mathcal{F}^{\Lambda}_{x_0}$ with destination $F^{\Lambda}_{x_0}(s') = s'(1) = c_n$ for some n, the paths s and s' are of the distance greater than ε in the *sup* metric.

Let U be the open $(\varepsilon/2)$-neighborhood of p in $\mathcal{F}_{x_0}$. By the Effros theorem there are homeomorphisms $h_n : X \to X$ converging to the identity such that $h_n(c) = c_n$. Since F_{x_0} is quasi-interior at p, the set $W = (F_{x_0}(U))^*$ is an open neighborhood of $p(1)$. The paths $r_n = h_n \circ q$ converge to q, and thus the *sup* distance from r_n to q is less than $\varepsilon/2$ and $r_n(1) \in W$ for almost all n. Fix such an n. Let V be the open $(\varepsilon/2)$-neighborhood of r_n in $\mathcal{F}_{c_n}$. Since $q \in \mathcal{F}^{Q}_{c}$ and h_n is a self-homeomorphism of X, it follows that $r_n = h_n \circ q$ is a member of $\mathcal{F}^{Q}_{c_n}$. Therefore, $(F_{c_n}(V))^*$ is a neighborhood of $r_n(1) = F_{c_n}(r_n)$. We have $r_n(1) \in (F_{x_0}(U))^* \cap (F_{c_n}(V))^* \neq \emptyset$. By Theorem 1.2 applied to the destination maps $F_{x_0} : \mathcal{F}_{x_0} \to X$ and $F_{c_n} : \mathcal{F}_{c_n} \to X$, there exist paths $p_0 \in \mathcal{F}^{Q}_{x_0}$ and $q_0 \in \mathcal{F}^{Q}_{c_n}$ such that

$p_0(1) = q_0(1)$. Thus, $s_0 = p_0 * q_0^{-1}$ is a path from x_0 to c_n, and s_0 is a member of $\mathcal{F}_{x_0}^{\Lambda}$. The distance from p to p_0 is less than $\varepsilon/2$ and from q to q_0 less than $\varepsilon/2 + \varepsilon/2$. Hence, the distance from s to s_0 is less than ε. This contradiction completes the proof. □

Theorem 2.7 (**Main Result**). *Each path connected homogeneous continuum is uniformly path connected.*

Proof. Let X be a homogeneous path connected continuum. We prove that X is a continuous image of the Cantor fan. This last condition is equivalent to the uniform path connectedness for continua [11].

In propositions 2.5 and 2.6 we have shown there is a topologically complete collection, $\mathcal{F}_{x_0}^{\Lambda}$, of paths in X with a fixed beginning at x_0 and with the destination map $F_{x_0}^{\Lambda} : \mathcal{F}_{x_0}^{\Lambda} \to X$ surjective and open. By Theorem 1.3 there is a compact set $Z \subset \mathcal{F}_{x_0}^{\Lambda}$ such that the restricted map $F_{x_0}^{\Lambda}|Z : Z \to X$ is surjective. Let $\mathcal{C}$ be the Cantor set. Since each compact metric space is a continuous image of $\mathcal{C}$, there is a surjective map $M : \mathcal{C} \to Z$. For $x \in \mathcal{C}$ and $t \in [0, 1]$, let $G(x, t) = (M(x))(t)$, which defines a continuous surjection $G : \mathcal{C} \times [0, 1] \to X$. Since $G(x, 0) = x_0$ for each $x \in \mathcal{C}$, the map G can be factored by the quotient map from $\mathcal{C} \times [0, 1]$ to an intermediate quotient space, $K = (\mathcal{C} \times [0, 1])/(\mathcal{C} \times \{0\})$. Thus, X is a continuous image of K. This last space, K, is homeomorphic to the Cantor fan. The proof is complete. □

3. Consequences

By Theorem 2.7 each path connected homogeneous continuum is a continuous image of the Cantor fan. It is known and easy to prove that non-locally connected homogeneous continua have open sets with uncountably many components. Thus, they admit maps onto the Cantor fan K by Bellamy's characterization of continuous pre-images of K [1]. Therefore, non-locally connected homogeneous path connected continua are continuously equivalent to the Cantor fan. A non-degenerate locally connected continuum is continuously equivalent to an arc by the Hahn-Mazurkiewicz theorem and the Urysohn lemma. Thus, we have the following.

Corollary 3.1. *Each non-degenerate homogeneous path connected continuum is either continuously equivalent to an arc or continuously equivalent to the Cantor fan.*

Corollary 3.2. *Each two non-degenerate homogeneous path connected continua are continuously equivalent if and only if either they both are locally connected or they both are non-locally connected.*

Two spaces X and Y are *continuously comparable* if there exists a continuous surjection either from X to Y or from Y to X.

Corollary 3.3. *Suppose X and Y are non-degenerate homogeneous path connected continua. There exists a continuous surjection from X to Y if and only if either X is non-locally connected or Y is locally connected. Each two homogeneous path connected continua are continuously comparable.*

Let $\mathcal{K}$ be a class of spaces. We say that a space X is a *common model* for $\mathcal{K}$ provided $X \in \mathcal{K}$ and, for each $Y \in \mathcal{K}$, there is a continuous surjection from X to Y.

Corollary 3.4. *The continuum $\mathbb{P}$ defined in* [13] *is a common model for all path connected homogeneous continua.*

A continuum X is *g-contractible* provided it admits a null-homotopic surjection $f : X \to X$. This concept was introduced in [1] by Bellamy. Every g-contractible continuum is uniformly path connected, but the converse does not always hold [9]. If two continua are continuously equivalent, then either both are g-contractible, or neither [9, p. 157]. Since an arc and the Cantor fan are contractible, they are g-contractible, and so are all continua continuously equivalent to either of them. Thus, Corollary 3.1 implies the following.

Corollary 3.5. *Each homogeneous path connected continuum is g-contractible.*

A continuum X has the *arc approximation property* provided every subcontinuum of X is the limit, in the sense of the Hausdorff distance, of path connected subcontinua of X. In homogeneous continua this property leads to a stronger property of being arc Kelley, which is useful in proofs and new constructions [5]. In [15] it has been shown that all isometrically homogeneous continua are arc Kelley.

Question 1. Does every homogeneous path connected continuum have the arc approximation property?

References

[1] David P. Bellamy, *The cone over the Cantor set-continuous maps from both directions* in Topology Conference (Proc. General Topology Conf., Emory Univ., Atlanta, Ga., 1970). Ed. J. W. Rogers, Jr. Atlanta, GA: Department of Mathematics, Emory University, 1970. 8–25.

[2] ———, *Short paths in homogeneous continua*, Topology Appl. **26** (1987), no. 3, 287–291.

[3] David P. Bellamy and Lewis Lum, *The cyclic connectivity of homogeneous arcwise connected continua*, Trans. Amer. Math. Soc. **266** (1981), no. 2, 389–396.

[4] J. Chaber and R. Pol, *Completeness* in Encyclopedia of General Topology. Ed. K. P. Hart, Jun-iti Nagata, and J. E. Vaughan. Amsterdam: Elsevier, 2004. 251–254.

[5] J. J. Charatonik, W. J. Charatonik, and J. R. Prajs, *Confluent mappings and arc Kelley continua*, Rocky Mountain J. Math. **38** (2008), no. 4, 1091–1115.

[6] J. J. Charatonik and T. Maćkowiak, *Around Effros' theorem*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 579–602.

[7] Edward G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55.

[8] Karl H. Hofmann and Sidney A. Morris, *The Structure of Compact Groups: A Primer for the Student–a Handbook for the Expert*. de Gruyter Studies in Mathematics, 25. Berlin: Walter de Gruyter & Co., 1998.

[9] Iwona Krzemińska and Janusz R. Prajs, *A non-g-contractible uniformly path connected continuum*, Topology Appl. **91** (1999), no. 2, 151–158.

[10] Krystyna Kuperberg, *A locally connected microhomogeneous nonhomogeneous continuum*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **28** (1980), no. 11-12, 627–630 (1981).

[11] Włodzimierz Kuperberg, *Uniformly pathwise connected continua* in Studies in Topology (Proc. Conf., Univ. North Carolina, Charlotte, N.C., 1974; dedicated to Math. Sect. Polish Acad. Sci.). Ed. Nick M. Stavrakas and Keith R. Allen. New York: Academic Press, 1975. 315–324.

[12] T. Maćkowiak and E. D. Tymchatyn, *Continuous Mappings on Continua. II.* Dissertationes Math. (Rozprawy Mat.) 225 (1984), 1–57.

[13] Janusz R. Prajs, *A homogeneous arcwise connected non-locally-connected curve*, Amer. J. Math. **124** (2002), no. 4, 649–675.

[14] ———, *Pseudo-path connected homogeneous continua*, Adv. Math. **273** (2015), 271–286.

[15] ———, *Isometrically homogeneous and topologically homogeneous continua*. To appear in Indiana Univ. Math. J.

[16] Keith Whittington, *Local connectedness in transformation groups*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 903–907.

California State University Sacramento, Department of Mathematics and Statistics, 6000 J Street, Sacramento, CA 95819, USA
E-mail address: prajs@csus.edu

University of Opole, Institute of Mathematics, Ul. Oleska 48, 45-052 Opole, Poland