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DYNAMICS ON LOCALLY COMPACT HAUSDORFF SPACES

M. ARCHANA, P. CHIRANJEEVI, AND V. KANNAN

ABSTRACT. Given a metric space X, it is natural to ask, "Which subsets of X arise as sets of periodic points of continuous self-maps on X?" Since most of the metric spaces have to contain a copy of ω^2, the following results of this paper partially answer this question.

(1) A subset S of ω^2 occurs as the set of periodic points for some continuous self-map on ω^2 if and only if $\overline{S} \setminus S$ is either empty or infinite.
(2) A subset S of ω^2 occurs as the set of periodic points for some self-homeomorphism on ω^2 if and only if $T \bigcap S^c$ is either empty or infinite for any (minimal) subset T of ω^2 which is invariant under all those homeomorphisms under which S is invariant.
(3) Every subset of $\mathbb{N}$ occurs as set of periods of periodic points for some self-homeomorphism on ω^2.

1. Introduction

There have been many papers characterizing the sets of periods of periodic points for various classes of self-maps, such as (i) continuous self-maps on the real line $\mathbb{R}$ (see [8]), (ii) polynomials on $\mathbb{C}$ (see [2]), (iii) toral automorphisms (see [13]), (iv) totally transitive maps on I (see [3]), (v) continuous self-maps of $\mathbb{R}^n$ (see [12]), (vi) additive cellular automata (see [16]), (vii) linear operators (see [1]), and (viii) degree one maps on S^1 (see [18]). Also, there have been some results giving partial information about the sets of periodic points for continuous self-maps on various sets (see

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[2], [4], [7], [11]) and a paper characterizing the sets of periodic points of toral automorphisms (see [20]). Now we consider the problem of characterizing the sets of periodic points of some dynamical systems, such as self-homeomorphisms on ω^2 and the continuous self-maps on ω^2.

The following observation adds importance to the characterization of the sets of periodic points of continuous self-maps on ω^2: If $S \subset Y \subset [0,1]$, where Y is homeomorphic to ω^2 such that $\overline{Y} \setminus Y$ is finite and S cannot occur as the set of periodic points for any continuous self-map on Y, then it cannot occur as the set of periodic points for any continuous self-map on $[0,1]$. So the characterization of sets of periodic points of continuous self-maps on ω^2 gives some information for the characterization of sets of periodic points of continuous self-maps on $[0,1]$.

Definitions and notation. A *dynamical system* is a pair (X, f) where X is a topological space and f is a continuous self-map on X. A point $x \in X$ is called a *periodic point* if $f^n(x) = x$ for some $n \in \mathbb{N}$ and the least such n is called the *period* of x. The set of all periodic points of f is denoted by $P(f)$ and the set of periods of periodic points of f is denoted by $Per(f)$. A point $x \in X$ is called a *fixed point* if $f(x) = x$. The set of all fixed points of f is denoted by $Fix(f)$. A subset S of X is called *f-invariant* if $f(S) = S$ and is called *forward f-invariant* if $f(S) \subset S$.

2. Periodic Points of Continuous Self-Maps on Locally Compact Hausdorff Spaces

Hereafter I denotes an interval in $\mathbb{R}$, not necessarily compact.

Theorem 2.1 ([5]). *If $S = P(f)$ for some continuous self-map f on a compact Hausdorff space X, then $\overline{S} \setminus S$ has to be either empty or infinite.*

Remark 2.2. The above theorem need not be true if X is not locally compact. For example, take $X = \{\frac{2m}{2n+1} : 0 \leq m \leq n\} \bigcup \{1\}$ and define $t : X \longrightarrow X$ by $t(x) = 1 - |1 - 2x|$ for all $x \in X$. Then $\overline{P(t)} \setminus P(t) = \{1\}$. (The continuous extension of this map t on $[0,1]$ is called the tent map.)

We prove later that the above result holds true even in the case of ω^2 which is a locally compact Hausdorff space but not compact. In this particular case, we prove that the converse is also true. *The truth of the above result for the case of a general locally compact Hausdorff space which is not compact is still open.*

Theorem 2.3 ([5]). *For any continuous self-map f on a Hausdorff space X, the set $P(f)$ has to be F_σ. Further, it has to be closed if $Per(f)$ is finite.*

Theorem 2.4 ([5]). *Let f be a continuous self-map on I. Then*

(1) *Every connected component of* $P(f)$ *is closed in* I.
(2) *Every element of* $P(f)^c$ *is a limit point of* $P(f)^c$.

Proposition 2.5. *If* f *is a continuous self-map on* I *such that* $P(f) = I$, *then* f^2 *is the identity.*

Proof. If $P(f) = I$, then f will be one-to-one and so it will be strictly monotonic. Therefore, f^2 will be strictly increasing for which every element is periodic. This concludes the result. □

Proposition 2.6. *Let* S *be a union of* n *pairwise disjoint closed subintervals of* $[0,1]$ *and let* $k \in \mathbb{N}$. *Then* $2^k - 1 \leq n < 2^{k+1} - 1$ *if and only if there exists a continuous self-map* f *on* $[0,1]$ *such that* $2^k \in Per(f), P(f) = S$ *and* $2^{k+1} \notin Per(g)$ *for any continuous self-map* g *on* $[0,1]$ *with* $P(g) = S$.

Proof. Use Proposition 2.5 and Theorem 2.4(1). □

Proposition 2.7 ([17])**.** *Any countable metric space without isolated points is homeomorphic to* $\mathbb{Q}$.

Proposition 2.8. *If* $S = P(f)$ *for some continuous self-map* f *on* I, *then* S^c *has to be either empty or uncountable.*

Proof. Since S^c has no isolated point, if it is nonempty countable, then it will be homeomorphic to $\mathbb{Q}$ and so it will not be G_δ, which is a contradiction to the fact that S is F_σ. Therefore, S^c is either empty or uncountable. □

Corollary 2.9. *If* f *is a continuous self-map on* I *such that* $I \bigcap \mathbb{Q}^c \subset P(f)$, *then* $P(f) = I$.

Proposition 2.10 ([5])**.** *If* S *is either a nonempty closed subset of* I *or a countable dense subset of* I, *then there exists a continuous self-map* f *on* I *such that* $P(f) = S$.

The following are some examples of subsets of $[0,1]$ which satisfy all the necessary conditions we have stated but not occurring as $P(f)$ for any continuous self-map f on $[0,1]$:

(1) $\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}; 1 < m < n\}$
(follows from Proposition 3.13 that will be proved in the next section).

(2) $\bigcup_{n\in\mathbb{N}}\{x \in [\frac{n-1}{n}, \frac{n}{n+1}] : (n+1)(nx-n+1)+1 \in \bigcup_{i\in\mathbb{N}}[2-\frac{1}{2i}, 2-\frac{1}{2i+1}]\}$.
(We identify any two elements in the same connected component of
$\overline{\bigcup_{n\in\mathbb{N}}\{x \in [\frac{n-1}{n}, \frac{n}{n+1}] : (n+1)(nx-n+1)+1 \in \bigcup_{i\in\mathbb{N}}[2-\frac{1}{2i}, 2-\frac{1}{2i+1}]\}}$
and consider the quotient topology. Then we use Proposition 3.13.)

(3) $([0, \frac{1}{2}) \bigcap \mathbb{Q}) \bigcup \{\frac{1}{2} + \frac{1}{2n} : n \in \mathbb{N}\}$.
(If it occurs as $P(f)$ for some continuous self-map f on $[0, 1]$, then $\{\frac{1}{2}+\frac{1}{2n} : n \in \mathbb{N}\}$ has to be forward f-invariant and so $\frac{1}{2}$ has to be periodic.)

The above examples motivate us to study the sets of periodic points of continuous self-maps on zero-dimensional and scattered spaces.

3. Zero Dimensional and Scattered Spaces

Definition 3.1. A subset V of a topological space X is called *clopen* if V is both closed and open in X.

Definition 3.2. A topological space is *zero-dimensional* if it has a base consisting of clopen sets.

Definition 3.3. A subset A of a topological space X is called a *retract* of X if there exists a continuous map $f : X \longrightarrow A$ such that $f(a) = a$ for all $a \in A$.

Proposition 3.4. *If A is a retract of X, then every continuous self-map f on A can be extended as a continuous map $g : X \longrightarrow A$.*

Proof. Take $g = f \circ f_1$ where $f_1 : X \longrightarrow A$ is a continuous map such that $f_1(a) = a$ for all $a \in A$. □

Proposition 3.5 ([14, p. 35])**.** *A separable metrizable space X is zero dimensional if and only if every closed subset of X is a retract of X.*

Corollary 3.6. *For every closed subset S of $I \bigcap \mathbb{Q}$, there exists a continuous self-map* f *on $I \bigcap \mathbb{Q}$ such that $P(f) = Fix(f) = S$.*

Definition 3.7. A topological space X is said to be *scattered* if every nonempty subset of X has an isolated point. Note that every well-ordered space is scattered.

Proposition 3.8 ([19])**.** *Every compact scattered space is zero dimensional.*

Proposition 3.9 ([15])**.** *Every countable compact Hausdorff space is well ordered.*

For a topological space X, we define recursively a transfinite sequence of subsets of X as follows:

$X_0 =$ the set of isolated points of X.

$X_\alpha =$ the set of isolated points of $X \setminus \bigcup_{i<\alpha} X_i$ for each nonzero ordinal number α.

Given a scattered space X, the least ordinal α such that $X_\alpha = \phi$ is called the *derived length of* X and it is denoted by $\delta(X)$. For $x \in X$, we define the *derived length of x in X* as the unique ordinal number α such that $x \in X_\alpha$ and it is denoted by $\delta(x, X)$.

Given a subset S of a scattered space X and ordinal numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, we define $S_X(\alpha_1, \alpha_2, \ldots, \alpha_n)$ recursively as follows:

$S_X(\alpha_1) = X_{\alpha_1}$

$S_X(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\overline{S \bigcap S_X(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})})_{\alpha_n}$.

Proposition 3.10 ([9]). *Let X be a compact scattered space and Y be a T_1 space such that Y is a closed continuous image of X, then $\delta(Y) \leq \delta(X)$.*

Theorem 3.11. *Let S be a dense subset of a compact scattered space X which occurs as $P(f)$ for some continuous self-map f on X such that $f^{-1}(y)$ is finite for all $y \in Y$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are ordinals, then $S_X(\alpha_1, \alpha_2, \ldots, \alpha_n) \bigcap S^c$ is either empty or infinite.*

Proof. Suppose that $X_\alpha \bigcap S^c$ is nonempty finite for some ordinal α. Then there exists $x \in X_\alpha$ such that $f(y) \neq x$ for any $y \in X_\alpha$. Let V be a clopen neighborhood of x such that $V \bigcap X_\alpha = \{x\}$. Then $f^{-1}(V)$ is a compact scattered subspace of X such that $\delta(f^{-1}(V)) < \delta(V)$, which is a contradiction to Proposition 3.10. Therefore, $X_\alpha \bigcap S^c$ is either empty or infinite. Fix the ordinals $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ for some $n > 1$. Then $S_X(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \bigcap S = P(f|_{\overline{S_X(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \bigcap S}})$, and therefore $S_X(\alpha_1, \alpha_2, \ldots, \alpha_n) \bigcap S^c$ is either empty or infinite. □

Proposition 3.12. *For any continuous surjective self-map f on a compact scattered space X, the set of highest derived length points is forward invariant.*

Proof. Let $x \in X$ be a point of highest derived length α such that $\delta(f(x), X) < \alpha$ and let n be the number of points of derived length α. Consider a separation (W_1, W_2) of X such that $x, f(x) \in W_1$ and $\delta(y, X) < \alpha$ for all $y \in W_1 \setminus \{x\}$. Then the cardinality of $\{y \in f(W_2) : \delta(y, f(W_2)) = \alpha\}$ is at most $n - 1$. Therefore, $\delta(f(W_1)) = \alpha$. Let V be a clopen neighborhood of $f(x)$ in X such that $\delta(V) < \alpha$ and $W_3 = f^{-1}(V) \bigcap W_1$. Then $\delta(W_3^c \bigcap W_1) < \alpha$ and so $\delta(f(W_3^c \bigcap W_1)) < \alpha$ and also $\delta(f(W_3)) < \alpha$, which implies that $\delta(f(W_1)) < \alpha$, which is a contradiction. Therefore, $\delta(f(x), X) = \alpha$. □

Proposition 3.13. *For any continuous surjective self-map f on a compact scattered space X, every point of highest derived length has to be periodic.*

Proof. Let $\alpha = sup\{\delta(x, X) : x \in X\}$ and let n be the number of points of derived length α. We need to prove that every point of derived length α is periodic. We proved the result for $n = 1$ and we will now prove for $n = 2$. The proof for general n is similar. Let $x, y \in X$ such that $\delta(x, X) =$

$\delta(y, X) = \alpha$ and $f(x) = f(y) = y$. Let V be a clopen neighborhood of y not containing x and $W = f^{-1}(V)$. Then $f(W)$ will be homeomorphic to X and so $f(W^c)$ has to be homeomorphic to X, which is a contradiction to $\delta(W^c) < \alpha$. □

Corollary 3.14. *If S is an infinite discrete subspace of a compact Hausdorff space X that occurs as $P(f)$ for some continuous self-map f on X, then $\overline{S}$ has to be uncountable.*

Now we indeed give an example of a continuous self-map on a compact Hausdorff space for which the set of periodic points is a discrete space whose closure is uncountable.

Let K denote the Cantor middle third set and let C be the set of midpoints of connected components of $I \setminus K$ where $I = [0, 1]$. Let $\tilde{0} = 2$, $\tilde{2} = 0$, and $\tilde{1} = 1$. Define $f : C \bigcup K \longrightarrow C \bigcup K$ by $f(\sum_{i=1}^{\infty} \frac{a_i}{3^i}) = \sum_{i=1}^{\infty} \frac{g(a_1 a_2 \dots a_n)}{3^i}$ where $\sum_{i=1}^{\infty} \frac{a_i}{3^i}$ is in the Cantor ternary form with $a_i = 1 \Rightarrow a_{i+1} = 1$ and $g(a_1 a_2 \dots a_n)$ is defined as follows:

$g(a_1) = \tilde{a_1}$ and

$$g(a_1 a_2 ... a_n) = \begin{cases} a_n & \text{if } a_i = 0 \text{ for some } i < n \\ \tilde{a_n} & \text{otherwise.} \end{cases}$$

Then f will be continuous with $P(f) = C$.

4. The Ordinal Space ω^2

Now we characterize the sets of periods and periodic points of self-homeomorphisms and continuous self-maps on ω^2. Homeomorphisms are considered in the first subsection and continuous maps in the next subsection.

4.1. Periodic points of self-homeomorphisms on ω^2.

In this section, we characterize the following:

(1) sets of periodic points of self-homeomorphisms on ω^2,
(2) sets of periods of periodic points of self-homeomorphisms on ω^2,
(3) the pairs (S, T) where $S \subset \omega^2$ and $T \subset \mathbb{N}$ such that $S = P(h)$ and $T = Per(h)$ for some self-homeomorphism h on ω^2.

Such a characterization of pairs has been given for the class of bounded linear operators on a Hilbert space in [6].

Definition 4.1. Given $n \in \mathbb{N}$, a subset $\{n_1, n_2, \dots, n_r\}$ of $\mathbb{N}$ is said to be a *partition set of n* if there exist positive integers $k_1, k_2, \dots, k_r$, which need not be distinct such that $n_1 k_1 + n_2 k_2 + \dots + n_r k_r = n$. By convention, we take the empty set to be the unique *partition set of 0* and any nonempty subset of $\mathbb{N}$ is taken as a *partition set of ∞*.

Hereafter, X denotes a space which is homeomorphic to ω^2 unless specified. Given a subset S of X, we define the sets $C_{i,S}$ for $i \in \{1, 2, ..., 8\}$ as follows:

(1) $C_{1,S} = S \cap X_0$
(2) $C_{2,S} = S^c \cap X_0$
(3) $C_{3,S} = (\overline{S \cap X_0} \setminus \overline{S^c \cap X_0}) \cap S \cap X_1$
(4) $C_{4,S} = \overline{S \cap X_0} \cap \overline{S^c \cap X_0} \cap S \cap X_1$
(5) $C_{5,S} = (\overline{S^c \cap X_0} \setminus \overline{S \cap X_0}) \cap S \cap X_1$
(6) $C_{6,S} = (\overline{S \cap X_0} \setminus \overline{S^c \cap X_0}) \cap S^c \cap X_1$
(7) $C_{7,S} = \overline{S \cap X_0} \cap \overline{S^c \cap X_0} \cap S^c \cap X_1$
(8) $C_{8,S} = (\overline{S^c \cap X_0} \setminus \overline{S \cap X_0}) \cap S^c \cap X_1$.

The nonempty sets in the above list give a partition of X. The significance of these eight sets is more evident in [10]. Now we prove that four of the above eight sets, viz., $C_{2,S}$, $C_{6,S}$, $C_{7,S}$, $C_{8,S}$, determine the existence of a self-homeomorphism h on X such that $P(h) = S$, and two of them, viz., $C_{6,S}$, $C_{7,S}$, determine the existence of a continuous self-map f on X such that $P(f) = S$.

Proposition 4.2. *If S is a subset of X, then for any self-homeomorphism h on X, the above eight sets are h-invariant provided S is h-invariant.*

Proof. For any self-homeomorphism h on X, since X_0 and X_1 are h-invariant and since the set of all h-invariant subsets of X is closed under closure, intersection, and set complementation, the above sets should be h-invariant whenever S is h-invariant. □

Proposition 4.3. *There exists a self-homeomorphism h on X such that $P(h) = \phi$.*

Proof. Let $X_1 = \{x_i : i \in \mathbb{Z}\}$. We may assume that $x_i \neq x_j$ for any $i, j \in \mathbb{Z}$ with $i \neq j$. For each $i \in \mathbb{Z}$, choose a sequence $(x_{ij})_{j\in\mathbb{N}}$ of distinct points of X converging to x_i such that $\{x_{ij} : i \in \mathbb{Z}, j \in \mathbb{N}\} = X_0$. Then the function $h : X \longrightarrow X$, defined by $h(x_i) = x_{i+1}$ and $h(x_{ij}) = x_{(i+1)j}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, is a homeomorphism such that $P(h) = \phi$. □

Proposition 4.4. *Given a subset T of $\mathbb{N}$, there exists a self-homeomorphism h on X such that $P(h) = X_0$ and $Per(h) = T$ if and only if T is infinite.*

Proof. If there exists a self-homeomorphism h on X such that $P(h) = X_0$ and $Per(h) = T$, then T has to be infinite since X_0 is not closed. Conversely, suppose that T is infinite, say $T = \{n_1, n_2, n_3, \ldots\}$ such that $n_i < n_{i+1}$ for all $i \in \mathbb{N}$. Let x_i and x_{ij} be as in the proof of Proposition 4.3. For each $j \in T$, let $S_j = \{x_{ij} : i \in \mathbb{Z}, |i| \leq \frac{j}{2}, i \neq \frac{-j}{2}\}$ and define a self-homeomorphism h_j on S_j by

$$h_j(x_{ij}) = \begin{cases} x_{(i+1)j} & \text{if } x_{(i+1)j} \in S_j \\ x_{-(i+1)j} & \text{if } x_{(i+1)j} \notin S_j \text{ and } i = \frac{j}{2} \\ x_{-ij} & \text{if } x_{(i+1)j} \notin S_j \text{ and } i \neq \frac{j}{2}. \end{cases}$$

Let h' be a self-homeomorphism on $X_0 \setminus \bigcup_{j\in T} S_j$ such that $P(h') = X_0 \setminus \bigcup_{j\in T} S_j$ and $Per(h') = T$. Define $h'' : X_1 \longrightarrow X_1$ by $h''(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. Then the map $h : X \longrightarrow X$, defined by

$$h(x) = \begin{cases} h_j(x) & \text{if } x \in S_j \\ h'(x) & \text{if } x \in X_0 \setminus \bigcup_{j\in T} S_j \\ h''(x) & \text{if } x \in X_1, \end{cases}$$

is a self-homeomorphism on X such that $P(h) = X_0$ and $Per(h) = T$. □

Proposition 4.5. *If K is a compact subset of X, then $C_{2,K}$ and $C_{8,K}$ are infinite and $C_{6,K}$ and $C_{7,K}$ are empty. Moreover, $C_{8,K}$ is a cofinite subset of X_1.*

Proof. If K is compact, then it is closed and so $C_{6,K}$ and $C_{7,K}$ are empty. Therefore, $C_{8,K} = K^c \bigcap X_1$. If $K \bigcap X_1$ is an infinite subset of the compact space K, then it has a limit point, which is a contradiction. Therefore, $K \bigcap X_1$ is finite and so $C_{8,K}$ is a cofinite subset of X_1. Finally, if $C_{2,K}$ is finite, then $C_{8,K}$ is empty, which is a contradiction. So $C_{2,K}$ is infinite. □

Proposition 4.6. *If Y is either the same as X or a compact subspace of X, then, for a subset T of $\mathbb{N}$, there exists a self-homeomorphism h on Y such that $P(h) = Y_1$ and $Per(h) = T$ if and only if T is a partition set of $|Y_1|$.*

Proof. The necessity part is trivial and we will now prove the sufficiency part. Suppose that T is a partition set of $|Y_1|$. If T is a nonempty finite set of cardinality k, let $T = \{n_1, n_2, ..., n_k\}$ with $n_i < n_j$ for $i < j$, and if T is infinite, let $T = \{n_1, n_2, n_3, \ldots\}$ with $n_i < n_j$ for $i < j$. Let I be the empty set or $\{1, 2, 3, ..., n\}$ or $\mathbb{N}$ such that $|I| = |Y_1|$. Let $Y_1 = \{y_i : i \in I\}$. For each $i \in I$, choose a sequence $(y_{ij})_{j\in\mathbb{N}}$ in Y_0 converging to y_i such that $y_{ij} = y_{kl}$ if and only if $i = k$ and $j = l$. We may assume that $Y_0 \setminus \{y_{ij} : i \in I, j \in \mathbb{N}\}$ is either empty or an infinite closed set.

Let h_1 be a self-bijection on Y_1 such that $P(h_1) = Y_1$ and $Per(h) = T$. Define $h_{01} : \{y_{ij} : i \in I, j \in \mathbb{N}\} \longrightarrow \{y_{ij} : i \in I, j \in \mathbb{N}\}$ by

$$h_{01}(y_{ij}) = \begin{cases} y_{k(j+2)} & \text{if } h_1(y_i) = y_k,\ j \text{ is odd} \\ y_{k(j-2)} & \text{if } h_1(y_i) = y_k,\ j \text{ is even},\ j \neq 2 \\ y_{k1} & \text{if } h_1(y_i) = y_k,\ j = 2. \end{cases}$$

If $Y_0 \setminus \{y_{ij} : i \in I, j \in \mathbb{N}\}$ is empty, let $h_0 = h_{01}$. If $Y_0 \setminus \{y_{ij} : i \in I, j \in \mathbb{N}\}$ is infinite, let this set be $\{z_n : n \in \mathbb{N}\}$ and define $h_{02} : \{z_n : n \in \mathbb{N}\} \longrightarrow \{z_n : n \in \mathbb{N}\}, h_0 : Y_0 \longrightarrow Y_0$ as

$$h_{02}(z_n) = \begin{cases} z_{n+2} & \text{if } n \text{ is odd} \\ z_{n-2} & \text{if } n \text{ is even, } n \neq 2 \\ z_1 & \text{if } n = 2 \end{cases}$$

$$h_0(x) = \begin{cases} h_{01}(x) & \text{if } x = y_{ij} \text{ for some } i \in I \text{ and } j \in \mathbb{N} \\ h_{02}(x) & \text{if } x = z_n \text{ for some } n \in \mathbb{N}. \end{cases}$$

Now the function $h : Y \longrightarrow Y$, defined by

$$h(x) = \begin{cases} h_0(x) & \text{if } x \in Y_0 \\ h_1(x) & \text{if } x \in Y_1, \end{cases}$$

is a self-homeomorphism on Y such that $P(h) = Y_1$ and $Per(h) = T$. $\square$

Proposition 4.7. *If Y is a compact subspace of X and $Y_0 = A \bigcup B$ such that A and B are disjoint and, for each $y \in Y_1$, there is a sequence of distinct terms in A and a sequence of distinct terms in B both converging to y, then, given a subset T of $\mathbb{N}$, there exists a self-homeomorphism h on Y such that $P(h) = Y_1 \bigcup A$ and $Per(h) = T$ if and only if T is a union of two subsets T_0 and T_1 of T such that T_1 is a partition set of $|Y_1|$ and all but finitely many elements of T_0 have divisors in T_1.*

Proof. Suppose that $P(h) = Y_1 \bigcup A$ and $Per(h) = T$ for some self-homeomorphism h on X. Let $T_0 = Per(h|_{Y_0})$ and $T_1 = Per(h|_{Y_1})$ so that $T = T_0 \bigcup T_1$. Then it is trivial that T_1 is a partition set of $|Y_1|$. Let $Y_1 = \{y_1, y_2, ..., y_n\}$ for some $n \in \mathbb{N}$ and, for every i with $1 \leq i \leq n$, let $(y_{ij})_{j \in \mathbb{N}}$ be a sequence in A converging to y_i such that $\{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\} = A$. Since f is continuous, the set $A' = \{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N},\ f(y_{ij}) \neq y_{kl}$ for any $1 \leq k \leq n, l \in \mathbb{N}$ where $f(y_i) = y_k\}$ is finite and the period of every element of A whose orbit is disjoint with A' has a divisor in T_1. This concludes the necessity part.

Now, conversely, suppose that T is a union of two subsets T_0 and T_1 of T such that T_1 is a partition set of $|Y_1|$ and all but finitely many elements of T_0 have divisors in T_1. Since $Y_1 \bigcup B$ is homeomorphic to Y, by Proposition 4.6, there exists a self-homeomorphism h_1 on $Y_1 \bigcup B$ such that $P(h_1) = Y_1$ and $Per(h_1) = T_1$. Let $Y_1 = \{y_1, y_2, ..., y_n\}$ for some $n \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, n\}$, choose a sequence $(y_{ij})_{j \in \mathbb{N}}$ in A converging to y_i such that $y_{ij} = y_{kl}$ if and only if $i = j$ and $k = l$. We may assume that $A \setminus \{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\}$ is either empty or an infinite closed set. Let h_{21} be a self-homeomorphism on $\{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\}$ defined by $h_{21}(y_{ij}) = y_{kj}$ where k is a unique element of $\{1, 2, ..., n\}$ such that $h_1(y_i) = y_k$ for all $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \mathbb{N}$. If $A \setminus \{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\}$ is empty, let $h_1 = h_{21}$. If $A \setminus \{y_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\}$ is infinite, define a self-bijection h_{22} on this set such that $P(h_{22}) \subset T$ and define $h_2 : A \longrightarrow A$ by

$$h_2(x) = \begin{cases} h_{21}(x) & \text{if } x = y_{ij} \text{ for some } i \in \{1, 2, 3, \dots, n\},\ j \in \mathbb{N} \\ h_{22}(x) & \text{if } x \neq y_{ij} \text{ for any } i \in \{1, 2, 3, \dots, n\},\ j \in \mathbb{N}. \end{cases}$$

Now the function $h : Y \longrightarrow Y$, defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in Y_1 \bigcup B \\ h_2(x) & \text{if } x \in A, \end{cases}$$

is a self-homeomorphism on Y such that $P(h) = Y_1 \bigcup A$ and $Per(h) = T$. □

Proposition 4.8. *If $X_0 = A \bigcup B$ such that A and B are disjoint and for each $x \in X_1$ there is a sequence of distinct terms in A and a sequence of distinct terms in B both converging to x, then, given a subset T of $\mathbb{N}$, there exists a self-homeomorphism h on X such that $P(h) = A$ and $Per(h) = T$ if and only if T is infinite.*

Proof. If there exists a self-homeomorphism h on X such that $P(h) = A$ and $Per(h) = T$, then, since A is not closed, T has to be infinite. Conversely, suppose that T is infinite. Since $X_1 \bigcup A$ and $X_1 \bigcup B$ are homeomorphic to ω^2, from the proofs of Proposition 4.4 and Proposition 4.3, we can define two self-homeomorphisms h_1 and h_2 on $X_1 \bigcup A$ and $X_1 \bigcup B$, respectively, which coincide on X_1 such that $P(h_1) = A$, $Per(h_1) = T$, and $P(h_2) = \phi$. Now the function $h : X \longrightarrow X$, defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in X_1 \bigcup A \\ h_2(x) & \text{if } x \in X_1 \bigcup B, \end{cases}$$

is a self-homeomorphism on X such that $P(h) = A$ and $Per(h) = T$. □

Proposition 4.9. *If f is a continuous bijective self-map on a topological space Y, then for any f-invariant subset S of Y, $S \bigcap P(f)^c$ is either empty or infinite.*

Proof. Since f is a continuous bijection, $y \in Y$ is periodic if and only if $f(y)$ is periodic, i.e., $P(f)^c$ is f-invariant. Further, if S is an f-invariant subset of Y, $S \bigcap P(f)^c$ is also f-invariant. Now if $S \bigcap P(f)^c$ is a nonempty finite set, then it has a periodic point, which is a contradiction. Therefore, $S \bigcap P(f)^c$ is either empty or infinite. □

Proposition 4.10. *If Y is a topological space and a subset S of Y occurs as $P(f)$ for some continuous bijective self-map f on Y, then S^c should be either empty or infinite.*

Proof. The proof follows directly from Proposition 4.9. □

The following theorem gives a characterization of sets of periods of periodic points of self-homeomorphisms on X.

Theorem 4.11. *Every subset of $\mathbb{N}$ occurs as $Per(h)$ for some self-homeomorphism h on X.*

Proof. Let $T \subset \mathbb{N}$. Take a subset S of X_0 such that it is closed in X and T is a partition set of $|S|$. Define a homeomorphism h_1 on S such that $Per(h_1) = T$. Observe that $X \setminus S$ is homeomorphic to X. So by Proposition 4.3, there exists a homeomorphism h_2 on $X \setminus S$ such that $Per(h_2) = \phi$. Now the function $h : X \longrightarrow X$, defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in S \\ h_2(x) & \text{if } x \in X \setminus S, \end{cases}$$

is a self-homeomorphism on X such that $Per(h) = T$. □

Proposition 4.12. *If $X_0 = A \bigcup B$ such that A and B are disjoint and, for each $x \in X_1$, there is a sequence of distinct terms in A and a sequence of distinct terms in B both converging to x, then, for every subset T of $\mathbb{N}$, there exists a self-homeomorphism h on X such that $P(h) = X_1 \bigcup A$ and $Per(h) = T$.*

Proof. The proof is similar to the proof of Proposition 4.6. □

The characterization of the pairs (S, T) such that $S = P(h)$ and $T = Per(h)$ for some self-homeomorphism h is the following.

Theorem 4.13. *Given a subset S of X and a subset T of $\mathbb{N}$, there exists a self-homeomorphism h on X such that $P(h) = S$ and $Per(h) = T$ if and only if the following conditions hold.*

(1) *$C_{2,S}$, $C_{6,S}$, $C_{7,S}$, and $C_{8,S}$ are either empty or infinite.*
(2) *For each $i \in \{1, 3, 4, 5\}$, there exists a subset T_i of T which is a partition set of $|C_{i,S}|$ such that $\bigcup_{i \in \{1,3,4,5\}} T_i = T$, and if S is compact, then all but finitely many elements of T_1 have divisors in $T_3 \bigcup T_4$.*
(3) *If T is finite, then S is closed.*

Proof. Suppose that $S = P(h)$ and $T = Per(h)$ for some self-homeomorphism h on X. Suppose $S = P(h)$ for some self-homeomorphism h on X. Since the sets $C_{2,S}$, $C_{6,S}$, $C_{7,S}$, and $C_{8,S}$ are h-invariant by Proposition 4.9, they should be either empty or infinite and so (1) follows. For each $i \in \{1, 3, 4, 5\}$, let $T_i = Per(h|C_{i,S}) \subset T$. Then the sets T_1, T_3, T_4, and T_5 satisfy the first part of (2). The proof of the second part of (2) is similar to the proof of Proposition 4.7. Condition (3) follows from Theorem 2.3.

Conversely, suppose that (1)–(3) hold true. We may assume that $X = \overline{\{m + \frac{1}{n} : m, n \in \mathbb{N}\}}$.

Let $W_3 = C_3 \bigcup \{x + \frac{1}{n} \in S : x \in C_3 \bigcup C_5 \bigcup C_8, n \in \mathbb{N}\}$
$W_4 = C_4 \bigcup \{x + \frac{1}{n} : x \in C_4, n \in \mathbb{N} \setminus \{1\}\}$
$W_5 = C_5 \bigcup \{x + \frac{1}{n} \in S^c : x \in C_5, n \in \mathbb{N} \setminus \{1\}\}$
$W_6 = C_6 \bigcup \{x + \frac{1}{n} \in S : x \in C_6, n \in \mathbb{N} \setminus \{1\}\}$

$W_7 = C_7 \bigcup \{x + \frac{1}{n} : x \in C_7, n \in \mathbb{N} \setminus \{1\}\}$
$W_8 = C_8 \bigcup \{x + \frac{1}{n} \in S^c : x \in C_8, n \in \mathbb{N} \setminus \{1\}\}$.

If $\{x + \frac{1}{n} \in S^c : x \in C_{3,S} \bigcup C_{6,S}\}$ is finite, we replace some W_i which is not disjoint with $C_{2,S}$ by $W_i \bigcup \{x + \frac{1}{n} \in S^c : x \in C_{3,S} \bigcup C_{6,S}\}$ and we take W_9 to be the empty set. Otherwise, we take $W_9 = \{x + \frac{1}{n} \in S^c : x \in C_{3,S} \bigcup C_{6,S}\}$. We can observe that each W_i is a clopen set in X and $\{W_i : 3 \leq i \leq 9,\ W_i \neq \phi\}$ is a partition of X. We can also observe that W_i is either empty or homeomorphic to ω^2 for all $i \in \{6, 7, 8\}$ and W_9 is either empty or infinite and does not contain any limit point. Also, $W_i \bigcap X_1$ is infinite for some $i \in \{3, 4, \ldots, 8\}$, say $W_3 \bigcap X_1$ is infinite. Take h_3 to be any self-homeomorphism on W_3 such that $P(h_3) = W_3$ and $Per(h_3) = T$. Since, for each $x \in W_4$, there is a sequence in $W_4 \bigcap S$ and a sequence in $W_4 \bigcap S^c$, both converging to x by Proposition 4.7, there exists a self-homeomorphism h_4 on W_4 such that $P(h_4) = W_4 \bigcap S$ and $Per(h_4) = T_4$. If W_5 is nonempty, then by Proposition 4.6, there exists a self-homeomorphism h_5 on W_5 such that $P(h_5) = W_5 \bigcap S$ and $Per(h_5) = T_5$. If W_6 is nonempty, then by Proposition 4.4, there exists a self-homeomorphism h_6 on W_6 such that $P(h_6) = W_6 \bigcap S$ and $Per(h_6) = T$. Since, for each $x \in W_7$, there is a sequence in $W_7 \bigcap S$ and a sequence in $W_7 \bigcap S^c$, both converging to x by Proposition 4.8, there exists a self-homeomorphism h_7 on W_7 such that $P(h_7) = W_7 \bigcap S$ and $Per(h_7) = T$. If W_8 is nonempty, then by Proposition 4.3, there exists a self-homeomorphism h_8 on W_8 such that $P(h_8) = \phi$. Let h_9 be a self-bijection on W_9 such that $P(h_9) = \phi$. Now the function $h : X \longrightarrow X$, defined by $h(x) = h_i(x)$ if $x \in W_i$ for $3 \leq i \leq 9$, is a homeomorphism such that $P(h) = S$ and $Per(h) = T$. □

The following theorem gives the characterization of the sets of periodic points of self-homeomorphisms on X.

Theorem 4.14. *A subset S of X occurs as the set of periodic points for some self-homeomorphism on X if and only if the sets $C_{2,S}$, $C_{6,S}$, $C_{7,S}$, and $C_{8,S}$ are either empty or infinite.*

Proof. If S is closed, take $T = \{1\}$. Otherwise, take $T = \mathbb{N}$. Then the proof follows from Theorem 4.13. □

Proposition 4.15 ([10])**.** *Let S be a subset of X and let T be a minimal subset T of X which is invariant under all those homeomorphisms under which S is invariant. Then $T = C_{i,S}$ for some $1 \leq i \leq 8$.*

Corollary 4.16. *A subset S of X occurs as the set of periodic points for some self-homeomorphism on X if and only if $T \bigcap S^c$ is either empty* *or infinite for any (minimal) subset T of X which is invariant under all those homeomorphisms under which S is invariant.*

Proof. The proof follows from Theorem 4.14 and Proposition 4.15. $\square$

Corollary 4.17. *If S is a subset of X such that $S = P(h)$ for some self-homeomorphism h on X, then for any compact subset K of X which is disjoint from S, there exists a self-homeomorphism h' on X such that $P(h') = S \bigcup K$.*

Proof. Suppose $S = P(h)$ for some self-homeomorphism h on X and K is a compact subset of X. Note that $C_{2,S\bigcup K} = (S \bigcup K)^c \bigcap X_0 = (S^c \bigcap X_0) \bigcap (K^c \bigcap X_0) = C_{2,S} \bigcap C_{2,K}$. So if $C_{2,S}$ is empty, then $C_{2,S\bigcup K}$ is empty, and, since S and K are disjoint, if $C_{2,S}$ is infinite, then $C_{2,S\bigcup K}$ is infinite. Now observe that K has only finitely many limit points, all of which are in K, and $C_{i,S\bigcup K} \subset C_{i,K}$ and $C_{i,K} \setminus C_{i,S\bigcup K} \subset K$ for all $i \in \{6, 7, 8\}$. Thus, if $C_{i,S}$ is empty, then $C_{i,S\bigcup K}$ is empty and if $C_{i,S}$ is infinite, then $C_{i,S\bigcup K}$ is infinite. So by Theorem 4.14, there exists a self-homeomorphism h' on X such that $P(h') = S \bigcup K$. $\square$

Remark 4.18. The above result may not be true if S and K are not disjoint.

Example 4.19. Let $X = \overline{\{m + \frac{1}{n} : m, n \in \mathbb{N}\}}$, $S = X \setminus \{1 + \frac{1}{n} : n \text{ is even}\}$, and $K = \overline{\{1 + \frac{1}{n} : n \in \mathbb{N} \setminus \{1, 2\}\}}$; then we observe that $C_{2,S} = \{1 + \frac{1}{n} : n \text{ is even}\}$ and $C_{6,S}$, $C_{7,S}$, and $C_{8,S}$ are empty. Also, we can observe that K is compact, but $S \bigcup K$ cannot occur as $P(h)$ for any self-homeomorphism h on X because $C_{2,S\bigcup K} = \{1 + \frac{1}{2}\}$.

Remark 4.20. If $S = P(h)$ for some self-homeomorphism h on X and K is a compact subset of X, then $S \setminus K$ may not occur as $P(h')$ for any self-homeomorphism h' on X.

Corollary 4.21. *Every compact subset of X arises as $P(h)$ for some self-homeomorphism* h *on X.*

Proof. The result follows directly from Proposition 4.5 and Theorem 4.14. The result also follows from Proposition 4.3 and Corollary 4.17. $\square$

Corollary 4.22. *If K is a nonempty compact subset of X, then K^c cannot occur as $P(h)$ for any self-homeomorphism* h *on X.*

Proof. Suppose $K^c = P(h)$ for any self-homeomorphism h on X. Since K is nonempty by Proposition 4.9, K should be infinite. Since K is compact, $K \bigcap X_1$ is a nonempty finite subset of X. So at least one of C_{6,K^c}, C_{7,K^c}, and C_{8,K^c} should be a nonempty finite subset of X, which is a contradiction. So K^c cannot occur as $P(h)$ for any self-homeomorphism h on X. $\square$

4.2. Periodic points of continuous self-maps on ω^2.

In this section, we characterize the following:

(1) sets of periodic points of continuous self-maps on ω^2,
(2) the pairs (S, T) where $S \subset \omega^2$ and $T \subset \mathbb{N}$ such that $S = P(f)$ and $T = Per(f)$ for some continuous self-map f on ω^2.

Proposition 4.23. *If f is a continuous self-map on X such that $P(f) = S$, then $C_{3,S} \bigcup C_{4,S}$ and $C_{1,S} \bigcup C_{5,S}$ are forward f-invariant.*

Proof. Let $x \in C_{3,S} \bigcup C_{4,S}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $C_{1,S}$ converging to x. If $f(x)$ does not belong to $C_{3,S} \bigcup C_{4,S}$, then it should belong to $C_{1,S}$. Therefore, $f(x_n) = f(x)$ for all but finitely many $n \in \mathbb{N}$ and so all but finitely many elements of the sequence (x_n) are nonperiodic, which is a contradiction. Therefore, $f(C_{3,S} \bigcup C_{4,S}) \subset C_{3,S} \bigcup C_{4,S}$. Since $f(S) = S$ and $f(C_{3,S} \bigcup C_{4,S}) \subset C_{3,S} \bigcup C_{4,S}$, we have $C_{1,S} \bigcup C_{5,S}$ is also forward f-invariant. □

Proposition 4.24. *Every continuous self-map f on a closed subset S of X can be extended as a continuous self-map g on X such that $P(g) = P(f)$.*

Proof. The proof follows from Proposition 3.5. □

The following theorem gives the characterization of the sets of periodic points of continuous self-maps on X.

Theorem 4.25. *Given a subset S of X, there exists a continuous self-map f on X such that $P(f) = S$ if and only if $\overline{S} \setminus S$ is either empty or infinite.*

Proof. Let $S = P(f)$ for some continuous self-map on X and let $x \in \overline{S} \setminus S$. By continuity of f, we know that $f(x) \in \overline{S}$. Now suppose that $f(x) \in S$. Take a sequence (y_n) in S converging to $f(x)$ such that $f(x_m) \neq y_n$ for any $m, n \in \mathbb{N}$ and any sequence of distinct elements in S converging to $f(x)$ contains infinitely many elements of the sequence (z_n) where (z_n) is a sequence defined by

$$z_n = \begin{cases} f(x_{\frac{n+1}{2}}) & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

If the period of $f(x)$ is k_1, then under the map f^{k_1}, except for finitely many, the image of every element in the sequence (z_n) will be in the sequence (z_n). Let $\mathcal{F} = \{z_n : f^{k_1}(z_n) \neq z_m \text{ for any } m \in \mathbb{N}\}$. Since $\{f^n(x) : x \in \mathcal{F}, n \in \mathbb{N} \bigcup \{0\}\}$ is finite under the map $f^{k_1+k_2}$, except for finitely many, the image of every element in the sequence (z_n) will be in the sequence (z_n) for every $k_2 \in \mathbb{N}$. Then all but finitely many elements of the sequence (x_n) are non-periodic, which is a contradiction. So $f(x) \notin S$.

Therefore, $\overline{S}\setminus S$ is forward f-invariant. Now if $\overline{S}\setminus S$ is nonempty finite, then it contains a periodic point, which is a contradiction to $P(f) = S$. So $\overline{S}\setminus S$ is either empty or infinite.

Now suppose that $\overline{S}\setminus S$ is either empty or infinite. We may assume that $X = \overline{\{m+\frac{1}{n} : m, n \in \mathbb{N}\}}$. Then, by Proposition 4.4, there exists a continuous self-map g_1 on $\overline{\{m+\frac{1}{n} \in S : m \in \overline{S}\setminus S, n \in \mathbb{N}\setminus\{1\}\}}$ such that $P(g_1) = \{m+\frac{1}{n} \in S : m \in \overline{S}\setminus S, n \in \mathbb{N}\setminus\{1\}\}$. Now the map $g : \overline{S} \longrightarrow \overline{S}$, defined by

$$g(x) = \begin{cases} g_1(x) & \text{if } x \in \overline{\{m+\frac{1}{n} \in S : m \in \overline{S}\setminus S, n \in \mathbb{N}\setminus\{1\}\}} \\ x & \text{if } x \notin \overline{\{m+\frac{1}{n} \in S : m \in \overline{S}\setminus S, n \in \mathbb{N}\setminus\{1\}\}}, \end{cases}$$

is a continuous self-map on $\bar{S}$ such that $P(g) = S$. So, by Proposition 4.24, there exists a continuous self-map $f$on X such that $P(f) = S$. □

The characterization of the pairs (S, T) such that $S = P(f)$ and $T = Per(f)$ for some continuous self-map f is the following theorem.

Theorem 4.26. *Given a subset S of X and a subset T of $\mathbb{N}$ there exists a continuous self-map f on X such that $P(f) = S$ and $Per(f) = T$ if and only if the following conditions hold:*

(1) *$\overline{S}\setminus S$ is either empty or infinite.*

(2) *There exist two subsets T_1 and T_2 (which need not be distinct) of T which are the partition sets of $|C_{3,S} \bigcup C_{4,S}|$ and $|C_{1,S} \bigcup C_{5,S}|$, respectively, such that $T_1 \bigcup T_2 = T$, and, if S is compact, then all but finitely many elements of T_2 have divisors in T_1.*

(3) *If T is finite, then S is closed.*

Proof. Suppose that $P(f) = S$ and $Per(f) = T$ for some continuous self-map f on X. (1) has been proved in Theorem 4.25. Let $T_1 = Per(f|_{C_{3,S} \bigcup C_{4,S}})$ and $T_2 = Per(f|_{C_{1,S} \bigcup C_{5,S}})$. Then T_1 and T_2 are the partition sets of $|C_{3,S} \bigcup C_{4,S}|$ and $|C_{1,S} \bigcup C_{5,S}|$, respectively, such that $T_1 \bigcup T_2 = T$. The second part of (2) is trivial if S is finite. Let S be an infinite compact subset of X. Then $S \bigcap X_1$ is nonempty finite, say $\{x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$ with $x_i \neq x_j$ for $i \neq j$. For every i with $1 \leq i \leq n$, let $(x_{ij})_{j\in\mathbb{N}}$ be a sequence in $S \bigcap X_0$ converging to x_i such that $\{x_{ij} : 1 \leq i \leq n, j \in \mathbb{N}\} = S \bigcap X_0$. Since f is continuous, the set $A = \{f^m(x_{ij}) : 1 \leq i \leq n, j, m \in \mathbb{N}, f(y_{ij}) = y_{kl}$ for any $1 \leq k \leq n, l \in \mathbb{N}$ where$f(y_i) = y_k\}$ is finite and the period of every element of $(S \bigcap X_0) \setminus A$ has a divisor in T_1, which proves the second part of (2). Condition (3) has been proved for a general Hausdorff space.

Conversely, suppose that (1)–(3) hold true. By Proposition 4.24, it is enough to prove that there exists a continuous self-map f on $\overline{S}$ such that $P(f) = S$ and $Per(S) = T$. The result is trivial if S is closed. If S

is not closed, then $\overline{S}$ will be homeomorphic to X. Let V and W be two closed disjoint subsets of $\overline{S}$ such that $\overline{S} = V \bigcup W$, $S \bigcap X_1 = V \bigcap X_1$, and $S^c \bigcap X_1 = W \bigcap X_1$. Then it can be observed that W is homeomorphic to X. Let h_1 and h_2 be the self-homeomorphisms on S_1 and S_2, respectively, such that $P(h_1) = V \bigcap S$, $Per(h_1) \subset T$, $P(h_2) = W \bigcap S$, and $Per(h_2) = T$. Then the map $h : \overline{S} \longrightarrow \overline{S}$, defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in V \\ h_2(x) & \text{if } x \in W, \end{cases}$$

is a self-homeomorphism on $\overline{S}$ such that $P(h) = S$ and $Per(h) = T$. $\square$

5. Some Open Problems

Question 5.1. Can $\overline{P(f)} \setminus P(f)$ be nonempty finite for some continuous self-map f on a locally compact Hausdorff space?

Question 5.2. Can $\overline{P(f)} \setminus P(f)$ be nonempty countable for some continuous self-map f on $[0,1]$?

Question 5.3. Can an infinite discrete subset of $[0,1]$ occur as $P(f)$ for some continuous self-map f on $[0,1]$?

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