http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 48, 2016

Pages 361–373

http://topology.nipissingu.ca/tp/

The Size of Multiple Points of Maps between Manifolds

(with an Appendix by Stepan Orevkov)

by

Daciberg L. Gonçalves

Electronically published on March 17, 2016

Topology Proceedings

Web: http://topology.auburn.edu/tp/

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 48 (2016)
Pages 361-373

E-Published on March 17, 2016

THE SIZE OF MULTIPLE POINTS OF MAPS BETWEEN MANIFOLDS (with an Appendix by Stepan Orevkov)

DACIBERG L. GONÇALVES

Abstract. Let $f : M \to N$ be a map between two connected manifolds of the same dimension. A point $x \in M$ is called a *dominating point for f* if $f^{-1}(f(x)) = \{x\}$; otherwise, it is called a *non-dominating point.* For M closed we give a criterion to decide if a given homotopy class of maps has the property that for all maps in the class the set of non-dominating points is dense. Also, we show that when the criterion holds, then the set of non-dominating points cannot be countable. The Appendix provides an example of a map $f : S^2 \to R^2$ such that the set of dominating points is dense (or, equivalently, the set of non-dominating points doesn't contain an open set). Some facts about the size of the dominating points are derived.

1. Introduction

In this work we will consider continuous maps between two manifolds M and N of the same dimension where the domain M is assumed to be closed and the target N can be arbitrary. Given a map $f : M \to N$ we say that $x \in M$ is a *dominating point for the map f* if $f^{-1}(f(x)) = \{x\}$; otherwise, it is called a *non-dominating point.* Very rarely we have that a map f is injective or, equivalently, the set of non-dominating points is

2010 *Mathematics Subject Classification.* Primary 55M25; Secondary 54C05, 54C10.

Key words and phrases. degree, multiple points, single points.

The visit of the author to the Laboratoire Emile Picard-Université Paul Sabatier was partially supported by FAPESP-Fundação de Amparo a Pesquisa do Estado de São Paulo, Projeto Temático Topologia Algébrica, Geométrica e Differential-2008/57607-6.

empty. We would like to study the set of dominating and non-dominating points. More precisely we discuss how big (in some sense) these sets are.

The study of dominating and non-dominating points is related to the study of the so-called deficient and non-deficient points. One may be able to profit from results about deficient and non-deficient points to obtain some results about the set of dominating and non-dominating points. Recall from [7] that for a map $f : M \to N$ between closed orientable manifolds a point $y \in N$ is called *deficient* if y belongs to the image of f and $\# f^{-1}(y) < |deg(f)|$. Denote by Δ_f the set of deficient points of f. We will use the results of [7] in our study.

In our context, since the manifolds are not necessarily orientable, we will need the equivalent notion of the classical $deg(f)$, defined for maps between closed and orientable manifolds of the same dimension, when the manifolds are not necessarily orientable and closed. So we will use the notion of absolute degree. For more details about the absolute degree, see [5], [4], [2], [6].

It is worthwhile to mention the remarkable works by Heinz Hopf [4], [5], where the concept of the absolute degree was established and many properties of maps, related to the degree, were explored. Notoriously, related to the present work, it follows from a result in [5] that for maps between 2-manifolds, the set of deficient points is discrete. See Corollary 3.2(d). For dimension greater than two see [3] for related results.

We are interested in the following two questions.

Question 1.1. In terms of the absolute degree of a map f, what can we say about the size of the non-dominating points of f?

Question 1.2. In terms of the absolute degree of a map f, how large can the set of the dominating points for maps homotopic to f be?

Recall that M and N are manifolds of the same dimension with M closed. The main results of this work, which are related to the questions above, are found in the following.

Theorem 2.2. *Let $[f]$ be the homotopy class of a map $f : M \to N$.*

(a) *If the absolute degree of the map f is 1, then there is a map $g \in [f]$ such that the set of non-dominating points is not dense in M.*

(b) *If the map f has absolute degree different from 1 then the set of non-dominating points of any map $g \in [f]$ is dense.*

Corollary 3.2. (a) *For any pair of manifolds M and N of the same dimension with M closed and a non-negative integer $d \neq 1$, the set of dominating points of any map $f : M \to N$ of absolute degree d cannot contain an open set.*

(b) *For each non-negative integer* $d \neq 1$, *there are manifolds* M *and* N *with* $dim(M) = dim(N) > 2$ *and a map* $f : M \to N$ *which has absolute degree* d *such that the set of dominating points is dense.*

(c) *There are manifolds* M *and* N *with* $dim(M) = dim(N) = 2$ *and a map* $f : M \to N$ *which has absolute degree* 0 *such that the set of dominating points is dense.*

(d) *For any map* $g : M \to N$ *with* $dim(M) = dim(N) = 2$ *and absolute degree of* $g > 1$, *the set of dominating points of* g *is never dense.*

Example 1.3 (Appendix)**.** *There is a map* $f : S^2 \to R^2$ *such that the set of dominating points is dense but does not contain an open set.*

Remark 1.4. (1) If $dimM = dimN = 1$, Corollary 3.2(b) for $d > 1$ does not hold because in this case the set of dominating points is empty by [7]. For $d = 0$ using elementary topology one can show that the set of dominating points contains at most two points. So the result also does not hold.

(2) Related to (b) when the absolute degree of f is 1, it is not clear if it is possible to have a map $g \in [f]$ such that the set of non-dominating points of g is dense.

The results of Theorem 2.2 and Corollary 3.2 lead us to the following more intricate question.

Classification Question 1.5. *Let* M *and* N *be two manifolds of the same dimension and let* d *be a nonnegative integer. Classify all homotopy classes of maps* $\alpha \in [M, N]$ *where* α *has absolute degree* d *for which there is a map* $f \in \alpha$ *such that the set of dominating points of* f *is dense.*

A similar question can be asked replacing the set of dominating points by non-dominating points. The degree very often does not classify the homotopy class. For example, consider maps $S_2 \to T$ from the orientable surface of genus 2 into the torus.

This note was motivated by a very simple application of the classical Borsuk-Ulam theorem for continuous maps from S^2 to R^2 and the antipodal map $A : S^2 \to S^2$. In more detail, let us consider the question, Does there exist a continuous map $f : S^2 \to R^2$ which is injective? As a result of the Borsuk-Ulam theorem, given any continuous map $f : S^2 \to R^2$ it follows that there is a point $x \in S^2$ such that $f(x) = f(A(x)) = f(-x)$. Therefore, the map f is never injective. Using the terminology introduced above, we can say that f admits at least two non-dominating points. Furthermore, because the Borsuk-Ulam theorem also holds for any free involution $\tau : S^2 \to S^2$ on S^2, this new setting provides possibly another point $x_1 \in S^2$ such that $f(x_1) = f(\tau(x_1))$. This opens the possibility that the number of non-dominating points is larger than two. The results of this work show that we know much more about the non-dominating points.

The manuscript is organized into two sections and an appendix (by Stepan Orekov) besides the introduction. In §2, we study the non-dominating points. The main result is Theorem 2.2. Also, we show that for maps between certain spaces, the set of non-dominating points is always uncountable. Section 3 is devoted to the set of dominating points. The main result is Corollary 3.2. In the Appendix an example is constructed where the set of dominating points is dense which is equivalent showing that the set of non-dominating points does not contain an open set.

2. The Set of Non-Dominating Points

Let $f : M \to N$ be a map between two manifolds of the same dimension with M closed. In this section we study the set of non-dominating points of f. We make use of the concepts of the geometric degree (see Definition 2.1) and the absolute degree (see [5], [4], [2], [6]), where we follow more closely the more modern presentation given in [2]. Then we state and prove our main results and illustrate some applications of the result.

Definition 2.1. The *geometric degree* $G(f)$ of f is defined as follows. If there is no disk D in $intN$ such that $f^{-1}(D)$ consists of a finite number of disks, each mapped homeomorphically onto D, we define $G(f) = \infty$. If such disks do exist, let $G(f)$ be the smallest integer such that $f^{-1}(D)$ has $G(f)$ components, each mapped homeomorphically onto D.

Theorem 2.2. *Let $[f]$ be the homotopy class of a map $f : M \to N$.*

(a) *If the absolute degree of the map f is* 1*, then there is a map $g \in [f]$ such that the set of non-dominating points is not dense in M.*

(b) *If the map f has absolute degree different from* 1*, then the set of non-dominating points of any map $g \in [f]$ is dense.*

Proof. For (a), let $A(f) = 1$. By [2, Theorem 4.1], we have a map g of geometric degree 1 which is homotopic to f. But from the definition of the geometric degree, it follows immediately that the set of non-dominating points of g is not dense.

For (b) we argue by contradiction. Suppose that the set of non-dominating points of f is not dense in M. This implies that there is a dominating point $x \in M$ of f and a closed neighborhood $\bar{U}$ of x such that $\bar{U}$ does not contain non-dominating points. Let $V \subset U$ be an Euclidean neighborhood of x such that $\bar{V} \subset U$. Therefore, the geometric degree of f is either 0 or 1. From [2, Theorem 4.1], it follows that it has to be the same as the absolute degree, so it is zero. Let $V \subset U$ be a Euclidean neighborhood of x such that $\bar{V} \subset U$. Since M is a closed manifold we have isomorphisms

$$H_n(M, Z_2) \to H_n(M, M - V, Z_2) \leftarrow H_n(\bar{V}, \bar{V} - U, Z_2),$$

where the second homomorphism is the excision isomorphism. The map f induces a commutative diagram

$$\begin{array}{ccc} H_n(M, Z_2)\to & H_n(M, M - V, Z_2)\leftarrow & H_n(\bar{V}, \bar{V} - V, Z_2) \\ \downarrow & \downarrow & \downarrow \\ H_n(N, Z_2)\to & H_n(N, N - V_1, Z_2)\leftarrow & H_n(\bar{V}_1, \bar{V}_1 - V_1, Z_2). \end{array}$$

The two horizontal homomorphisms on the top and the last two vertical homomorphisms on the right are isomorphisms, so this implies that the absolute degree of f is congruent to one, which is a contradiction, and the result follows. □

If case N is not closed, then we easily state the following.

Corollary 2.3. *Let $f : M \to N$ be an arbitrary map where M and N are manifolds of the same dimension with M a closed manifold and N not a closed manifold. Then the set of the non-dominating points of f is dense.*

Proof. The proof follows promptly from the fact that the absolute degree of f is 0 since N has top cohomology trivial. □

Here we provide a few examples which are either related to or illustrate Theorem 2.2 above.

(1) For the identity map $id : M \to M$ the set of non-dominating points is empty.

(2) Take the orientable surface S_h of genus h and the map from S_h to the surface S_g of genus g, $h \geq g$ which pinches $h - g$ handle of S_h. Then the points of the complement of the handles of S_h are dominating points. Therefore, the set of non-dominating points is not dense. That pinch map has degree 1.

(3) Consider any map f from the orientable surface S_h (of genus h) to the orientable surface S_k (of genus k). If $h < k$, then the set of non-dominating points of f is dense. This follows because, from [6], all such maps have degree zero.

(4) For some pair M and N of manifolds of the same dimension, the problem of deciding which integers can be realized as the degree of some map from M to N, has been studied extensively, in particular, when the manifolds have dimension 3. By looking at those results we can provide more examples of pairs where, for all maps, the set of non-dominating points is dense. More specifically, there are pairs of lens spaces which do not admit a map of degree one, for example, if $M = S^3$ (the 3-sphere) and $N = RP^3$. Therefore, for any map $S^3 \to RP^3$, the set of non-dominating points is dense.

To have a better understanding of the set of non-dominating points in case the degree of the map f is different from ± 1, two natural questions arise:

(I) Does the set of non-dominating points always contains an open set?
(II) Can the set of non-dominating points be countable?

The answer to the first question is no and an example is given in the Appendix. The answer to the second question, at least in the case of maps from S^n into R^n, is also negative which we will show. The statement generalizes for a few other similar cases, but we do not know the answer in general.

Proposition 2.4. *For every map $f : S^n \to R^n$, the set of non-dominating points is not countable.*

Proof. If f is a constant map the result is certainly true. So let us assume that f is not constant. So the projection of $f(S^n)$ to one of the axes is not a point. Call this axes x. Let p be a leftmost and q a rightmost point of the projection of $f(S^n)$ on the axis-x. Suppose that there is a hyperplane L perpendicular to the axis x which separates p and q such that all points of $X = f^{-1}(L)$ are dominating. Otherwise, all hyperplanes L perpendicular to the axis x contain a non-dominating point and the result follows since the cardinality of the vertical hyperplanes is uncountable. Then X is homeomorphic to a closed subset $f(X) \subset L = R^{n-1} \subset S^{n-1}$. Using Poincaré duality (see [1, Ch. VIII, Proposition 7.2]), we have that $\check{H}^{n-1}(f(X), Z) = H_0(S^{n-1}, S^{n-1} - f(X); Z) = 0$ where the last equality follows because $H_0(S^{n-1} - f(X)) \to H_0(S^{n-1}; Z) = Z$ is surjective, and X separates the sphere. But we claim that this is impossible. To see this, since $X \subset S^n$ is a closed set, using Poincaré duality (see [1, Ch. VIII, Proposition 7.2]), we have that $\check{H}^{n-1}(X, Z)$ is isomorphic to $H_1(S^n, S^n - X, Z)$. The long exact sequence of the pair $(S^n, S^n - X, Z)$ provides the short exact sequence

$$0 \to H_1(S^n, S^n - X; Z) \to H_0(S^n - X, Z) \to H_0(S^n, Z) \to 0.$$

Since $H_1(S^n, S^n - X; Z) = 0$ and $H_0(S^n, Z) = Z$, from the short exact sequence above, it follows that $H_0(S^n - X, Z) = Z$, which is a contradiction since $S^n - X$ is not path-connected. So the result follows. □

3. The Set of Dominating Points

Here we show some results for maps between two manifolds relative to the set of the dominating points, where we make use of the results from [7] and from the previous section.

We begin by recalling some results from [7] which are related to the study of the dominating points. The study of dominating and non-dominating points is related to the study of the so-called deficient and non-deficient points. We will use some results about deficient and non-deficient points to obtain some results about the set of dominating and non-dominating points. Recall that for a map $f : M \to N$ between closed orientable manifolds a point $y \in N$ is called *deficient* if y belongs to the image of f and $\#f^{-1}(y) < |deg(f)|$. Denote by Δ_f the set of deficient points of f. Clearly, if $|deg(f)| \leq 1$, then Δ_f is empty. In [7] an example is constructed of a map $f : S^q \to S^q$ such that $|deg(f)| = d$, $f^{-1}(\Delta_f)$ is a dense subset, and the restriction of f is a homeomorphism from $f^{-1}(\Delta_f)$ to Δ_f for each pair of integers $q \geq 3$ and $d \geq 2$. In [7] for each pair of integers $q \geq 3$ and $d \geq 2$ an example is constructed of a map $f : S^q \to S^q$ such that $|deg(f)| = d$, $f^{-1}(\Delta_f)$ is a dense subset, and the restriction of f is a homeomorphism from $f^{-1}(\Delta_f)$ to Δ_f. Therefore, the examples above are examples where the set of dominating points is dense. For $deg(f) = \pm 1$ let f be the identity and the map which changes the sign of one coordinate, respectively. They are examples of maps where the set of dominating points is dense. So it remains the question for $deg(f) = 0$.

Proposition 3.1. *Let S_1 and S_2 be two surfaces with S_1 a closed surface. There is a map $f : S_1 \to S_2$ of degree zero such that the set of dominating points is dense.*

Proof. Let us consider the natural projection $p : S_1 \to S^2$ such that the preimage of the north pole is the boundary of a polygon used to define the surface S_1. Now consider the map $h : S^2 \to R^2$, constructed in the Appendix, an embedding $\iota : R^2 \to S_2$, and finally the composite $\iota \circ h \circ p : S_1 \to S_2$. The set of dominating points of h is dense since the set of non-dominating points does not contain an open set. So the composite shows the result. $\square$

A related and more subtle question is to ask, If for given manifolds M and N of the same dimension, for which the homotopy class of maps from M to N of degree zero, can we find a map g in the class such that the set of dominating points is dense? One may try to construct such a map g using a variation of the example in the Appendix.

Now we come to the main result of the section.

Corollary 3.2. (a) *For any pair of manifolds M and N of the same dimension with M closed and a non-negative integer $d \neq 1$, the set of dominating points of any map $f : M \to N$ of absolute degree d cannot contain an open set.*

(b) *For each non-negative integer $d \neq 1$, there are manifolds M and N with $dim(M) = dim(N) > 2$ and a map $f : M \to N$ which has absolute degree d such that the set of dominating points is dense.*

(c) *There are manifolds M and N with $dim(M) = dim(N) = 2$ and a map $f : M \to N$ which has absolute degree 0 such that the set of dominating points is dense.*

(d) *For any map $g : M \to N$ with $dim(M) = dim(N) = 2$ and absolute degree of $g > 1$, the set of dominating points of g is never dense.*

This corollary is an easy consequence of the Appendix, Theorem 2.2, and known results in the literature.

Proof. Part (a) is a direct consequence of Theorem 2.2.

Part (b) follows immediately from the main result of [7].

Part (c) follows from Proposition 3.1.

Part (d) follows from [5]. □

Remark 3.3. Corollary 3.2 motivates the following question: For which pairs of manifolds M and N does the conclusion of (b) hold?

Acknowledgments. This work began during the visit of the author to the Laboratoire Emile Picard-Université Paul Sabatier. He thanks the Topology group there for the invitation and the hospitality. The author also thanks S. Spiez for helpful discussions and S. Orevkov not only for providing the Appendix and fruitful discussions but also for pointing out reference [7]. Also, he would like to thank the referee for his/her careful reading and comments.

References

[1] Albrecht Dold, *Lectures on Algebraic Topology.* (1972). Classics in Mathematics. New York-Berlin: Springer-Verlag, 1995.

[2] D. B. A. Epstein, *The degree of a map*, Proc. London Math. Soc. (3) **16** (1966), 369–383.

[3] Matti Honkapohja, *Degree and point-inverses of mappings on spheres*, Ann. Acad. Sci. Fenn. Ser. A I **447** (1969), 1–5,

[4] Heinz Hopf, *Zur Topologie der Abbildungen von Mannigfaltigkeiten: Erster Teil. neue Darstellung der Theorie des Abbildungsgrades für topologische Mannigfaltigkeiten*, Math. Ann. **100** (1928), no. 1, 579–608.

[5] ______, *Zur Topologie der Abbildungen von Mannigfaltigkeiten: Zweiter Teil Klasseninvarianten von Abbildungen*, Math. Ann. **102** (1930), no. 1, 562–623.

[6] Hellmuth Kneser, *Die kleinste Bedeckungszahl innerhalb einer Klasse von Flächenabbildungen*, Math. Ann. **103** (1930), no. 1, 347–358.

[7] John J. Walsh, *Mappings with dense deficiency set*, Michigan Math. J. **28** (1981), no. 1, 117–127.

APPENDIX

by Stepan Orevkov

Example. *There is a continuous mapping $f : S^2 \to R^2$ whose set of dominating points is dense in S^2.*

Using the Cantor function, we construct a continuous mapping $f : S^2 \to \mathbb{R}^2$ such that $\{p \in S^2 \mid f^{-1}(f(p)) = \{p\}\}$ – the set of dominating points of f – is dense in S^2. The image of f in our example is an infinite binary tree. This construction answers a question posed to me by Daciberg Gonçalves.

1. **Preliminaries**

Let $B = \{0,1\}^\infty$ be the set of all binary sequences $(b^{(1)}, b^{(2)}, \dots)$ where $b^{(i)} \in \{0,1\}$, $i = 1,2,\dots$ and only finite numbers of $b^{(i)}$ are nonzero. For $b \in B$, we define its *length* as $\operatorname{len}(b) = \max\{\, n \mid b^{(n)} = 1 \,\}$ and we set $B_n = \{\, b \in B \mid \operatorname{len}(b) = n \,\}$. If $b = (b^{(1)}, b^{(2)}, \dots)$ is a binary sequence of length n, we shall represent it by a word (without any delimiters) $b^{(1)} \dots b^{(n)}$, i.e., we shall write just 0101 instead of $(0,1,0,1,0,0,\dots)$. Thus, we have $B_0 = \varnothing$,

$B_1 = \{1\}$, $B_2 = \{01, 11\}$, $B_3 = \{001, 011, 101, 111\}$, etc., and we have $B = \bigcup_{n=1}^\infty B_n$.

For $b \in B_n$, let $y(b)$ be the binary number

$$y(b) = 0\,.\,b^{(1)} b^{(2)} \cdots = \sum_{k \ge 1} b^{(k)}/2^k$$

and let $t(b)$ be the ternary number

$$t(b) = 2 \times 0\,.\,b^{(1)} b^{(2)} \cdots = \sum_{k \ge 1} 2b^{(k)}/3^k.$$

Let $F : [0,1] \to [0,1]$ be a Cantor function, i.e., the monotone function uniquely determined by the condition that

$$F(t(b)) = F(t(b) - 3^{-n}) = y(b) \quad \text{for } b \in B_n$$

(see Figure 1). For $b \in B_n$, let I_b be the closed interval

$$I_b = F^{-1}(y(b)) = [t(b) - 3^{-n}\,,\, t(b)]$$

(see Figure 1). Let $\mathbf{B} = \bigcup_{m=1}^\infty \mathbf{B}_m$ where $\mathbf{B}_m = \{(b_1, \dots, b_m) \mid b_i \in B\,\}$. We shall identify $\mathbf{B}_1$ with B and $\operatorname{len}(\vec{b}) = \operatorname{len}(b_1) + \dots + \operatorname{len}(b_m)$. For $\vec{b} = (b_1, \dots, b_m) \in \mathbf{B}_m$, we denote $\vec{b}' = (b_1, \dots, b_{m-1}) \in \mathbf{B}_{m-1}$. We write $\vec{b}_1 \prec \vec{b}_2$ if $\vec{b}_1$ is an initial segment of $\vec{b}_2$, i.e., $\vec{b}_1 = (b_1, \dots, b_{m_1})$ and $\vec{b}_2 = (b_1, \dots, b_{m_1}, b_{m_1+1}, \dots, b_{m_2})$.

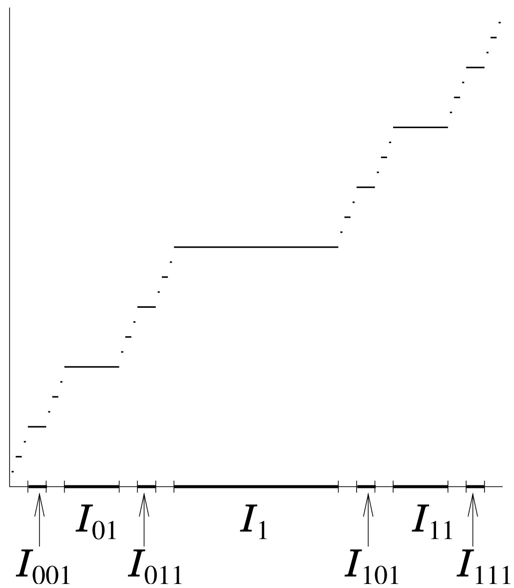

FIGURE 1

2. **Construction of Annuli**

Let $\mathbb{D}$ be the closed unit disk in $\mathbb{C}$. For $b \in B$, let us denote the annulus $\{z \in D : |z| \in I_b\} = \{z \in D : F(|z|) = y(b)\}$ by A_b. Let $\mathbb{U} = \{U_1, U_2, \dots\}$ be some countable base of the standard topology in $\mathbb{D}$. For any $\vec{b} = (b_1, \dots, b_m) \in \mathbf{B}$, we define an annulus $A_{\vec{b}}$, a distinguished point $p_{\vec{b}}$ in it, and a mapping $\varphi_{\vec{b}} : \mathbb{D} \to A_{\vec{b}}$. We shall define them inductively, first, for $\vec{b}$ with $\operatorname{len}(\vec{b}) = 1$, then for all $\vec{b}$ with $\operatorname{len}(\vec{b}) = 2$, then for all $\vec{b}$ with $\operatorname{len}(\vec{b}) = 3$, etc.

If $\vec{b} = (b_1) \in \mathbf{B}_1$, then we set $A_{\vec{b}} = A_{b_1}$.

If $A_{\vec{b}}$ is already defined, then we choose $p_{\vec{b}}$ as any point in $\operatorname{Int} A_{\vec{b}} \cap U_k$ where k is the minimal number such that $\operatorname{Int} A_{\vec{b}} \cap U_k$ is non-empty and U_k was not used on previous steps.

If $p_{\vec{b}}$ is already defined, then we define $\varphi_{\vec{b}} : \mathbb{D} \to A_{\vec{b}}$ as a continuous map such that

- $\varphi_{\vec{b}}(0) = p_{\vec{b}}$,
- $\varphi_{\vec{b}}(\mathbb{D}) = A_{\vec{b}}$,
- $\varphi_{\vec{b}}$ maps $\operatorname{Int} \mathbb{D}$ homeomorphically onto a dense open subset of $A_{\vec{b}}$.

If $\varphi_{\vec{b}'}$ is already defined, then we set $A_{\vec{b}} = \varphi_{\vec{b}'}(A_{b_m})$. We have depicted some of the annuli $A_{\vec{b}}$ in Figure 2.

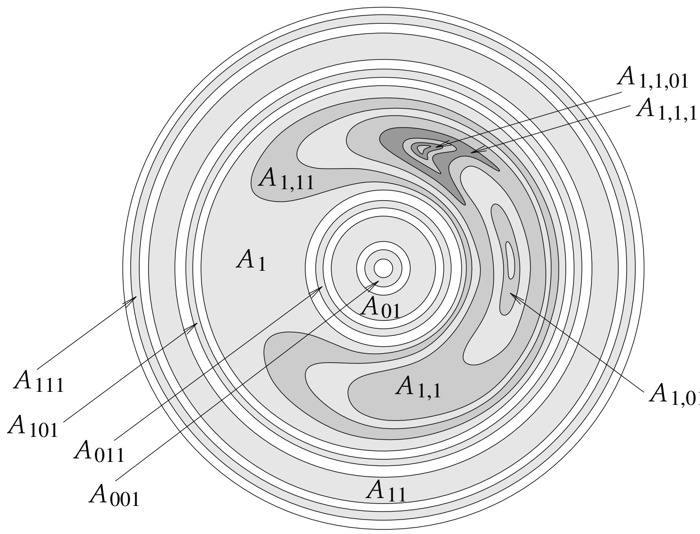

FIGURE 2

Let us set $A_m = \bigcup_{\vec{b}\in\mathbf{B}_m} \operatorname{Int} A_{\vec{b}}$, $A = \bigcap_{m=1}^{\infty} A_m$, and $P = \{p_{\vec{b}} \mid \vec{b} \in \mathbf{B}\}$.

Remark 2.1. Using conformal mappings, we can choose $\varphi_{\vec{b}}$ in a canonical way. Namely, we can set $\varphi = \varphi_1^{-1} \circ \varphi_2$ where φ_1 is the conformal mapping of $\operatorname{Int} A_{\vec{b}}$ onto $A_r = \{ z : r < |z| < 1\}$ such that $\varphi_1(p_{\vec{b}}) \in [r, 1]$ (r is uniquely determined by $A_{\vec{b}}$) and φ_2 is a conformal mapping of $\operatorname{Int}\mathbb{D}$ onto $A_r \setminus [-1, -r]$ such that $\varphi_2(0) = \varphi_1(p_{\vec{b}})$.

Lemma 2.2. *A and P are dense in $\mathbb{D}$.*

Proof. The fact that A is dense in $\mathbb{D}$ is an immediate consequence from Baire's theorem.

Let us prove by induction that each U_k contains a point of P. Suppose we know already that this is true for $U_1, \dots, U_{k-1}$.

Since A is dense, there exists a point z in $A \cap U_k$. It belongs to each A_m; hence, for any $m = 1, 2, \dots$, there is $\vec{b}_m \in \mathbf{B}_m$ such that $z \in \operatorname{Int} A_{\vec{b}_m}$. Let m be the minimal number such that $p_{\vec{b}_m}$ was not yet defined at the moment when all of $U_1, \dots, U_{k-1}$ had been used. Then $p_{\vec{b}_m}$ must be chosen in $\operatorname{Int} A_{\vec{b}_m} \cap U_k$ because it is non-empty (it contains z). $\square$

3. **Construction of a Fractal Tree**

Let us define an infinite tree T embedded into $\mathbb{R}^2$ as follows. Let $I = [0, 1]$ and let $\lambda_0 : I \to \mathbb{R}^2$ be a non-constant linear mapping, say, $\lambda_0(t) =$

$(t, 0)$. For any $\vec{b} \in \mathbf{B}$, we shall define a linear mapping $\lambda_{\vec{b}}$ inductively as follows. If $m = 0$ (i.e., $\vec{b}$ is empty), we set $\lambda_{\vec{b}} = \lambda_0$. If $\vec{b} = (b_1, \ldots, b_m)$ and $\lambda_{\vec{b}'}$ is already defined, then we set $\lambda_{\vec{b}}(t) = (1-t)e_{\vec{b}} + ta_{\vec{b}}$ where $a_{\vec{b}} = \lambda_{\vec{b}'}(y(b_m))$, the segment $\lambda_{\vec{b}}(I) = [a_{\vec{b}}, e_{\vec{b}}]$ is orthogonal to the segment $\lambda_{\vec{b}'}(I)$ (the direction is not so important, we can choose it, for instance, as in Figure 3), and the length of the segment $\lambda_{\vec{b}}(I)$ is $3^{-\operatorname{len}(\vec{b})}$ (recall that $\operatorname{len}(\vec{b}) = \operatorname{len}(b_1) + \cdots + \operatorname{len}(b_m)$). Let $T = \bigcup_{\vec{b} \in \mathbf{B}} \lambda_{\vec{b}}(I)$ (see Figure 3). We shall call the points $a_{\vec{b}}$ and $e_{\vec{b}}$ the *nodes* and the *ends* of T, respectively. Let us denote the branch at $a_{\vec{b}}$ by $T_{\vec{b}}$, i.e., $T_{\vec{b}} = \bigcup_{\vec{b} \prec \vec{b}_1} \lambda_{\vec{b}_1}(I)$. By construction, $T_{\vec{b}} \subset \Delta_{\vec{b}}$ where $\Delta_{\vec{b}}$ is the triangle with vertices $a_{\vec{b}}$, $e_{\vec{b}}$, and $e_{\vec{b},1}$. In Figure 3, we depict the triangles Δ_1 and $\Delta_{1,11}$, i.e., the triangles $\Delta_{\vec{b}}$ for $\vec{b} = (1) \in \mathbf{B}_1$ and for $\vec{b} = (1, 11) \in \mathbf{B}_2$.

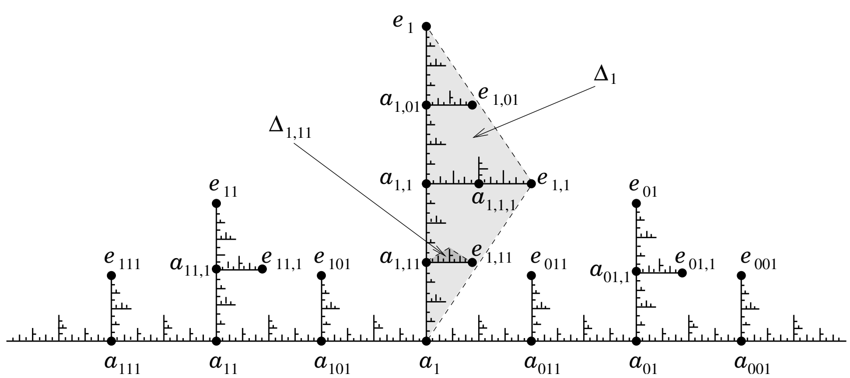

Figure 3

One can check that $\Delta_{\vec{b}_1} \supset \Delta_{\vec{b}_2}$ if $\vec{b}_1 \prec \vec{b}_2$ and $\Delta_{\vec{b}_1} \cap \Delta_{\vec{b}_2} = \varnothing$ otherwise. This implies that the segments of T meet each other only at nodes (in particular, the ends cannot lie on other segments).

4. Construction of the Mapping

Let us define $f : \mathbb{D} \to T$ as $f = \lim_{m \to \infty} f_m$ where the mappings f_m are inductively constructed as follows.

Let $f_0(z) = \lambda_0(F(|z|))$, where λ_0 has been defined in the previous subsection. Then f_0 is continuous and it contracts each annulus A_b into the node a_b. Suppose that f_{m-1} is already constructed. Then we set

$$f_m(z) = \begin{cases} \lambda_{\vec{b}}\big(F\big(|\varphi_{\vec{b}}^{-1}(z)|\big)\big) & \text{if } z \in \operatorname{Int} A_{\vec{b}} \text{ for } \vec{b} \in \mathbf{B}_m \\ f_{m-1}(z) & \text{otherwise.} \end{cases}$$

It is clear that if $\vec{b} \in \mathbf{B}_m$, then f_m maps $A_{\vec{b}}$ onto the segment $\lambda_{\vec{b}}(I) = [a_{\vec{b}}, e_{\vec{b}}]$ so that $\partial A_{\vec{b}}$ is mapped to the node $a_{\vec{b}}$, the distinguished point $p_{\vec{b}}$ is mapped to the end $e_{\vec{b}}$, and each annulus $A_{\vec{b},b}$ is contracted to the node $a_{\vec{b},b}$.

Using the fact that $f_m(\partial A_{\vec{b}}) = f_{m-1}(\partial A_{\vec{b}}) = a_{\vec{b}}$ for $\vec{b} \in \mathbf{B}_m$, it is easy to prove by induction that each f_m is continuous.

Let us show that f is continuous. Indeed, when we pass from f_{m_1} to f_{m_2}, we modify f_{m_1} on each $A_{\vec{b}}$, where $\vec{b} \in \mathbf{B}_{m_1}$, replacing the value $a_{\vec{b}}$ by values lying in $T_{\vec{b}} \subset \Delta_{\vec{b}}$ and the diameter of $\Delta_{\vec{b}}$ tends to zero as $m \to \infty$. Thus, $\{f_m\}$ is a Cauchy sequence in the metric of uniform convergence.

In fact, f can be characterized as the continuous mapping $\mathbb{D} \to \mathbb{R}^2$ uniquely defined either by the condition $f(\partial A_{\vec{b}}) = a_{\vec{b}}$ for any $\vec{b} \in \mathbf{B}$ or by the condition that $f(A_{\vec{b}}) = T_{\vec{b}}$ for any $\vec{b} \in \mathbf{B}$.

Since f is constant on $\partial\mathbb{D}$, it can be considered as a continuous mapping of the sphere obtained from $\mathbb{D}$ by contracting the boundary. Let E be the set of ends of T, i.e., $E = \{\, e_{\vec{b}} \,|\, \vec{b} \in \mathbf{B}\,\}$. It is clear that each point of E has only one preimage and $f^{-1}(E) = P$ is dense in the sphere.

References are on page 368.

(Gonçalves) Departamento de Matemática; IME-USP; Caixa Postal 66281; Ag. Cidade de São Paulo; CEP: 05314-970 - São Paulo - SP - Brasil
E-mail address: dlgoncal@ime.usp.br

(Orevkov) Institut de Mathématiques de Toulouse; Université Paul Sabatier; 118 Route de Narbonne; 31400 Toulouse France
E-mail address: stepan.orevkov@math.univ-toulouse.fr