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by

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ADDING A CONVERGENT SEQUENCE

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ABSTRACT. Let p be a non-isolated point in a space X. Suppose that no sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ converges to p. We investigate in what circumstances can a cardinal-preserving forcing add a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.

1. INTRODUCTION

Let $\langle X, \tau \rangle$ be a topological space and let $\mathbb{P}$ be a notion of forcing. Let $\mathbf{V}$ be a ground model and let $\mathbf{V}^{\mathbb{P}}$ be the forcing extension of $\mathbf{V}$ by $\mathbb{P}$. We define in $\mathbf{V}^{\mathbb{P}}$ a topological space $\langle X, \tau^{\mathbb{P}} \rangle$ such that $\tau^{\mathbb{P}} = \{\bigcup S : S \subseteq \tau\}$; that is, $\tau^{\mathbb{P}}$ is the topology on X generated by τ in $\mathbf{V}^{\mathbb{P}}$. We observe that in general $\tau \subsetneqq \tau^{\mathbb{P}}$ because new open sets are added by $\mathbb{P}$. Also we note that τ serves as a base for $\tau^{\mathbb{P}}$.

Let p be a non-isolated point in a space $\langle X, \tau \rangle$. Suppose that in $\mathbf{V}$ no sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ converges to p. We investigate in what circumstances can we add a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ which converges to p in the space $\langle X, \tau^{\mathbb{P}} \rangle$ by a cardinal-preserving forcing $\mathbb{P}$.

First let us illustrate an example where a forcing adds a convergent sequence.

Example 1.1. There exist a space $\langle X, \tau \rangle$, a point $p \in X$ and a forcing $\mathbb{P}$ with the countable chain condition (ccc) such that:

(1) in $\langle X, \tau \rangle$, no sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ converges to p, and
(2) in $\langle X, \tau^{\mathbb{P}} \rangle$, there is a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.

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Proof. Let $X = \{p\} \cup (\omega \times \omega)$. Each point in $\omega \times \omega$ is isolated and a basic open neighborhood of p has the form:

$$U(f, n) := \{p\} \cup \{\langle i, j\rangle : i \geq n \text{ and } j \geq f(i)\}$$

for some $f \in {}^{\omega}\omega$ and $n < \omega$. It is easy to see that no sequence in $\omega \times \omega$ converges to p. Now we add a sequence which converges to p. Let $\mathbb{P}$ be a ccc forcing which adjoins a dominating real g; that is, for every $f \in {}^{\omega}\omega \cap \mathbf{V}$, $f(i) < g(i)$ for all but finitely many $i < \omega$. (See, for example, [1] Definition 3.1.9.) To show that the sequence $\{\langle i, g(i)\rangle : i < \omega\}$ converges to p, fix a neighborhood $U(f, n)$ of p. Then there is an $m \geq n$ such that for all $i \geq m$, $f(i) < g(i)$ and so $\langle i, g(i)\rangle \in U(f, n)$. Thus, in $\langle X, \tau^{\mathbb{P}}\rangle$, the sequence $\{\langle i, g(i)\rangle : i < \omega\}$ converges to p. □

2. A point in the closure of a countable set

In Example 1.1, the point p is in the closure of a countable set; that is, $p \in \overline{\omega \times \omega}$. In fact, if a point is in the closure of a countable set, then it is always possible to produce a sequence converging to the point by a cardinal-preserving forcing. To prove this fact, we use the following partially ordered set defined by D. Booth.

Definition 2.1. ([3]) Let C be a countable infinite set and let λ be a cardinal. Suppose $\mathcal{F} = \{F_\xi \subseteq C : \xi < \lambda\}$ is a non-principal filter on C. Define

$$\mathbb{P}(\mathcal{F}) = \{\langle s, E\rangle : s \in [C]^{<\omega} \text{ and } E \in [\lambda]^{<\omega}\}.$$

Order $\mathbb{P}(\mathcal{F})$ by: $\langle s, E\rangle \leq \langle s', E'\rangle$ iff $s \supseteq s', E \supseteq E'$ and $s - s' \subseteq \bigcap_{\xi \in E'} F_\xi$.

Proposition 2.2. *Let $\mathbb{P}(\mathcal{F})$ be as in Definition 2.1. $\mathbb{P}(\mathcal{F})$ has the following properties:*

(1) *$\mathbb{P}(\mathcal{F})$ has the countable chain condition (ccc).*
(2) *For $n < \omega$ and $\xi < \lambda$, $D_{n\,\xi} := \{\langle s, E\rangle : |s| \geq n,\ \xi \in E\}$ is a dense subset of $\mathbb{P}(\mathcal{F})$.*
(3) *For a filter $G \subseteq \mathbb{P}(\mathcal{F})$ such that $G \cap D_{n\,\xi} \neq \emptyset$ for all $n < \omega$ and $\xi < \lambda$, let $A = \bigcup\{s : \langle s, E\rangle \in G\}$; then $|A| = \aleph_0$ and $A \setminus F_\xi$ is finite for all $\xi < \lambda$.*

Using the partially ordered set defined above, we prove the theorem below, which says that we can add a sequence which converges to p if p is in the closure of a countable set.

Theorem 2.3. *Let $\langle X, \tau\rangle$ be a space and let $p \in X$. Suppose that there is a countable set $C \subseteq X \setminus \{p\}$ such that $p \in \overline{C}$. Then there is a ccc forcing $\mathbb{P}$ such that in $\langle X, \tau^{\mathbb{P}}\rangle$, there exists a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.*

Proof. Let C be as above and let $\mathcal{N}_p$ be a neighborhood base at p. Then $\mathcal{F} := \{U \cap C : U \in \mathcal{N}_p\}$ is a non-principal filter on C. Enumerate $\mathcal{F} = \{F_\xi : \xi < \lambda\}$. Let G be a generic subset of $\mathbb{P}(\mathcal{F})$ and let $A = \bigcup\{s : \langle s, E\rangle \in G\}$. To show that $A = \{a_n : n < \omega\}$ converges to p, fix $U \in \mathcal{N}_p$. (Note that in $\langle X, \tau^{\mathbb{P}}\rangle$, $\mathcal{N}_p$ still serves as a neighborhood base at p.) Then $U \cap C = F_\xi$ for some $\xi < \lambda$, and as in Proposition 2.2(3), F_ξ contains all but finitely many members of A. $\square$

Corollary 2.4. *Let $\langle X, \tau\rangle$ be a separable space and let p be a non-isolated point in X. Then there is a ccc forcing $\mathbb{P}$ such that in $\langle X, \tau^{\mathbb{P}}\rangle$, there exists a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.*

3. A point not in the closure of a countable set

Now we consider the case where a point p in a space X is not in the closure of a countable set; that is, for every countable set $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$. We show that it is consistent relative to large cardinals that no cardinal-preserving forcing can add a sequence that converges to p. Let us prove a lemma first.

Lemma 3.1. *Let $\mathbb{P}$ be a cardinal-preserving forcing. Suppose that A is an infinite countable set in $\mathbf{V}^{\mathbb{P}}$ such that for every countable set C in $\mathbf{V}$, $A \cap C$ is finite. Let*

$$\lambda = \min\{|B|^{\mathbf{V}} : B \in \mathbf{V} \textit{and } A \subseteq B\},$$

where $|B|^{\mathbf{V}}$ is the cardinality of B taken in $\mathbf{V}$. Then λ is an uncountable cardinal in $\mathbf{V}^{\mathbb{P}}$ and the cofinality of λ in $\mathbf{V}^{\mathbb{P}}$ is ω.

Proof. By the definition of the set A, λ is an uncountable cardinal in $\mathbf{V}$. Since $\mathbb{P}$ is cardinal-preserving, λ is an uncountable cardinal in $\mathbf{V}^{\mathbb{P}}$ as well. Fix $B \in \mathbf{V}$ such that $A \subseteq B$ and $|B|^{\mathbf{V}} = \lambda$. Enumerate $A = \{a_n : n < \omega\}$ and $B = \{b_\xi : \xi < \lambda\}$. For each $n < \omega$, pick $\xi_n < \lambda$ such that $a_n = b_{\xi_n}$. To show that the set $\{\xi_n : n < \omega\}$ is cofinal in λ, fix $\lambda' < \lambda$. By the definition of λ, we have $A \nsubseteq \{b_\xi : \xi < \lambda'\}$, and so $\{\xi_n : n < \omega\} \nsubseteq \lambda'$. Thus, the cofinality of λ in $\mathbf{V}^{\mathbb{P}}$ is ω. $\square$

We use the Covering Lemma for the Dodd-Jensen core model.

Theorem 3.2. ([7] Theorem 1.2) *Assume that there is no inner model with a measurable cardinal, and let K be the Dodd-Jensen core model. Then for any set A of ordinals, there is a set $B \in K$ such that $B \supseteq A$ and $|B| = |A| + \aleph_1$.*

We show that under the large cardinal hypothesis in Theorem 3.2, no cardinal-preserving forcing adds a sequence which converges to p if p is not in the closure of a countable set.

Theorem 3.3. *Assume that there is no inner model with a measurable cardinal. Suppose that $\langle X, \tau \rangle$ is a space, $p \in X$ and for every countable set $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$. Then in $\langle X, \tau^{\mathbb{P}} \rangle$ for any cardinal-preserving forcing $\mathbb{P}$, there is no sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.*

Proof. Assume on the contrary that there is a cardinal-preserving forcing $\mathbb{P}$ such that in $\langle X, \tau^{\mathbb{P}} \rangle$, there exists a sequence $A = \{a_n : n < \omega\} \subseteq X \setminus \{p\}$ which converges to p. We shall show that the set $A \in \mathbf{V}^{\mathbb{P}}$ does not satisfy Theorem 3.2. To do so, let $K^{\mathbf{V}^{\mathbb{P}}}$ be the Dodd-Jensen core model constructed in $\mathbf{V}^{\mathbb{P}}$ and fix $B \in K^{\mathbf{V}^{\mathbb{P}}}$ such that $A \subseteq B$. By the assumption, for every countable set $C \in \mathbf{V}$ with $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$, and so $p \notin \overline{A \cap C}$. Since $A = \{a_n : n < \omega\}$ converges to p, $A \cap C$ must be finite. The set A satisfies the hypothesis of Lemma 3.1. So, as in Lemma 3.1, we let $\lambda = \min\{|B'|^{\mathbf{V}} : B' \in \mathbf{V}, A \subseteq B'\}$. Since $K^{\mathbf{V}^{\mathbb{P}}} = K$ ([5] Theorem 35.6(iv)) and $B \in K^{\mathbf{V}^{\mathbb{P}}}$, we have $B \in K$, and so $B \in \mathbf{V}$. Since $A \subseteq B$, we have $|B|^{\mathbf{V}} \geq \lambda$. Since $\mathbb{P}$ preserves cardinals, we have $|B|^{\mathbf{V}^{\mathbb{P}}} \geq \lambda$. (If $|B|^{\mathbf{V}^{\mathbb{P}}} < \lambda$, then λ would not be a cardinal in $\mathbf{V}^{\mathbb{P}}$.) By Lemma 3.1, λ is an uncountable cardinal of cofinality ω in $\mathbf{V}^{\mathbb{P}}$, and so $\lambda > \aleph_1$. Thus in $\mathbf{V}^{\mathbb{P}}$, we have

$$|B|^{\mathbf{V}^{\mathbb{P}}} \geq \lambda > \aleph_1 = \aleph_0 + \aleph_1 = |A|^{\mathbf{V}^{\mathbb{P}}} + \aleph_1.$$

This violates Theorem 3.2. □

Now we characterize spaces where some cardinal-preserving forcing adds a sequence that converges to a point which is not in the closure of a countable set. In the following proposition, $cf^{\mathbf{V}}(\lambda)$ denotes the cofinality of λ taken in $\mathbf{V}$, and $cf^{\mathbf{V}^{\mathbb{P}}}(\lambda)$ denotes the cofinality of λ taken in $\mathbf{V}^{\mathbb{P}}$.

Proposition 3.4. *Suppose that $\langle X, \tau \rangle$ is a space, $p \in X$ and $\mathbb{P}$ is a cardinal-preserving forcing such that:*

1. *in $\langle X, \tau \rangle$, for every countable set $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$, and*
2. *in $\langle X, \tau^{\mathbb{P}} \rangle$, there is a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.*

Then, in $\mathbf{V}$, there is a set $B \subseteq X \setminus \{p\}$ such that $p \in \overline{B}$ and either:

1. *$cf^{\mathbf{V}}(|B|)$ is weakly inaccessible, or*
2. *$cf^{\mathbf{V}}(|B|) = \omega$.*

Proof. Let $A = \{a_n : n < \omega\}$. First we observe that for every countable set $C \in \mathbf{V}$ with $C \subseteq X \setminus \{p\}$, $A \cap C$ is finite. (If $A \cap C$ is infinite, then we would have $p \in \overline{C}$.) So the set A satisfies the hypothesis of Lemma 3.1 and we let $\lambda = \min\{|B|^{\mathbf{V}} : B \in \mathbf{V}, A \subseteq B\}$ as in Lemma 3.1.

Fix $B \in \mathbf{V}$ so that $B \subseteq X \setminus \{p\}$, $A \subseteq B$ and $|B| = \lambda$. Since $A \subseteq B$, we have $p \in \overline{B}$. We assume that $cf^{\mathbf{V}}(\lambda) > \omega$ and show that $cf^{\mathbf{V}}(\lambda)$ is a weakly inaccessible (= uncountable, regular and limit) cardinal. Since $cf^{\mathbf{V}}(\lambda) > \omega$, $cf^{\mathbf{V}}(\lambda)$ is uncountable, and by definition, a cofinality is a regular cardinal. So it remains to show that $cf^{\mathbf{V}}(\lambda)$ is a limit cardinal. We clearly have $cf^{\mathbf{V}^{\mathbb{P}}}(cf^{\mathbf{V}}(\lambda)) \leq cf^{\mathbf{V}^{\mathbb{P}}}(\lambda)$, and by Lemma 3.1, $cf^{\mathbf{V}^{\mathbb{P}}}(\lambda) = \omega$; therefore, $cf^{\mathbf{V}^{\mathbb{P}}}(cf^{\mathbf{V}}(\lambda)) = \omega$. This means that $cf^{\mathbf{V}}(\lambda)$ is a limit cardinal in $\mathbf{V}^{\mathbb{P}}$, which implies that $cf^{\mathbf{V}}(\lambda)$ is a limit cardinal in $\mathbf{V}$ as well. □

We give two examples (Example 3.6 and Example 3.7) where a point p in a space X is not in the closure of a countable set, yet a cardinal-preserving forcing adds a sequence which converges to p. These two examples correspond to the two possibilities in the conclusion of Proposition 3.4 ($cf^{\mathbf{V}}(|B|)$ is either weakly inaccessible or ω). Because of Theorem 3.3, we need to assume the existence of a measurable cardinal in an inner model to construct these examples. Let us introduce notions of forcing that we use in the examples and state their properties.

Fact 3.5.
(1) For a cardinal κ, *the Cohen forcing*, $Fn(\kappa, 2)$, adjoins κ-many Cohen reals and preserves cardinals (and cofinalities). ([6] VII Definition 5.1)
(2) For a measurable cardinal κ, *the Prikry forcing*, $Pr(\kappa)$, adds a countable set which is cofinal in κ and preserves cardinals. ([5] Theorem 21.10)
(3) For an increasing sequence $\{\kappa_n : n < \omega\}$ of measurable cardinals, let $\kappa_\omega = \sup\{\kappa_n : n < \omega\}$. *The diagonal Prikry forcing*, $Pr(\kappa_n, n < \omega)$, adds a sequence $\langle a_n : n < \omega \rangle$ in $\prod_{n<\omega} \kappa_n$ such that for every sequence $\langle x_n : n < \omega \rangle \in \mathbf{V} \cap \prod_{n<\omega} \kappa_n$, there is an $m < \omega$ such that for all $i > m$, we have $a_i > x_i$. In particular, if we regard $A := \{a_n : n < \omega\}$ as a subset of κ_ω, then for every $S \in \mathbf{V}$ with $S \subseteq \kappa_\omega$ and $|S| < \kappa_\omega$, $A \cap S$ is finite. $Pr(\kappa_n, n < \omega)$ preserves cardinals (and cofinalities). ([4] Definition 1.31, Theorem 1.38)
(4) Let $\kappa_1 = \aleph_1$ and let $\{\kappa_n : 1 < n < \omega\}$ be an increasing sequence of measurable cardinals. Let $\kappa_\omega = \sup\{\kappa_n : n < \omega\}$. *The product of Levy Collapse*, $\prod_n Lv(\kappa_{n+1}, \kappa_n)$, preserves cardinals $\geq \kappa_\omega$ and for a $\prod_n Lv(\kappa_{n+1}, \kappa_n)$-generic filter G, $\mathbf{V}[G] \models$ "$\kappa_n = \aleph_n$ for each $n \geq 1$, and $\kappa_\omega = \aleph_\omega$." ([6] VIII Exercises (F3))

An idea of the example below comes from [2] (Theorem (c)).

Example 3.6. Assume that there is a measurable cardinal in $\mathbf{V}$. Then there exists a generic extension $\mathbf{W}$ of $\mathbf{V}$ such that in $\mathbf{W}$ there exist a space $\langle X, \tau \rangle$, a point $p \in X$ and a cardinal-preserving forcing $\mathbb{P}$ such that:

(1) in $\langle X, \tau\rangle$, for every countable set $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$,
(2) in $\langle X, \tau^{\mathbb{P}}\rangle$, there is a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p, and
(3) $B := X \setminus \{p\}$ satisfies the conclusion of Proposition 3.4 with $cf^{\mathbf{W}}(|B|)$ being a weakly inaccessible cardinal but not a strongly inaccessible cardinal.

Proof. Let κ be a measurable cardinal in $\mathbf{V}$. We use two forcings $Pr(\kappa)$ (Fact 3.5(2)) and $Fn(\kappa, 2)$ (Fact 3.5(1)). Let G be a $Pr(\kappa)$-generic filter over $\mathbf{V}$. In $\mathbf{V}[G]$, we force with $Fn(\kappa, 2)$ and obtain an $Fn(\kappa, 2)$-generic filter H over $\mathbf{V}[G]$. $Fn(\kappa, 2)$ is the set of all finite partial functions from κ to 2, and so $Fn(\kappa, 2)$, which is defined in $\mathbf{V}[G]$, actually belongs to $\mathbf{V}$. Consequently, by [6] (VIII Theorem 1.4), G is $Pr(\kappa)$-generic over $\mathbf{V}[H]$, and $\mathbf{V}[G][H] = \mathbf{V}[H][G]$.

Let $\mathbf{W} = \mathbf{V}[H]$ and $\mathbb{P} = Pr(\kappa)$. In $\mathbf{W}$, we define a space $\langle X, \tau\rangle$ such that $X = \{p\} \cup \kappa$, each point in κ is isolated and a basic open neighborhood of p has the form $\{p\} \cup \{\xi : \gamma < \xi < \kappa\}$ for some $\gamma < \kappa$. We shall show that $\mathbf{W}$, $\mathbb{P}$ and $\langle X, \tau\rangle$ are as required.

To show that $\mathbb{P} = Pr(\kappa)$ is cardinal-preserving in $\mathbf{W}$, we note that in $\mathbf{V}$, $Pr(\kappa)$ is cardinal-preserving and in $\mathbf{V}[G]$, $Fn(\kappa, 2)$ is cardinal-preserving. Therefore, in $\mathbf{V}[G][H] = \mathbf{W}[G]$, all cardinals are preserved.

We work in $\mathbf{W}$. By the definition of the space X, p is not in the closure of any countable subset of $X \setminus \{p\}$. By forcing with $\mathbb{P} = Pr(\kappa)$, we get a $Pr(\kappa)$-generic filter G, which adds a countable set $\{a_n : n < \omega\}$ cofinal in κ; the sequence $\{a_n : n < \omega\}$ converges to p. We have $\kappa = 2^\omega$ (in $\mathbf{W} = \mathbf{V}[H]$), so κ is not a strongly inaccessible cardinal, but it is still weakly inaccessible. If we let $B = X \setminus \{p\}$, then $p \in \overline{B}$ and $cf^{\mathbf{W}}(|B|) = cf^{\mathbf{W}}(\kappa) = \kappa$. □

Here is the second example.

Example 3.7. Assume that there exist countably many measurable cardinals in $\mathbf{V}$. Then there exists a generic extension $\mathbf{W}$ of $\mathbf{V}$ such that in $\mathbf{W}$ there exist a space $\langle X, \tau\rangle$, a point $p \in X$ and a cardinal-preserving forcing $\mathbb{P}$ such that:

(1) in $\langle X, \tau\rangle$, for every countable set $C \subseteq X \setminus \{p\}$, we have $p \notin \overline{C}$,
(2) in $\langle X, \tau^{\mathbb{P}}\rangle$, there is a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p, and
(3) $B := X \setminus \{p\}$ satisfies the conclusion of Proposition 3.4 with $|B| = \aleph_\omega$.

Proof. Let $\kappa_1 = \aleph_1$ and let $\{\kappa_n : 1 < n < \omega\}$ be an increasing sequence of measurable cardinals. Let $\kappa_\omega = \sup\{\kappa_n : n < \omega\}$. We use two forcings $Pr(\kappa_n, n < \omega)$ (Fact 3.5(3)) and $\prod_n Lv(\kappa_{n+1}, \kappa_n)$ (Fact 3.5(4)).

Let G be a $Pr(\kappa_n, n < \omega)$-generic filter over $\mathbf{V}$. In $\mathbf{V}[G]$, we force with $\prod_n Lv(\kappa_{n+1}, \kappa_n)$ and obtain a $\prod_n Lv(\kappa_{n+1}, \kappa_n)$-generic filter H over $\mathbf{V}[G]$. $\prod_n Lv(\kappa_{n+1}, \kappa_n)$ consists of functions from a subset of κ_n of cardinality $< \kappa_n$ to κ_{n+1} for some $n \in \omega$ ([6] VIII Exercises (F1)), and $Pr(\kappa_n, n < \omega)$ does not add a subset of a set of cardinality $< \kappa_\omega$ ([4] Lemma 1.35). Therefore, $\prod_n Lv(\kappa_{n+1}, \kappa_n)$, which is defined in $\mathbf{V}[G]$, actually belongs to $\mathbf{V}$. Using [6] (VIII Theorem 1.4), we conclude that G is $Pr(\kappa_n, n < \omega)$-generic over $\mathbf{V}[H]$, and $\mathbf{V}[G][H] = \mathbf{V}[H][G]$.

Let $\mathbf{W} = \mathbf{V}[H]$ and $\mathbb{P} = Pr(\kappa_n, n < \omega)$. By Fact 3.5(4), we have in $\mathbf{W}$, $\kappa_n = \aleph_n$ for each $n \geq 1$, and $\kappa_\omega = \aleph_\omega$. In $\mathbf{W}$, we define a space $\langle X, \tau \rangle$ such that $X = \{p\} \cup \aleph_\omega$, each point in $\aleph_\omega$ is isolated, and U is a neighborhood of p if $p \in U$ and $|\aleph_\omega \setminus U| < \aleph_\omega$. We shall show that $\mathbf{W}$, $\mathbb{P}$ and $\langle X, \tau \rangle$ are as required.

To show that $\mathbb{P} = Pr(\kappa_n, n < \omega)$ is cardinal-preserving in $\mathbf{W}$, we note that in $\mathbf{V}$, $Pr(\kappa_n, n<\omega)$ preserves cardinals and in $\mathbf{V}[G]$, $\prod_n Lv(\kappa_{n+1}, \kappa_n)$ collapses cardinals strictly between κ_n and κ_{n+1} for $n \geq 1$. Therefore, in $\mathbf{V}[G][H] = \mathbf{W}[G]$, the only cardinals that are collapsed are the ones strictly between κ_n and κ_{n+1} for $n \geq 1$, and this is already the case in $\mathbf{W}$.

We work in $\mathbf{W}$. By the definition of the space X, no countable subset of $X \setminus \{p\}$ contains p in its closure. According to Fact 3.5(3), forcing with $\mathbb{P} = Pr(\kappa_n, n < \omega)$ adds a countable set $A := \{a_n : n < \omega\}$ such that for every $S \in \mathbf{W}$ with $S \subseteq \kappa_\omega$ and $|S| < \kappa_\omega$, $A \cap S$ is finite. Since $\kappa_\omega = \aleph_\omega$, this means that for every $S \in \mathbf{W}$ with $S \subseteq \aleph_\omega$ and $|S| < \aleph_\omega$, $A \cap S$ is finite. If U is a neighborhood of p, then $|\aleph_\omega \setminus U| < \aleph_\omega$, and so $A \cap (\aleph_\omega \setminus U)$ is finite. Hence, U contains all but finitely many members of A. Thus, the sequence $\{a_n : n < \omega\}$ converges to p. $\square$

At the end of this section, we give a ZFC example of a space where a cardinal-preserving forcing cannot add a convergent sequence.

Example 3.8. Let λ be an uncountable cardinal. There exist a space $\langle X, \tau \rangle$ and a point $p \in X$ such that if $B \subseteq X \setminus \{p\}$ and $p \in \overline{B}$, then $|B| = \lambda$, and no cardinal-preserving forcing adds a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ that converges to p.

Proof. Let $X = \{p\} \cup (\lambda \times \omega_1)$. Each point in $\lambda \times \omega_1$ is isolated, and a basic open neighborhood U of p has the following form: For some $S \subseteq \lambda$ with $|\lambda \setminus S| < \lambda$,

$$U := \{p\} \cup \bigcup \{\{\xi\} \times (\eta_\xi, \omega_1) : \xi \in S \text{ and } \eta_\xi < \omega_1\}.$$

It is easy to see that if $B \subseteq X \setminus \{p\}$ and $p \in \overline{B}$, then $|B| = \lambda$. Suppose that a sequence $\{a_n : n < \omega\} \subseteq X \setminus \{p\}$ is added by a cardinal-preserving forcing. Then we have $\{a_n : n < \omega\} \subseteq \lambda \times [0, \gamma)$ for some $\gamma < \omega_1$ because ω_1 is preserved. Thus, $\{a_n : n < \omega\}$ does not converges to p. □

4. Adding a convergent sequence to every point

In this section, we consider adding a sequence which converges to x for every x in a space X. In order to do so, we weaken the condition that a forcing is cardinal-preserving to the condition that a forcing preserves cardinals $\leq \kappa$ for some regular cardinal $\kappa > 2^{|X|}$. With this weaker condition, we can produce a sequence which converges to x for every $x \in X$ as long as x is in the closure of a countable set.

Theorem 4.1. *Let $\langle X, \tau \rangle$ be a space and let κ be a regular cardinal such that $\kappa > 2^{|X|}$. Suppose that for each $x \in X$, there is a countable set $C_x \subseteq X \setminus \{x\}$ such that $x \in \overline{C_x}$. Then there is a forcing $\mathbb{P}$ which preserves cardinals $\leq \kappa$ such that in $\langle X, \tau^{\mathbb{P}} \rangle$, there exists for each $x \in X$ a sequence $\{a_n(x) : n < \omega\} \subseteq X \setminus \{x\}$ that converges to x.*

Proof. Let $\lambda = 2^{|X|}$ and fix a regular cardinal κ such that $\lambda < \kappa$. For each $x \in X$, fix a neighborhood base $\mathcal{N}_x$ at x such that $|\mathcal{N}_x| \leq \lambda$. Let $\mathbb{P}_1 = Fn(\kappa, 2, \kappa)$, the set of all functions f such that $dom(f) \subseteq \kappa$, $ran(f) \subseteq 2$ and $|f| < \kappa$ ([6] VII Definition 6.1). $\mathbb{P}_1$ preserves cardinals $\leq \kappa$, and in $\mathbf{V}^{\mathbb{P}_1}$, $2^{<\kappa} = \kappa$ ([6] VII Exercises (G3)). Let $\dot{\mathbb{P}}_2$ be a $\mathbb{P}_1$-name for the partially ordered set for constructing a model of Martin's Axiom (MA) ([6] VIII Theorem 6.3). Then $1 \Vdash_{\mathbb{P}_1}$ "$\dot{\mathbb{P}}_2$ has the countable chain condition," and so the two-step iteration $\mathbb{P}_1 * \dot{\mathbb{P}}_2$ preserves cardinals $\leq \kappa$. In $\mathbf{V}^{\mathbb{P}_1 * \dot{\mathbb{P}}_2}$, MA holds and $2^\omega = \kappa$.

We work in $\mathbf{V}^{\mathbb{P}_1 * \dot{\mathbb{P}}_2}$ and show that for every $x \in X$, there is a sequence converging to x. Fix $x \in X$ and take a countable set $C_x \subseteq X \setminus \{x\}$ such that $x \in \overline{C_x}$. Let $\mathcal{F}_x = \{U \cap C_x : U \in \mathcal{N}_x\}$ and enumerate $\mathcal{F}_x = \{F_\xi : \xi < \lambda\}$. Let $\mathbb{P}(\mathcal{F}_x)$ be as in Definition 2.1. In $\mathbf{V}^{\mathbb{P}_1 * \dot{\mathbb{P}}_2}$, MA holds and $\lambda < \kappa = (2^\omega)^{\mathbf{V}^{\mathbb{P}_1 * \dot{\mathbb{P}}_2}}$, and so there exists a filter $G \subseteq \mathbb{P}(\mathcal{F}_x)$ such that $G \cap D_{n\,\xi} \neq \emptyset$ for all $n < \omega$ and $\xi < \lambda$, where $D_{n\,\xi}$ is as in Proposition 2.2(2). Let $A = \bigcup\{s : \langle s, E \rangle \in G\}$. To show that $A = \{a_n(x) : n < \omega\}$ converges to x, fix $U \in \mathcal{N}_x$. Then $U \cap C_x = F_\xi$ for some $\xi < \lambda$. By Proposition 2.2(3), F_ξ contains all but finitely many members of A. □

Corollary 4.2. *Let $\langle X, \tau \rangle$ be a separable space and let κ be a regular cardinal such that $\kappa > 2^{|X|}$. Then there is a forcing $\mathbb{P}$ which preserves cardinals $\leq \kappa$ such that in $\langle X, \tau^{\mathbb{P}} \rangle$, for every non-isolated point $x \in X$, there is a sequence $\{a_n(x) : n < \omega\} \subseteq X \setminus \{x\}$ that converges to x.*

References

[1] Tomek Bartoszyński and Haim Judah, *Set Theory: On the structure of the real Line*. A K Peters, Ltd., Wellesley, MA, 1995.

[2] Andreas Blass, *Small extensions of models of set theory*, Axiomatic set theory (Boulder, Colo., 1983), 35–39, Contemp. Math., **31**, Amer. Math. Soc., Providence, RI, 1984.

[3] David Booth, *Ultrafilters on a countable set*, Ann. Math. Logic 2, (1970/1971), no. 1, 1–24.

[4] Moti Gitik, *Prikry-Type Forcings*, Handbook of Set Theory, 1351–1447, Springer, Dordrecht, 2010.

[5] Thomas Jech, *Set Theory*. The third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[6] Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs*. Studies in Logic and the Foundation of Mathematics, 102. Amsterdam-New York: North-Holland Publishing Co., 1980.

[7] William J. Mitchell, *The covering lemma*, Handbook of Set Theory, 1497-1594, Springer, Dordrecht, 2010.

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