http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 49, 2017

Pages 41–64

http://topology.nipissingu.ca/tp/

Straight homotopy invariants

by

Semën Podkorytov

Electronically published on April 28, 2016

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 49 (2017)
Pages 41-64

E-Published on April 28, 2016

STRAIGHT HOMOTOPY INVARIANTS

SEMËN PODKORYTOV

Abstract. Let X and Y be spaces and M be an abelian group. A homotopy invariant $f\colon [X,Y]\to M$ is called straight if there exists a homomorphism $F\colon L(X,Y)\to M$ such that $f([a])=F(\langle a\rangle)$ for all $a\in C(X,Y)$. Here $\langle a\rangle\colon \langle X\rangle\to\langle Y\rangle$ is the homomorphism induced by a between the abelian groups freely generated by X and Y and $L(X,Y)$ is a certain group of "admissible" homomorphisms. We show that all straight invariants can be expressed through a "universal" straight invariant of homological nature.

1. Introduction

We define straight homotopy invariants of maps and give their characterization, which reduces them to the classical homology theory.

The group $L(X,Y)$. For a set X, let $\langle X\rangle$ be the (free) abelian group with the basis $X^\sharp\subseteq\langle X\rangle$ endowed with the bijection $X\to X^\sharp$, $x\mapsto <x>$. For sets X and Y, let $L(X,Y)\subseteq \operatorname{Hom}(\langle X\rangle,\langle Y\rangle)$ be the subgroup generated by the homomorphisms u such that $u(X^\sharp)\subseteq Y^\sharp\cup\{0\}$. (Elements of $L(X,Y)$ are the homomorphisms bounded with respect to the ℓ_1-norm.) A map $a\colon X\to Y$ induces the homomorphism $\langle a\rangle\in L(X,Y)$, $\langle a\rangle(<x>)=<a(x)>$.

Straight homotopy invariants. Let X and Y be spaces. Let $C(X,Y)$ be the set of continuous maps $X\to Y$ and $[X,Y]$ be the set of their homotopy classes. For $a\in C(X,Y)$, let $[a]\in[X,Y]$ be the homotopy class of a. Let M be an abelian group, and $f\colon [X,Y]\to M$ be a map (a homotopy invariant). The invariant f is called *straight* if there exists a homomorphism $F\colon L(X,Y)\to M$ such that $f([a])=F(\langle a\rangle)$ for all $a\in C(X,Y)$.

2010 *Mathematics Subject Classification.* 55N10.
Key words and phrases. Ordinary homology, homotopy invariant of finite degree.

The main invariant $h\colon [X,Y] \to [SX,SY]$**.** For a space X, let SX be its singular chain complex. Let X and Y be spaces. Let $[SX,SY]$ be the group of chain homotopy classes of morphisms $SX \to SY$. There is a (non-naturally) split exact natural sequence

$$0 \longrightarrow \prod_{i\in\mathbb{Z}} \mathrm{Ext}(H_{i-1}X, H_iY) \longrightarrow [SX,SY] \longrightarrow \prod_{i\in\mathbb{Z}} \mathrm{Hom}(H_iX, H_iY) \longrightarrow 0$$

("the universal coefficient theorem", cf. [12, Theorem 5.5.3]). For $a \in C(X,Y)$, let $Sa\colon SX \to SY$ be the induced morphism and $[Sa] \in [SX,SY]$ be its chain homotopy class. The invariant $h\colon [X,Y] \to [SX,SY]$, $[a] \mapsto [Sa]$, is called *main*.

The main result.

Theorem 1.1. *Let X be a space homotopy equivalent to a compact CW-complex, Y be a space homotopy equivalent to a CW-complex, $h\colon [X,Y] \to [SX,SY]$ be the main invariant, M be an abelian group, and $f\colon [X,Y] \to M$ be an invariant. The invariant f is straight if and only if there exists a homomorphism $d\colon [SX,SY] \to M$ such that $f = d \circ h$.*

Proof. The theorem follows from Propositions 7.3 and 12.2. $\square$

The theorem says that the main invariant is a "universal" straight invariant. For divisible M, it was known in an equivalent form [7, Theorem II]. In this case, the sufficiency ("if") follows easily from an appropriate form of the Dold–Thom theorem (see § 7). Any abelian group is a subgroup of a divisible one. Straightness, however, is sensitive to the codomain of the invariant. The Brouwer degree $b\colon [S^3, \mathbb{R}P^3] \to \mathbb{Z}$ takes even values only. Thus we have the lift $b'\colon [S^3, \mathbb{R}P^3] \to 2\mathbb{Z}$. It follows from Theorem 1.1 that b is a straight invariant and b' is not.

The hypotheses about the homotopy type of X and Y are essential, see §§ 13, 14. In § 15, we consider K-straight invariants taking values in modules over a commutative ring K (by definitions, straight $=$ $\mathbb{Z}$-straight).

On the definition. If M is divisible, the group $L(X,Y)$ in the definition of a straight invariant can be replaced by $\mathrm{Hom}(\langle X\rangle, \langle Y\rangle)$ because any homomorphism $L(X,Y) \to M$ extends to $\mathrm{Hom}(\langle X\rangle, \langle Y\rangle)$ in this case. In general, this replacement is inadequate. For example, let $X = Y = S^1$. Then the Brouwer degree $b\colon [X,Y] \to \mathbb{Z}$ is a straight invariant by Theorem 1.1 (or Corollary 6.8). At the same time, every homomorphism $F\colon \mathrm{Hom}(\langle X\rangle, \langle Y\rangle) \to \mathbb{Z}$ factors through the restriction homomorphism $\mathrm{Hom}(\langle X\rangle, \langle Y\rangle) \to \mathrm{Hom}(\langle T\rangle, \langle Y\rangle)$ for some finite set $T \subseteq X$ [2, § 94]. Thus F cannot give rise to a non-constant homotopy invariant.

The task done in this paper was to choose the domain of F in the definition of a straight invariant in such a way that we could find a simple homological characterization for arbitrary M.

Related notions. The notion of straight invariant can be generalized as follows. Declare an invariant $f\colon [X,Y] \to M$ to have *degree* at most r if there exists a homomorphism $F\colon L(X^r, Y^r) \to M$ such that $f([a]) = F(\langle a^r \rangle)$ for all $a \in C(X,Y)$. Here $a^r\colon X^r \to Y^r$ is the rth Cartesian power of a. Clearly, invariants of degree at most 1 are precisely straight ones. Similar (and equivalent for M divisible) notions were considered in [10, 8, 9, 11]. Finite-degree invariants distinguish non-homotopic maps under certain conditions [11].

Instead of homotopy invariants of continuous maps, one can consider isotopy invariants of smooth embeddings of one fixed smooth manifold in another. Their degree can be defined in the same way. At least for divisible M, finite-degree invariants $\mathrm{Emb}(S^1, \mathbb{R}^3) \to M$ are precisely Vassiliev knot invariants [8, 13].

We do not study finite-degree invariants in this paper.

2. Notation

The question mark. The expression $[?]$ denotes the map $a \mapsto [a]$ between sets indicated in the context. We similarly use $\langle ? \rangle$, etc. This notation is also used for functors.

Sets and abelian groups. For a set X, let $c_X\colon X \to \langle X \rangle$ be the canonical map $x \mapsto {<}x{>}$. For $v \in \langle X \rangle$ and $x \in X$, let $v/x \in \mathbb{Z}$ be the coefficient of ${<}x{>}$ in v. For an abelian group G, a map $a\colon X \to G$ gives rise to the homomorphism $a^+\colon \langle X \rangle \to G$, ${<}x{>} \mapsto a(x)$. G^X is the group of maps $X \to G$.

Simplicial sets. For simplicial sets U and V, let $\mathrm{Si}(U,V)$ be the set of simplicial maps and $[U,V]$ be the set of their homotopy classes (two simplicial maps are homotopic if they are connected by a sequence of homotopies). The functor $\langle ? \rangle$ takes simplicial sets to simplicial abelian groups degreewise. There is the canonical simplicial map $c_U\colon U \to \langle U \rangle$. For a simplicial abelian group Z, a simplicial map $s\colon U \to Z$ gives rise to the simplicial homomorphism $s^+\colon \langle U \rangle \to Z$. For a simplicial set T, a simplicial map $s\colon U \to V$ induces the maps $s^T_\#\colon \mathrm{Si}(T,U) \to \mathrm{Si}(T,V)$, $s_T^\#\colon \mathrm{Si}(V,T) \to \mathrm{Si}(U,T)$, $s^T_*\colon [T,U] \to [T,V]$, and $s_T^*\colon [V,T] \to [U,T]$. This notation is also used in the topological case.

3. Induced straight invariants

Lemma 3.1. *Let X, $\tilde{X}$, $\tilde{Y}$, and Y be spaces, $r\colon X \to \tilde{X}$ and $s\colon \tilde{Y} \to Y$ be continuous maps, M be an abelian group and $f\colon [X,Y] \to M$ be a straight invariant. Then the invariant $\tilde{f}\colon [\tilde{X},\tilde{Y}] \to M$, $\tilde{f}([\tilde{a}]) = f([s \circ \tilde{a} \circ r])$, $\tilde{a} \in C(\tilde{X},\tilde{Y})$, is straight.*

Proof. There is a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a \rangle)$, $a \in C(X,Y)$. We have the commutative diagram

$$\begin{tikzcd}
C(\tilde{X},\tilde{Y}) \arrow[rr, "\langle ? \rangle"] \arrow[d, "{[?]}"'] \arrow[dr, "K"] & & L(\tilde{X},\tilde{Y}) \arrow[d, "T"'] \arrow[dd, bend left=60, "\tilde{F}"] \\
 & C(X,Y) \arrow[r, "\langle ? \rangle"] \arrow[d, "{[?]}"'] & L(X,Y) \arrow[d, "F"'] \\
{[\tilde{X},\tilde{Y}]} \arrow[r, "k"] \arrow[rr, bend right, "\tilde{f}"'] & {[X,Y]} \arrow[r, "f"] & M,
\end{tikzcd}$$

where the maps K and k and the homomorphism T are induced by the pair (r,s) (that is, $K(\tilde{a}) = s \circ \tilde{a} \circ r$, $k([\tilde{a}]) = [s \circ \tilde{a} \circ r]$, $T(\tilde{u}) = \langle s \rangle \circ \tilde{u} \circ \langle r \rangle$), and $\tilde{F} = F \circ T$. Thus $\tilde{f}$ is straight. $\square$

4. The main invariant $h\colon [|U|,|V|] \to [S|U|,S|V|]$

The geometric realization $|Z|$ of a simplicial abelian group Z has the structure of an abelian group. $|Z|$ is a topological abelian group if Z is countable; in general, it is a group of the category of compactly generated Hausdorff spaces. For a simplicial set T, $C(|T|,|Z|)$ and $[|T|,|Z|]$ are abelian groups with respect to pointwise addition. Clearly, $\mathrm{Si}(T,Z)$ and $[T,Z]$ are also abelian groups.

Lemma 4.1. *Let U and V be simplicial sets. Then there exists a commutative diagram*

$$\begin{tikzcd}
{[U,V]} \arrow[rr, "(c_V)_*^U"] \arrow[dd, "i"'] & & {[U,\langle V \rangle]} \arrow[dl, "e"] \arrow[dd, "j"] \\
 & {[S|U|,S|V|]} \arrow[dr, "E"] & \\
{[|U|,|V|]} \arrow[ur, "h"] \arrow[rr, "|c_V|_*^{|U|}"] & & {[|U|,|\langle V \rangle|]},
\end{tikzcd}$$

where $i\colon [s] \mapsto [|s|]$ *(the map induced by the geometric realization map),* j *is similar,* h *is the main invariant, and* e, E *are some isomorphisms.*

This is a version of the Dold–Thom theorem [3, § 4.K].

Proof. Let $\triangle$ be the singularization functor. For a simplicial set T, let $k_T\colon T \to \triangle|T|$ be the canonical weak equivalence. If T is a simplicial abelian group, k_T is a simplicial homomorphism. We have the commutative diagram

$$\begin{tikzcd}
V \arrow[rr, "c_V"] \arrow[dd, "k_V"'] & & \langle V\rangle \arrow[dl, "\langle k_V\rangle"] \arrow[dd, "k_{\langle V\rangle}"] \\
& \langle \triangle|V|\rangle \arrow[dr, "m"] & \\
\triangle|V| \arrow[ur, "c_{\triangle|V|}"] \arrow[rr, "\triangle|c_V|"] & & \triangle|\langle V\rangle|,
\end{tikzcd}$$

where $m = (\triangle|c_V|)^+$. $k_{\langle V\rangle}$, $\langle k_V\rangle$, and thus m are weak equivalences. Consider the commutative diagram

$$\begin{tikzcd}
{[U,V]} \arrow[rr, "(c_V)^U_*"] \arrow[dd, "(k_V)^U_*"'] \arrow[ddd, bend right, "i"'] & & {[U,\langle V\rangle]} \arrow[dl, "\langle k_V\rangle^U_*"] \arrow[dd, "(k_{\langle V\rangle})^U_*"] \arrow[ddd, bend left, "j"] \\
& {[U,\langle \triangle|V|\rangle]} \arrow[dr, "m^U_*"] & \\
{[U,\triangle|V|]} \arrow[ur, "(c_{\triangle|V|})^U_*"] \arrow[rr, "(\triangle|c_V|)^U_*"] & & {[U,\triangle|\langle V\rangle|]} \\
{[|U|,|V|]} \arrow[u, "p"] \arrow[rr, "|c_V|^{|U|}_*"] & & {[|U|,|\langle V\rangle|]}, \arrow[u, "q"']
\end{tikzcd}$$

where the upper part is the result of applying the functor $[U, ?]$ to the previous diagram and p and q are the standard adjunction bijections for the functors $|?|$ and $\triangle$. $\langle k_V\rangle^U_*$, m^U_*, and q are isomorphisms.

We will find an isomorphism $P\colon [S|U|, S|V|] \to [U, \langle\triangle|V|\rangle]$ such that $P \circ h = (c_{\triangle|V|})^U_* \circ p$. Then it will be enough to set $e = P^{-1} \circ \langle k_V\rangle^U_*$ and $E = q^{-1} \circ m^U_* \circ P$.

For a simplicial set T, let AT be its chain complex, so that $(AT)_n = \langle T_n \rangle$ for $n \geqslant 0$, $(AT)_n = 0$ for $n < 0$, and, for $n \geqslant 1$, the differential $\partial\colon (AT)_n \to (AT)_{n-1}$ is given by

$$\partial = \sum_{r=0}^{n} (-1)^r \langle d_r \rangle,$$

where $d_r\colon T_n \to T_{n-1}$ are the face maps. Then $SX = A\triangle X$ for any space X. A simplicial map $s\colon T \to \langle W \rangle$ gives rise to the morphism $v\colon AT \to AW$, $v_n = s_n^+$, $n \geqslant 0$. This rule yields an isomorphism $D\colon [T, \langle W \rangle] \to [AT, AW]$ (the Dold–Kan correspondence). We set $T = \triangle|U|$ and $W = \triangle|V|$. Consider the commutative diagram

$$\begin{tikzcd}
{[|U|,|V|]} \arrow[r, "b"] \arrow[d, "h"'] \arrow[rr, bend left, "p"] & {[\triangle|U|, \triangle|V|]} \arrow[r, "(k_U)^*_{\triangle|V|}"] \arrow[d, "(c_{\triangle|V|})_*^{\triangle|U|}"'] & {[U, \triangle|V|]} \arrow[d, "(c_{\triangle|V|})_*^{U}"] \\
{[A\triangle|U|, A\triangle|V|]} \arrow[rr, bend right, "P"'] & {[\triangle|U|, \langle \triangle|V| \rangle]} \arrow[l, "D"'] \arrow[r, "(k_U)^*_{\langle\triangle|V|\rangle}"] & {[U, \langle \triangle|V| \rangle],}
\end{tikzcd}$$

where the map b is given by the functor $\triangle$ and $P = (k_U)^*_{\langle\triangle|V|\rangle} \circ D^{-1}$. Since $(k_U)^*_{\langle\triangle|V|\rangle}$ is an isomorphism, P is an isomorphism too. $\square$

5. Nöbeling–Bergman theory

By a *ring* we mean a (non-unital) commutative ring; *subring* is understood accordingly. The following facts follow from [5, Theorem 2 and its proof], cf. [2, § 97].

Lemma 5.1. *Let E be a torsion-free ring generated by idempotents. Then E is a free abelian group.* $\square$

An example: the ring $B(X)$ of bounded functions $X \to \mathbb{Z}$, where X is an arbitrary set.

Lemma 5.2. *Let E be a torsion-free ring and $F \subseteq E$ be a subring, both generated by idempotents. Then the abelian group E/F is free.* $\square$

For $F = 0$, this is Lemma 5.1.

6. Maps to a space with addition

Let X be a space and T be a Hausdorff space.

For a set $V \subseteq T$, we introduce the homomorphism $s_V\colon L(X,T) \to \mathbb{Z}^X$, $s_V(u)(x) = I_V^+(u({<}x{>}))$, $x \in X$, where $I_V\colon T \to \mathbb{Z}$ is the indicator function of the set V.

The subgroup $R \subseteq L(X,T)$. For $p \in X$, $q \in T$, let $R(p,q) \subseteq L(X,T)$ be the subgroup of homomorphisms u such that, for any sufficiently small (open) neighbourhood V of q, the function $s_V(u)$ is constant in some neighbourhood of p. Let $R \subseteq L(X,T)$ be the intersection of the subgroups $R(p,q)$, $p \in X$, $q \in T$.

Lemma 6.1. *For $a \in C(X,T)$, we have $\langle a \rangle \in R$.*

Proof. Take $p \in X$, $q \in T$. We show that $\langle a \rangle \in R(p,q)$. If $a(p) = q$, then, for any neighbourhood V of q, we take the neighbourhood $U = a^{-1}(V)$ of p and get $s_V(\langle a \rangle)|_U = 1$. Otherwise, choose disjoint neighbourhoods W of q and W_1 of $a(p)$. Consider the neighbourhood $U = a^{-1}(W_1)$ of p. For any $V \subseteq W$, we have $s_V(\langle a \rangle)|_U = 0$. $\square$

Lemma 6.2. *The abelian group $L(X,T)/R$ is free.*

Proof. Let O_T be the set of open sets in T. Consider the ring $E = B(X \times X \times O_T)$. For $p \in X$, $q \in T$, let $I(p,q) \subseteq E$ be the ideal of functions f such that, for any sufficiently small neighbourhood V of q, the function $X \to \mathbb{Z}$, $x \mapsto f(p,x,V)$, vanishes in some neighbourhood of p. Let $I \subseteq E$ be the intersection of the ideals $I(p,q)$, $p \in X$, $q \in T$. The ring E/I is torsion-free and generated by idempotents. By Lemma 5.1, E/I is a free abelian group. Consider the homomorphism $k \colon L(X,T) \to E$, $k(u)(p,x,V) = s_V(u)(x) - s_V(u)(p)$, $p,x \in X$, $V \in O_T$, $u \in L(X,T)$. We have $k^{-1}(I(p,q)) = R(p,q)$ and thus $k^{-1}(I) = R$. Therefore, k induces a monomorphism $L(X,T)/R \to E/I$. It follows that the abelian group $L(X,T)/R$ is free. $\square$

The set Q and the homomorphisms $e(D,a)$. Let Q be the set of pairs (D,a), where $D \subseteq X$ is a closed set and $a \in C(D,T)$. For $(D,a) \in Q$, introduce the homomorphism $e(D,a) \in L(X,T)$,

$$e(D,a)(\text{<}x\text{>}) = \begin{cases} \text{<}a(x)\text{>} & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

$x \in X$.

Lemma 6.3. *Let $(D,a) \in Q$, $p \in X$, and $q \in T$. If $e(D,a) \notin R(p,q)$, then $p \in D$ and $a(p) = q$.*

Proof. Put $u = e(D,a)$. *The case $p \notin D$.* Consider the neighbourhood $U = X \setminus D$ of p. We have $s_V(u)|_U = 0$ for any $V \subseteq T$. Thus $u \in R(p,q)$. *The case $p \in D$, $a(p) \neq q$.* Choose disjoint neighbourhoods W of q and W_1 of $a(p)$. There is a neighbourhood U of p such that $a(D \cap U) \subseteq W_1$. We have $s_V(u)|_U = 0$ for any $V \subseteq W$. Thus $u \in R(p,q)$. $\square$

The subgroup $K \subseteq L(X,T)$. Let $K \subseteq L(X,T)$ be the subgroup generated by $e(D,a)$, $(D,a) \in Q$.

Lemma 6.4. *The abelian group $L(X,T)/K$ is free.*

Proof. Consider the monomorphism $j\colon L(X,T) \to B(X \times T)$, $j(u)(x,t) = u({<}x{>})/t$. For $(D_i, a_i) \in Q$, $i = 1,2$, we have $j(e(D_1,a_1))j(e(D_2,a_2)) = j(e(D,a))$, where $D = \{x \in D_1 \cap D_2 : a_1(x) = a_2(x)\}$ and $a = a_1|_D = a_2|_D$. In particular, $j(e(D,a))$, $(D,a) \in Q$, are idempotents. Therefore, $j(K)$ is a subring generated by idempotents. By Lemma 5.2, the abelian group $B(X \times T)/j(K)$ is free. Since j induces a monomorphism $L(X,T)/K \to B(X \times T)/j(K)$, the abelian group $L(X,T)/K$ is free. □

Lemma 6.5. *The abelian group $L(X,T)/(K \cap R)$ is free.*

Proof. The quotients in the chain $L(X,T) \supseteq K \supseteq K \cap R$ are free: $L(X,T)/K$ by Lemma 6.4, and $K/(K \cap R)$ as a subgroup of $L(X,T)/R$, which is free by Lemma 6.2. □

The homomorphism $G\colon L(X,T) \to T^X$. Let T have the structure of an abelian group such that, $(*)$ for any closed set $D \subseteq X$, the set $C(D,T)$ becomes an abelian group with respect to pointwise addition[1]. T^X denotes the abelian group of all maps $X \to T$. Consider the homomorphism $G\colon L(X,T) \to T^X$, $G(u)(x) = r(u({<}x{>}))$, $x \in X$, $u \in L(X,T)$, where $r = \mathrm{id}^+\colon \langle T \rangle \to T$.

Lemma 6.6. $G(K \cap R) \subseteq C(X,T)$.

Proof. Take $u \in K \cap R$. We show that $G(u) \in C(X,T)$. Since $u \in K$, we have

$$u = \sum_{i \in I} u_i, \qquad u_i = k_i e(D_i, a_i),$$

where I is a finite set, $k_i \in \mathbb{Z}$, and $(D_i, a_i) \in Q$. For $J \subseteq I$, put

$$u_J = \sum_{i \in J} u_i, \qquad D_J = \bigcap_{i \in J} D_i \subseteq X$$

(so $D_\varnothing = X$) and

$$b_J = \sum_{i \in J} k_i a_i|_{D_J} \in C(D_J, T), \qquad k_J = \sum_{i \in J} k_i.$$

[1]The condition $(*)$ is satisfied if T is a topological abelian group or if $X = |U|$ and $T = |Z|$, where U is a simplicial set and Z is a simplicial abelian group.

Take $p \in X$. We verify that $G(u)$ is continuous at p. Put $N = \{i \in I : p \notin D_i\}$. For $q \in T$, put $I(q) = \{i \in I : p \in D_i,\ a_i(p) = q\}$. We have

$$u = u_N + \sum_{q \in T} u_{I(q)}$$

(almost all summands are zero). Clearly, $G(u_N)$ vanishes in some neighbourhood of p. Take $q \in T$. It suffices to show that $G(u_{I(q)})$ is continuous at p. Put $t_0 = G(u_{I(q)}) \in T$. We have $t_0 = k_{I(q)}q$. Let W be a neighbourhood of t_0. We seek a neighbourhood U of p such that $G(u_{I(q)})(U) \subseteq W$.

Put $E = \{J \subseteq I(q) : k_J = k_{I(q)}\}$. For $J \in E$, we have $p \in D_J$ and $b_J(p) = t_0$. There is a neighbourhood U_1 of p such that $b_J(D_J \cap U_1) \subseteq W$ for all $J \in E$.

By Lemma 6.3, $u_i \in R(p, q)$ for $i \in I \setminus I(q)$. Since $u \in R(p, q)$, we have $u_{I(q)} \in R(p, q)$. Therefore, there is a neighbourhood $V \subseteq T$ of q such that the function $s_V(u_{I(q)})$ is constant in some neighbourhood U_2 of p.

There is a neighbourhood U_3 of p such that $a_i(D_i \cap U_3) \subseteq V$ for all $i \in I(q)$. For $x \in X$, put $J(x) = \{i \in I(q) : x \in D_i\}$. For $x \in U_2 \cap U_3$, we have $k_{J(x)} = s_V(u_{I(q)})(x) = s_V(u_{I(q)})(p) = k_{I(q)}$, i. e. $J(x) \in E$.

Set $U = U_1 \cap U_2 \cap U_3$. Take $x \in U$. We have $G(u_{I(q)})(x) = b_{J(x)}(x) \in W$ because $J(x) \in E$. $\square$

Lemma 6.7. *There exists a homomorphism $g\colon L(X, T) \to C(X, T)$ such that $g(\langle a \rangle) = a$ for all $a \in C(X, T)$.*

Proof. We have $G(\langle a \rangle) = a$ for all $a \in T^X$. Since $G(K \cap R) \subseteq C(X, T)$ (by Lemma 6.6) and the abelian group $L(X, T)/(K \cap R)$ is free (by Lemma 6.5), there is a homomorphism $g\colon L(X, T) \to C(X, T)$ such that $g(u) = G(u)$ for $u \in K \cap R$. For $a \in C(X, T)$, we have $\langle a \rangle \in K$ (because $\langle a \rangle = e(X, a)$) and $\langle a \rangle \in R$ (by Lemma 6.1). We get $g(\langle a \rangle) = G(\langle a \rangle) = a$. $\square$

Corollary 6.8. *Suppose that* $(*)$ $[X, T]$ *is an abelian group with respect to pointwise addition*[2]. *Then the invariant* $\mathrm{id}\colon [X, T] \to [X, T]$ *is straight.*

Proof. By Lemma 6.7, there is a homomorphism $g\colon L(X, T) \to C(X, T)$ such that $g(\langle a \rangle) = a$ for all $a \in C(X, T)$. Consider the homomorphism $F\colon L(X, T) \to [X, T]$, $u \mapsto [g(u)]$. For $a \in C(X, T)$, we have $[a] = [g(\langle a \rangle)] = F(\langle a \rangle)$. $\square$

7. Sufficiency in Theorem 1.1

The proof of sufficiency in Theorem 1.1 relies on Corollary 6.8. If the group M is divisible, it is easy to use Lemma 7.1 instead (then the stuff of §§ 5, 6 is needless).

---

[2] See footnote 1.

Lemma 7.1 (cf. [10, Lemma 1.2])**.** *Let X and T be spaces, where T has the structure of an abelian group such that* $(*)$ *the sets $C(X,T)$ and $[X,T]$ become abelian groups with respect to pointwise addition*[3]*. Let M be a divisible abelian group and $f\colon [X,T] \to M$ be a homomorphism. Then f is a straight invariant.*

Proof. Consider the homomorphism $G\colon L(X,T) \to T^X$, $G(u)(x) = r(u({<}x{>}))$, $x \in X$, $u \in L(X,T)$, where $r = \mathrm{id}^+\colon \langle T\rangle \to T$. Let $D \subseteq L(X,T)$ be the subgroup generated by the homomorphisms $\langle a\rangle$, $a \in C(X,T)$. Clearly, $G(\langle a\rangle) = a$ for $a \in C(X,T)$. Therefore, $G(D) \subseteq C(X,T)$. Consider the homomorphism $F_0\colon D \to M$, $u \mapsto f([G(u)])$. Since M is divisible, there is a homomorphism $F\colon L(X,T) \to M$ such that $F|_D = F_0$. For $a \in C(X,T)$, we have $f([a]) = f([G(\langle a\rangle)]) = F_0(\langle a\rangle) = F(\langle a\rangle)$. □

Claim 7.2. *Let U and V be simplicial sets. Then the main invariant $h\colon [|U|,|V|] \to [S|U|,S|V|]$ is straight.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
{[|U|,|V|]} \arrow[r, "h"] \arrow[dr, "|c_V|_*^{|U|}"'] & {[S|U|,S|V|]} \arrow[d, "E"] \\
 & {[|U|,|\langle V\rangle|]},
\end{tikzcd}$$

where E is the isomorphism from Lemma 4.1. By Corollary 6.8, the invariant $\mathrm{id}\colon [|U|,|\langle V\rangle|] \to [|U|,|\langle V\rangle|]$ is straight. Therefore, by Lemma 3.1, the invariant $|c_V|_*^{|U|}$ is straight. Since E is an isomorphism, h is also straight. □

Proposition 7.3. *Let X be a space and Y be a space homotopy equivalent to a CW-complex. Then the main invariant $h\colon [X,Y] \to [SX,SY]$ is straight.*

Proof. There are homology equivalences $r\colon |U| \to X$ and $s\colon Y \to |V|$, where U and V are simplicial sets. Consider the commutative diagram

$$\begin{tikzcd}
{[X,Y]} \arrow[r, "h"] \arrow[d, "k"'] & {[SX,SY]} \arrow[d, "l"] \\
{[|U|,|V|]} \arrow[r, "\tilde{h}"] & {[S|U|,S|V|]},
\end{tikzcd}$$

[3]See footnote 1.

where $\tilde{h}$ is the main invariant and the map k as well as the isomorphism l are induced by the pair (r, s). By Claim 7.2, $\tilde{h}$ is straight. By Lemma 3.1, the invariant $\tilde{h} \circ k$ is straight. Since $h = l^{-1} \circ \tilde{h} \circ k$, h is also straight. $\square$

8. The superposition $Z\colon \langle \mathrm{Si}(U, V)\rangle_0 \to \mathrm{Si}(U, \langle V\rangle_0)$

For a set X, let $\langle X\rangle_0 \subseteq \langle X\rangle$ be the kernel of the homomorphism $\langle X\rangle \to \mathbb{Z}$, $<x> \mapsto 1$. We apply the functor $\langle ?\rangle_0$ to simplicial sets degreewise.

Let U and V be simplicial sets. The canonical simplicial map $c = c_V\colon V \to \langle V\rangle$ gives rise to the map $c^U_\#\colon \mathrm{Si}(U, V) \to \mathrm{Si}(U, \langle V\rangle)$ and the homomorphism $(c^U_\#)^+\colon \langle \mathrm{Si}(U, V)\rangle \to \mathrm{Si}(U, \langle V\rangle)$. We have the commutative diagram

$$\begin{tikzcd}
\langle \mathrm{Si}(U,V)\rangle_0 \arrow[r, "Z"] \arrow[d] & \mathrm{Si}(U, \langle V\rangle_0) \arrow[d] \\
\langle \mathrm{Si}(U,V)\rangle \arrow[r, "(c^U_\#)^+"] & \mathrm{Si}(U, \langle V\rangle),
\end{tikzcd}$$

where the vertical arrows are induced by the canonical inclusion $\langle ?\rangle_0 \to \langle ?\rangle$ and Z is a new homomorphism called the *superposition*.

9. Surjectivity of the superposition

Our aim here is Lemma 9.1. We follow [10, §§ 12, 13].

Extension of simplicial maps. For $n \geqslant 0$, let Δ^n be the combinatorial standard n-simplex (a simplicial set) and $\partial\Delta^n$ be its boundary.

Let W be a contractible fibrant simplicial set. For each $n \geqslant 0$, choose a map $e_n\colon \mathrm{Si}(\partial\Delta^n, W) \to \mathrm{Si}(\Delta^n, W)$ such that $e_n(q)|_{\partial\Delta^n} = q$ for any $q \in \mathrm{Si}(\partial\Delta^n, W)$.

Let U be a simplicial set. For each simplicial subset $A \subseteq U$, we introduce the map $E_A\colon \mathrm{Si}(A, W) \to \mathrm{Si}(U, W)$, $x \mapsto t$, where $t|_A = x$ and $t \circ p = e_n(t \circ p|_{\partial\Delta^n})$ for the characteristic map $p\colon \Delta^n \to U$ of each nondegenerate simplex outside A. Clearly,

(1) $E_A(x)|_A = x$;
(2) $E_A(x)|_B = E_{A\cap B}(x|_{A\cap B})|_B$,

where $A, B \subseteq U$ are simplicial subsets and $x \in \mathrm{Si}(A, W)$.

The ring $\langle Q\rangle$ and its identity I. Let Q be the system of simplicial subsets of U consisting of all subsets isomorphic to Δ^n, $n \geqslant 0$, and the empty subset. Suppose that the simplicial set U is *polyhedral*, i. e. Q is its cover closed under intersection, and *compact*, i. e. generated by a finite number of simplices. Q is finite.

We introduce multiplication in $\langle Q\rangle$ by putting ${<}A{>}{<}B{>} = {<}A\cap B{>}$ for $A, B \in Q$. The ring $\langle Q\rangle$ has an identity I. Indeed, the homomorphism $e\colon \langle Q\rangle \to \mathbb{Z}^Q$,

$$e({<}A{>})(B) = \begin{cases} 1 & \text{if } A \supseteq B, \\ 0 & \text{otherwise,} \end{cases}$$

$A, B \in Q$, is an isomorphism ("an upper unitriangular matrix") preserving multiplication. Therefore, $I = e^{-1}(1)$ is an identity.

The homomorphism $K\colon \mathrm{Si}(U, \langle W\rangle_0) \to \langle \mathrm{Si}(U, W)\rangle_0$. For a simplicial set T, let $Z_T\colon \langle \mathrm{Si}(T, W)\rangle_0 \to \mathrm{Si}(T, \langle W\rangle_0)$ be the superposition. For simplicial sets $T \supseteq A$, let $r^T_A\colon \mathrm{Si}(T, W) \to \mathrm{Si}(A, W)$ and $s^T_A\colon \mathrm{Si}(T, \langle W\rangle_0) \to \mathrm{Si}(A, \langle W\rangle_0)$ be the restriction maps. s^T_A is a homomorphism. If $T = U$, we omit the corresponding sub/superscript in this notation.

Note that Z_A is an isomorphism for $A \in Q$. Consider the map $k\colon Q \to \mathrm{Hom}(\mathrm{Si}(U, \langle W\rangle_0), \langle \mathrm{Si}(U, W)\rangle_0)$, $A \mapsto \langle E_A\rangle_0 \circ Z_A^{-1} \circ s_A$:

$$k(A)\colon \mathrm{Si}(U, \langle W\rangle_0) \xrightarrow{s_A} \mathrm{Si}(A, \langle W\rangle_0) \xrightarrow{Z_A^{-1}} \langle \mathrm{Si}(A, W)\rangle_0 \xrightarrow{\langle E_A\rangle_0} \langle \mathrm{Si}(U, W)\rangle_0.$$

Put $K = k^+(I)$.

Lemma 9.1. *The diagram*

$$\begin{tikzcd} & \langle \mathrm{Si}(U,W)\rangle_0 \arrow[d, "Z"] \\ \mathrm{Si}(U,\langle W\rangle_0) \arrow[ur, "K"] \arrow[r, "\mathrm{id}"] & \mathrm{Si}(U,\langle W\rangle_0) \end{tikzcd}$$

is commutative.

Proof. Take $A, B \in Q$. We have the commutative diagram

$$\begin{tikzcd} & \mathrm{Si}(A,\langle W\rangle_0) \arrow[r, "Z_A^{-1}"] \arrow[dd, "s^A_C"] & \langle \mathrm{Si}(A,W)\rangle_0 \arrow[r, "\langle E_A\rangle_0"] \arrow[dd, "\langle r^A_C\rangle_0"] & \langle \mathrm{Si}(U,W)\rangle_0 \arrow[d, "\langle r_B\rangle_0"] \\ \mathrm{Si}(U,\langle W\rangle_0) \arrow[ur, "s_A"] \arrow[dr, "s_C"'] & & & \langle \mathrm{Si}(B,W)\rangle_0 \\ & \mathrm{Si}(C,\langle W\rangle_0) \arrow[r, "Z_C^{-1}"] & \langle \mathrm{Si}(C,W)\rangle_0 \arrow[r, "\langle E_C\rangle_0"] & \langle \mathrm{Si}(U,W)\rangle_0, \arrow[u, "\langle r_B\rangle_0"'] \end{tikzcd}$$

where $C = A \cap B$ (commutativity of the "pentagon" follows from the property (2) of the family E). Therefore, $\langle r_B\rangle_0 \circ k(A) = \langle r_B\rangle_0 \circ k(A\cap B)$. Therefore, $\langle r_B\rangle_0 \circ k^+(X) = \langle r_B\rangle_0 \circ k^+(X{<}B{>})$ for $X \in \langle Q\rangle$. We have $\langle r_B\rangle_0 \circ K = \langle r_B\rangle_0 \circ k^+(I) = \langle r_B\rangle_0 \circ k^+(I{<}B{>}) = \langle r_B\rangle_0 \circ k^+({<}B{>}) = \langle r_B\rangle_0 \circ k(B) = \langle r_B\rangle_0 \circ \langle E_B\rangle_0 \circ Z_B^{-1} \circ s_B = Z_B^{-1} \circ s_B$, because $r_B \circ E_B = \mathrm{id}$ by property (1) of the family E. We get $s_B \circ Z \circ K = Z_B \circ \langle r_B\rangle_0 \circ K = s_B$. Since B is arbitrary, $Z \circ K = \mathrm{id}$. □

10. A cocartesian square

Let U be a compact polyhedral simplicial set and V be a fibrant simplicial set. The canonical simplicial map $c = c_V : V \to \langle V\rangle$ induces the maps $c^U_{\#} \colon \mathrm{Si}(U,V) \to \mathrm{Si}(U,\langle V\rangle)$ and $c^U_* \colon [U,V] \to [U,\langle V\rangle]$. Consider the commutative square of abelian groups and homomorphisms

$$\begin{tikzcd} \langle \mathrm{Si}(U,V)\rangle \arrow[r, "(c^U_{\#})^+"] \arrow[d, "\langle p\rangle"'] & \mathrm{Si}(U,\langle V\rangle) \arrow[d, "q"] \\ \langle [U,V]\rangle \arrow[r, "(c^U_*)^+"] & [U,\langle V\rangle], \end{tikzcd}$$

where $p = [?] \colon \mathrm{Si}(U,V) \to [U,V]$ and $q = [?]$ (the projections).

Lemma 10.1. *This square is cocartesian.*

Proof. Since $\langle p\rangle$ and q are epimorphisms, it suffices to show that $\operatorname{Ker} q = (c^U_{\#})^+(\operatorname{Ker}\langle p\rangle)$.

Suppose we have a decomposition

$$V = \coprod_{i\in I} V_i.$$

Consider the commutative diagram

$$\begin{tikzcd} \bigoplus_{i\in I}\langle \mathrm{Si}(U,V_i)\rangle \arrow[rrr, "\bigoplus_{i\in I}((c_i)^U_{\#})^+"] \arrow[dr] \arrow[ddd, "\bigoplus_{i\in I}\langle p_i\rangle"'] & & & \bigoplus_{i\in I}\mathrm{Si}(U,\langle V_i\rangle) \arrow[dl, "E"] \arrow[ddd, "\bigoplus_{i\in I} q_i"] \\ & \langle \mathrm{Si}(U,V)\rangle \arrow[r, "(c^U_{\#})^+"] \arrow[d, "\langle p\rangle"'] & \mathrm{Si}(U,\langle V\rangle) \arrow[d, "q"] & \\ & \langle [U,V]\rangle \arrow[r, "(c^U_*)^+"] & [U,\langle V\rangle] & \\ \bigoplus_{i\in I}\langle [U,V_i]\rangle \arrow[rrr, "\bigoplus_{i\in I}((c_i)^U_*)^+"] \arrow[ur] & & & \bigoplus_{i\in I}[U,\langle V_i\rangle], \arrow[ul, "e"'] \end{tikzcd}$$

where c_i, p_i, and q_i are similar to c, p, and q (respectively) and the slanting arrows are induced by the inclusions $V_i \to V$. Since U is compact, E and e are isomorphisms. Therefore, is suffices to show that $\operatorname{Ker} q_i = ((c_i)^U_{\#})^+(\operatorname{Ker}\langle p_i\rangle)$ for each $i \in I$. This reduction allows us to assume that V is 0-connected.

Consider the commutative diagram

$$\begin{tikzcd}
\langle \mathrm{Si}(U,V)\rangle_0 \arrow[rrr, "Z"] \arrow[dr, "I"] \arrow[ddd, "\langle p\rangle_0"'] & & & \mathrm{Si}(U,\langle V\rangle_0) \arrow[dl, "j^U_\#"'] \arrow[ddd, "q_0"] \\
& \langle \mathrm{Si}(U,V)\rangle \arrow[r, "(c^U_\#)^+"] \arrow[d, "\langle p\rangle"'] & \mathrm{Si}(U,\langle V\rangle) \arrow[d, "q"] & \\
& \langle [U,V]\rangle \arrow[r, "(c^U_*)^+"] & [U,\langle V\rangle] & \\
\langle [U,V]\rangle_0 \arrow[rrr, "z"] \arrow[ur, "i"] & & & {[U,\langle V\rangle_0],} \arrow[ul, "j^U_*"']
\end{tikzcd}$$

where $q_0 = [?]$ (the projection), Z is the superposition, z is the homomorphism such that the outer square is commutative, I and i are the inclusion homomorphisms, and $j\colon \langle V\rangle_0 \to \langle V\rangle$ is the inclusion simplicial homomorphism. Clearly, $\operatorname{Ker} q = j^U_\#(\operatorname{Ker} q_0)$. Therefore, it suffices to show that $\operatorname{Ker} q_0 = Z(\operatorname{Ker}\langle p\rangle_0)$.

Since V is fibrant and 0-connected, there is a surjective simplicial map $f\colon W \to V$, where W is a contractible fibrant simplicial set. Consider the commutative diagram

$$\begin{tikzcd}
\langle \mathrm{Si}(U,W)\rangle_0 \arrow[r, "\tilde Z"] \arrow[d, "\langle f^U_\#\rangle_0"'] & \mathrm{Si}(U,\langle W\rangle_0) \arrow[d, "(\langle f\rangle_0)^U_\#"] \\
\langle \mathrm{Si}(U,V)\rangle_0 \arrow[r, "Z"] \arrow[d, "\langle p\rangle_0"'] & \mathrm{Si}(U,\langle V\rangle_0) \arrow[d, "q_0"] \\
\langle [U,V]\rangle_0 \arrow[r, "z"] & {[U,\langle V\rangle_0],}
\end{tikzcd}$$

where the map $f^U_\#\colon \mathrm{Si}(U,W) \to \mathrm{Si}(U,V)$ and the simplicial homomorphism $\langle f\rangle_0\colon \langle W\rangle_0 \to \langle V\rangle_0$ are induced by f and $\tilde Z$ is the superposition. Since $\langle f\rangle_0$ is surjective, it is a fibration. Therefore, $\operatorname{Ker} q_0 \subseteq \operatorname{Im}(\langle f\rangle_0)^U_\#$. By Lemma 9.1, $\tilde Z$ is surjective. Since W is contractible, $\operatorname{Im}\langle f^U_\#\rangle_0 \subseteq \operatorname{Ker}\langle p\rangle_0$. Therefore, $\operatorname{Ker} q_0 \subseteq Z(\operatorname{Ker}\langle p\rangle_0)$. The reverse inclusion is obvious. □

11. The homomorphism $P : \mathrm{Si}(U,\langle V\rangle) \to L(|U|,|V|)$

For $n \geqslant 0$, let $\mathbf{\Delta}^n$ be the geometric standard n-simplex and $\mathring{\mathbf{\Delta}}^n$ be its interior. For a simplicial set U and a point $z \in \mathbf{\Delta}^n$, there is a canonical map $z_U\colon U_n \to |U|$. The map $\mathbf{\Delta}^n \times U_n \to |U|$, $(z,u) \mapsto z_U(u)$, is the canonical pairing of geometric realization.

Let U and V be simplicial sets. We define a homomorphism $\tilde{P}$: $\mathrm{Si}(U, \langle V\rangle) \to \mathrm{Hom}(\langle |U|\rangle, \langle |V|\rangle)$. For $t \in \mathrm{Si}(U, \langle V\rangle)$ and $x \in |U|$, $x = z_U(u)$, where $z \in \mathbf{\Delta}^n$ and $u \in U_n$ ($n \geqslant 0$), put $\tilde{P}(t)(<\!x\!>) = \langle z_V\rangle(t_n(u))$:

$$u \in U_n \xrightarrow{t_n} \langle V\rangle_n = \langle V_n\rangle \xrightarrow{\langle z_V\rangle} \langle |V|\rangle.$$

$\tilde{P}$ is well-defined.

Suppose that U is compact.

Lemma 11.1. $\operatorname{Im}\tilde{P} \subseteq L(|U|, |V|)$.

Proof. Let $U_n^\times \subseteq U_n$ ($n \geqslant 0$) be the set of non-degenerate simplices. For $u \in U_n^\times$ ($n \geqslant 0$), we define a homomorphism $I_u\colon \langle V_n\rangle \to L(|U|, |V|)$. For $v \in V_n$, $x \in |U|$, put

$$I_u(<\!v\!>)(<\!x\!>) = \begin{cases} <\!z_V(v)\!> & \text{if } x = z_U(u) \text{ for } z \in \mathring{\mathbf{\Delta}}^n, \\ 0 & \text{otherwise.} \end{cases}$$

This equality is preserved if we replace $<\!v\!>$ by $w \in \langle V_n\rangle$ and $<\!z_V(v)\!>$ by $\langle z_V\rangle(w)$. It suffices to show that

$$\tilde{P}(t) = \sum_{n \geqslant 0,\, u \in U_n^\times} I_u(t_n(u)), \qquad t \in \mathrm{Si}(U, \langle V\rangle).$$

Evaluating each side at $<\!x\!>$, $x = z_U(u)$, where $z \in \mathring{\mathbf{\Delta}}^n$ and $u \in U_n^\times$ ($n \geqslant 0$), we get $\langle z_V\rangle(t_n(u))$. □

Lemma 11.1 allows us to introduce the homomorphism $P : \mathrm{Si}(U, \langle V\rangle) \to L(|U|, [V|)$, $P(t) = \tilde{P}(t)$.

Lemma 11.2. *The diagram*

$$\begin{tikzcd} \mathrm{Si}(U, V) \arrow[r, "c^U_\#"] \arrow[d, "|?|"'] & \mathrm{Si}(U, \langle V\rangle) \arrow[d, "P"] \\ C(|U|, |V|) \arrow[r, "\langle ?\rangle"] & L(|U|, |V|), \end{tikzcd}$$

where $c = c_V\colon V \to \langle V\rangle$ is the canonical simplicial map, is commutative.

Proof. For $s \in \mathrm{Si}(U, V)$ and $x \in |U|$, $x = z_U(u)$, where $z \in \mathbf{\Delta}^n$ and $u \in U_n$ ($n \geqslant 0$), we have $(P \circ c^U_\#)(s)(<\!x\!>) = P(c \circ s)(<\!x\!>) = \langle z_V\rangle((c \circ s)_n(u)) = <\!z_V(s_n(u))\!> = <\!|s|(z_U(u))\!> = <\!|s|(x)\!> = \langle |s|\rangle(<\!x\!>)$. □

12. Necessity in Theorem 1.1

Claim 12.1. *Let U be a compact polyhedral simplicial set, V be a fibrant simplicial set, $h\colon [|U|,|V|] \to [S|U|,S|V|]$ be the main invariant, M be an abelian group, and $f\colon [|U|,|V|] \to M$ be a straight invariant. Then there exists a homomorphism $d\colon [S|U|,S|V|] \to M$ such that $f = d \circ h$.*

Proof. Since f is straight, there is a homomorphism $F\colon L(|U|,|V|) \to M$ such that $f([a]) = F(\langle a\rangle)$ for $a \in C(|U|,|V|)$. Consider the diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
\langle C(|U|,|V|)\rangle \arrow[rrr, "k^+"] \arrow[ddd, "\langle r\rangle"'] & & & L(|U|,|V|) \arrow[ddd, "F"] \\
& \langle \mathrm{Si}(U,V)\rangle \arrow[ul, "\langle I\rangle"'] \arrow[r, "(c^U_\#)^+"] \arrow[d, "\langle p\rangle"'] & \mathrm{Si}(U,\langle V\rangle) \arrow[ur, "P"'] \arrow[d, "q"] & \\
& \langle [U,V]\rangle \arrow[r, "(c^U_*)^+"] \arrow[dl, "\langle i\rangle"'] & {[U,\langle V\rangle]} \arrow[dr, dashed, "\tilde d"] & \\
\langle [|U|,|V|]\rangle \arrow[rrr, "f^+"] & & & M.
\end{tikzcd}$$

Here the inner square is as in § 10, $r = [?]\colon C(|U|,|V|) \to [|U|,|V|]$ (the projection), $k = \langle ?\rangle\colon C(|U|,|V|) \to L(|U|,|V|)$, $I = |?|\colon \mathrm{Si}(U,V) \to C(|U|,|V|)$ (the geometric realization map), $i\colon [U,V] \to [|U|,|V|]$, $[s] \mapsto [|s|]$, and P is as in § 11. By Lemma 11.2, the upper trapezium is commutative. The solid arrows are defined and form a commutative subdiagram. Since the inner square is cocartesian by Lemma 10.1, the dashed arrow $\tilde d$ is well-defined by the condition of commutativity of the diagram.

Consider the diagram

$$\begin{tikzcd}
\langle [U,V]\rangle \arrow[rr, "(c^U_*)^+"] \arrow[dd, "\langle i\rangle"'] & & {[U,\langle V\rangle]} \arrow[dl, "\tilde d"'] \arrow[dd, "e"] \\
& M & \\
\langle [|U|,|V|]\rangle \arrow[rr, "h^+"] \arrow[ur, "f^+"] & & {[S|U|,S|V|],} \arrow[ul, "d"']
\end{tikzcd}$$

where e is the isomorphism from Lemma 4.1 and $d = \tilde d \circ e^{-1}$. The square is commutative by Lemma 4.1. We have $\tilde d \circ (c^U_*)^+ = f^+ \circ \langle i\rangle$. Since V is fibrant, i is a bijection, and thus $\langle i\rangle$ is an isomorphism. We get $f^+ = d \circ h^+$ (so the diagram is commutative). Therefore, $f = d \circ h$. $\square$

Proposition 12.2. *Let X be space homotopy equivalent to a compact CW-complex, Y be a space, $h\colon [X,Y] \to [SX,SY]$ be the main invariant, M be an abelian group, and $f\colon [X,Y] \to M$ be a straight invariant. Then there exists a homomorphism $d\colon [SX,SY] \to M$ such that $f = d \circ h$.*

Proof. There are a homotopy equivalence $r\colon X \to |U|$ and a weak homotopy equivalence $s\colon |V| \to Y$, where U is a compact polyhedral simplicial set and V is a fibrant simplicial set. We construct the commutative diagram

$$\begin{tikzcd}
{[|U|,|V|]} \arrow[rr, "\tilde h"] \arrow[dd, "k"'] \arrow[dr, "\tilde f"] & & {[S|U|,S|V|]} \arrow[dl, "\tilde d"'] \arrow[dd, "l"] \\
& M & \\
{[X,Y]} \arrow[rr, "h"] \arrow[ur, "f"] & & {[SX,SY].} \arrow[ul, "d"']
\end{tikzcd}$$

Here the bijection k and the isomorphism l are induced by the pair (r,s) and $\tilde h$ is the main invariant. The square is commutative. By Lemma 3.1, the invariant $\tilde f = f \circ k$ is straight. By Claim 12.1, there is a homomorphism $\tilde d$ such that $\tilde f = \tilde d \circ \tilde h$. Set $d = \tilde d \circ l^{-1}$. Since k is a bijection, we get $f = d \circ h$ (so the diagram is commutative). $\square$

13. Three counterexamples

The Hawaiian ear-ring. Let us show that the hypothesis about the homotopy type of Y in Theorem 1.1 and Proposition 7.3 is essential. Let X be the one-point compactification of the ray $\mathbb{R}_+ = (0,\infty)$ (a circle) and Y be that of the space $\mathbb{R}_+ \setminus \mathbb{N}$ (the Hawaiian ear-ring [3, Example 1.25]). We define a map $m \in C(X,Y)$ by putting

$$m(x) = \left[\frac{x+1}{2}\right] + (-1)^{[x/2]}\{-x\}$$

for $x \in \mathbb{R}_+ \setminus \mathbb{N}$. Here $[t]$ and $\{t\}$ are the integral and the fractional (respectively) parts of a number $t \in \mathbb{R}$. The element of $\pi_1(Y,\infty)$ represented by the loop m is the (reasonably understood) infinite product of commutators

$$(*) \qquad \prod_{p=0}^{\infty} [u_{2p}, u_{2p+1}],$$

where u_q is the element realized by the closure of the interval $(q, q+1)$. Let $e \in H_1(X)$ be the standard generator. As in [4, p. 76], we get that the element $m_*(e) \in H_1(Y)$ has infinite order.

Therefore, there is a homomorphism $k\colon H_1(Y) \to \mathbb{Q}$ such that $k(m_*(e)) = 1$. We define a homomorphism $d\colon [SX, SY] \to \mathbb{Q}$ by putting $d([v]) = k(v_*(e))$ for a morphism $v\colon SX \to SY$. Let $h\colon [X, Y] \to [SX, SY]$ be the main invariant. We show that *the invariants $d \circ h$ and thus h are not straight.*

For $y \in Y$ and $i = 0, 1$, put $y_{(i)} \in Y$ equal to ∞ if $i = 1$ and to y otherwise. For $i, j = 0, 1$, we define a map $r_{ij} \in C(Y, Y)$. For $y \in \mathbb{R}_+ \setminus \mathbb{N}$, we put $r_{ij}(y)$ equal to $y_{(j)}$ if $[y]$ is odd and to $y_{(i)}$ otherwise. For elements z_{ij}, $i, j = 0, 1$, of an abelian group, put $\vee_{ij} z_{ij} = z_{00} - z_{10} - z_{01} + z_{11}$. Clearly, $\vee_{ij}\langle r_{ij}\rangle = 0$ in $L(Y, Y)$. Put $a_{ij} = r_{ij} \circ m \in C(X, Y)$. We get $\vee_{ij}\langle a_{ij}\rangle = 0$ in $L(X, Y)$. Therefore, $\vee_{ij} f([a_{ij}]) = 0$ for any straight invariant f. We show that this is false for the invariant $d \circ h$. We have $a_{00} = m$; the map a_{11} is constant. It is easy to see that the maps a_{10} and a_{01} are null-homotopic (this "follows formally" from the presentation $(*)$ and the equalities $r_{10\,*}(u_{2p}) = r_{01\,*}(u_{2p+1}) = 1$). We get $\vee_{ij}(d \circ h)([a_{ij}]) = (d \circ h)([m]) = k(m_*(e)) = 1$. □

Using [1, Theorem 2], one can make the spaces X and Y simply-connected in this example.

The Warsaw circle. Let us show that the hypothesis about the homotopy type of X in Theorem 1.1 and Proposition 12.2 is essential and cannot be replaced by the weaker assumption that X is weakly homotopy equivalent to a compact CW-complex. Let X be the Warsaw circle [3, Exercise 7 in § 1.3] and Y be the unit circle in $\mathbb{C}$. Y is a topological abelian group. The group $[X, Y]$ is non-zero by [3, Exercise 7 in § 1.3, Proposition 1.30] and torsion-free by [6, Theorem 1 in § 56-III]. Therefore, there is a non-zero homomorphism $f\colon [X, Y] \to \mathbb{Q}$. By Lemma 7.1, f is a straight invariant. Since X is weakly homotopy equivalent to a point [3, Exercise 10 in § 4.1] and Y is 0-connected, the main invariant $h\colon [X, Y] \to [SX, SY]$ is constant. Therefore *there exists no homomorphism $d\colon [SX, SY] \to \mathbb{Q}$ such that $f = d \circ h$.* □

An infinite discrete space. Let us show that the word "compact" in the hypothesis about the homotopy type of X in Theorem 1.1 and Proposition 12.2 is essential (see also § 14).

Note that, for an infinite set X, the subgroup $B(X) \subseteq \mathbb{Z}^X$ is not a direct summand because the group $\mathbb{Z}^X$ is reduced and the group $\mathbb{Z}^X / B(X)$ is divisible and non-zero.

Let X and Y be discrete spaces, X infinite and $Y = \{y_0, y_1\}$. Introduce the function $k\colon Y \to \mathbb{Z}$, $y_i \mapsto i$, $i = 0, 1$. Consider the invariant $f\colon [X, Y] \to B(X)$, $[a] \mapsto k \circ a$, $a \in C(X, Y)$.

The invariant f is straight because, for the homomorphism $F\colon L(X,Y) \to B(X)$, $F(u)(x) = k^+(u(\text{<}x\text{>}))$, $x \in X$, $u \in L(X,Y)$, we have $f([a]) = F(\langle a \rangle)$, $a \in C(X,Y)$.

Let $h\colon [X,Y] \to [SX,SY]$ be the main invariant. We show that *there exists no homomorphism* $d\colon [SX,SY] \to B(X)$ *such that* $f = d \circ h$. Assume that there is such a d.

Consider the homomorphism $l\colon \mathbb{Z}^X \to \mathrm{Hom}(\langle X \rangle, \langle Y \rangle)$, $l(v)(\text{<}x\text{>}) = v(x)(\text{<}y_1\text{>} - \text{<}y_0\text{>})$, $x \in X$, $v \in \mathbb{Z}^X$. We have $l(f([a])) = \langle a \rangle - \langle a_0 \rangle$, $a \in C(X,Y)$, where $a_0\colon X \to Y$, $x \mapsto y_0$. Clearly, there is an isomorphism $e\colon \mathrm{Hom}(\langle X \rangle, \langle Y \rangle) \to [SX,SY]$ such that $e(\langle a \rangle) = h([a])$, $a \in C(X,Y)$. Consider the composition

$$r\colon \mathbb{Z}^X \xrightarrow{l} \mathrm{Hom}(\langle X \rangle, \langle Y \rangle) \xrightarrow{e} [SX,SY] \xrightarrow{d} B(X).$$

For $a \in C(X,Y)$, we have $r(f([a])) = (d \circ e \circ l \circ f)([a]) = d(e(\langle a \rangle - \langle a_0 \rangle)) = d(h([a]) - h([a_0])) = f([a]) - f([a_0]) = f([a])$. Since the elements $f([a])$, $a \in C(X,Y)$, generate $B(X)$, we get $r|_{B(X)} = \mathrm{id}$, which is impossible. $\square$

14. Invariants of maps $\mathbb{R}P^\infty \to \mathbb{R}P^\infty$

Here we show that the word "compact" in the hypothesis about the homotopy type of X in Theorem 1.1 and Proposition 12.2 is essential even if M is divisible. (Possibly, if M is divisible and/or Y is (simply-) connected, the hypothesis about the homotopy type of X can be replaced by the weaker assumption that X is homotopy equivalent to a finite-dimensional CW-complex.)

Let X and Y be spaces. A set $E \subseteq X$ is called *Y-representative* if any maps $a, b \in C(X,Y)$ equal on E are homotopic. X is called *Y-unitary* if any finite cover of X contains a Y-representative set.

Lemma 14.1. *Let M be a divisible group. If X is Y-unitary, then any invariant $f\colon [X,Y] \to M$ is straight.*

Proof. Introduce the maps $r = [?]\colon C(X,Y) \to [X,Y]$ (the projection) and $k = \langle ? \rangle\colon C(X,Y) \to L(X,Y)$. We seek a homomorphism F giving the commutative diagram

$$\begin{array}{ccc} \langle C(X,Y) \rangle & \xrightarrow{k^+} & L(X,Y) \\ \Big\downarrow{\scriptstyle \langle r \rangle} & & \Big\downarrow{\scriptstyle F} \\ \langle [X,Y] \rangle & \xrightarrow{f^+} & M. \end{array}$$

Since M is divisible, it suffices to show that $\mathrm{Ker}\, k^+ \subseteq \mathrm{Ker} \langle r \rangle$. Take an element $w \in \mathrm{Ker}\, k^+$. We show that $w \in \mathrm{Ker} \langle r \rangle$. There are a finite set I, a map $l\colon I \to C(X,Y)$, and an element $v \in \langle I \rangle$ such that $\langle l \rangle(v) = w$.

Put $a_i = l(i)$, $i \in I$. For an equivalence d on I, let $p_d\colon I \to I/d$ be the projection. Let N be the set of equivalences d on I such that $\langle p_d\rangle(v) = 0$ in $\langle I/d\rangle$.

Take $x \in X$. Consider the equivalence $d(x) = \{(i,j) : a_i(x) = a_j(x)\}$ on I. We show that $d(x) \in N$. We have the commutative diagrams

$$\begin{tikzcd} I \arrow[r, "l"] \arrow[d, "p_{d(x)}"'] & C(X,Y) \arrow[d, "e_x"] \\ I/d(x) \arrow[r, "l_x"] & Y, \end{tikzcd} \qquad \begin{tikzcd} \langle C(X,Y)\rangle \arrow[r, "k^+"] \arrow[d, "\langle e_x\rangle"'] & L(X,Y) \arrow[dl, "h_x"] \\ \langle Y\rangle, & \end{tikzcd}$$

where the map l_x is defined by the condition of commutativity of the diagram, e_x is the map of evaluation at x, and h_x is the homomorphism of evaluation at $<x>$. We get $\langle l_x\rangle(\langle p_{d(x)}\rangle(v)) = \langle e_x\rangle(\langle l\rangle(v)) = \langle e_x\rangle(w) = h_x(k^+(w)) = 0$. Since l_x is injective, we get $\langle p_{d(x)}\rangle(v) = 0$, which is what we promised.

For an equivalence d on I, put $E_d = \{x \in X : (i,j) \in d \Rightarrow a_i(x) = a_j(x)\}$. Since $x \in E_{d(x)}$ for any $x \in X$, the family E_d, $d \in N$, is a cover of X. Since X is Y-unitary, E_d is Y-representative for some $d \in N$. For $(i,j) \in d$, the maps a_i and a_j are equal on E_d and thus homotopic. Therefore, there is a map m giving the commutative diagram

$$\begin{tikzcd} I \arrow[r, "l"] \arrow[d, "p_d"'] & C(X,Y) \arrow[d, "r"] \\ I/d \arrow[r, "m"] & {[X,Y]}. \end{tikzcd}$$

We get $\langle r\rangle(w) = \langle r\rangle(\langle l\rangle(v)) = \langle m\rangle(\langle p_d\rangle(v)) = 0$ because $d \in N$. $\square$

Hereafter, let X and Y be homeomorphic to $\mathbb{R}P^\infty$.

Lemma 14.2. *X is Y-unitary.*

Proof. Let $H^\bullet$ be the $\mathbb{Z}_2$-cohomology. Let $g \in H^1X$ and $h \in H^1Y$ be the non-zero classes.

We show that $(*)$ a set $E \subseteq X$ is Y-representative if $g|_U \neq 0$ for any neighbourhood U of E. If maps $a, b \in C(X,Y)$ are equal on E, they are homotopic on some neighbourhood U of E. Then $a^*(h)|_U = b^*(h)|_U$. Since $g|_U \neq 0$, the homomorphism $?|_U\colon H^1X \to H^1U$ is injective. Therefore, $a^*(h) = b^*(h)$. Since Y is a $\mathcal{K}(\mathbb{Z}_2,1)$ space, a and b are homotopic, as needed.

We show that X is Y-unitary. Assume that $X = E_1 \cup \ldots \cup E_n$, where the sets E_i are not Y-representative. By $(*)$, each E_i has a neighbourhood U_i with $g|_{U_i} = 0$. Since $U_1 \cup \ldots \cup U_n = X$, we get $g^n = 0$, which is false. $\square$

We have $[X, Y] = \{u_0, u_1\}$, where u_0 is the class of a constant map and u_1 is that of a homeomorphism. Consider the invariant $f\colon [X, Y] \to \mathbb{Q}$, $u_i \mapsto i$, $i = 0, 1$. By Lemmas 14.2 and 14.1, f is straight. Let $h\colon [X, Y] \to [SX, SY]$ be the main invariant. Using the isomorphism

$$[SX, SY] \longrightarrow \prod_{i\in\mathbb{Z}} \mathrm{Hom}(H_iX, H_iY), \qquad [v] \mapsto v_*,$$

we get $2h(u_0) = 2h(u_1)$. Therefore, *there exists no homomorphism* $d\colon [SX, SY] \to \mathbb{Q}$ *such that* $f = d \circ h$. $\square$

15. K-STRAIGHT INVARIANTS

Let K be a unital ring. *K-modules* are unital.

K-module $L_K(X, Y)$. For a set X, let $\langle X\rangle_K$ be the (free) K-module with the basis $X^\sharp_K \subseteq \langle X\rangle_K$ endowed with the bijection $X \to X^\sharp_K$, $x \mapsto {<}x{>}_K$. For sets X and Y, let $L_K(X, Y) \subseteq \mathrm{Hom}_K(\langle X\rangle_K, \langle Y\rangle_K)$ be the K-submodule generated by the K-homomorphisms u such that $u(X^\sharp_K) \subseteq Y^\sharp_K \cup \{0\}$. A map $a\colon X \to Y$ induces a K-homomorphism $\langle a\rangle_K \in L_K(X, Y)$, $\langle a\rangle_K({<}x{>}_K) = {<}a(x){>}_K$.

K-straight invariants. Let X and Y be spaces and M be a K-module. An invariant $f\colon [X, Y] \to M$ is called *K-straight* if there exists a K-homomorphism $\tilde F\colon L_K(X, Y) \to M$ such that $f([a]) = \tilde F(\langle a\rangle_K)$ for all $a \in C(X, Y)$.

Proposition 15.1. *An invariant $f\colon [X, Y] \to M$ is K-straight if and only if it is straight.*

Proof is given in § 16.

The K-main invariant $\tilde h\colon [X, Y] \to [S_KX, S_KY]_K$. Let S_KX be the K-complex of singular chains of X with coefficients in K and $[S_KX, S_KY]_K$ be the K-module of K-chain homotopy classes of K-morphisms $S_KX \to S_KY$. For $a \in C(X, Y)$, let $S_Ka\colon S_KX \to S_KY$ be the induced K-morphism and $[S_Ka]_K \in [S_KX, S_KY]_K$ be its K-chain homotopy class. The invariant $\tilde h\colon [X, Y] \to [S_KX, S_KY]_K$, $[a] \mapsto [S_Ka]_K$, is called *K-main.*

Theorem 15.2. *Suppose that X is homotopy equivalent to a compact CW-complex and Y is homotopy equivalent to a CW-complex. An invariant $f\colon [X, Y] \to M$ is K-straight if and only if there exists a K-homomorphism $\tilde d\colon [S_KX, S_KY]_K \to M$ such that $f = \tilde d \circ \tilde h$.*

Proof is given in § 16. For $K = \mathbb{Z}$, this is Theorem 1.1.

16. K-STRAIGHT INVARIANTS: PROOFS

Let X and Y be sets. We define a homomorphism $e\colon L(X,Y) \to L_K(X,Y)$. For $u \in L(X,Y)$, let $e(u)$ be the K-homomorphism giving the commutative diagram

$$\begin{tikzcd}
\langle X\rangle \arrow[r, "u"] \arrow[d, "i_X"'] & \langle Y\rangle \arrow[d, "i_Y"] \\
\langle X\rangle_K \arrow[r, "e(u)"] & \langle Y\rangle_K,
\end{tikzcd}$$

where i_X is the homomorphism $<x> \mapsto <x>_K$ and i_Y is similar.

For an abelian group A, a K-module M, and a homomorphism $t\colon A \to M$, we introduce the K-homomorphism $t^{(K)}\colon K \otimes A \to M$, $1 \otimes a \mapsto t(a)$.

Lemma 16.1. *$e^{(K)}\colon K \otimes L(X,Y) \to L_K(X,Y)$ is a K-isomorphism.*

Proof. For $w \in \langle Y\rangle_K$ and $y \in Y$, let $w/y \in K$ be the coefficient of $<y>_K$ in w. For $v \in L_K(X,Y)$ and $k \in K\setminus\{0\}$, we introduce the homomorphism $v_k \in L(X,Y)$,

$$v_k(<x>) = \sum_{y\in Y: v(<x>_K)/y=k} <y>, \qquad x \in X.$$

It is not difficult to verify that the map $d\colon L_K(X,Y) \to K \otimes L(X,Y)$,

$$d(v) = \sum_{k\in K\setminus\{0\}} k \otimes v_k,$$

is a K-homomorphism. Using this, we get $e^{(K)} \circ d = \mathrm{id}$ and $d \circ e^{(K)} = \mathrm{id}$. $\square$

Proof of Proposition 15.1. Necessity. Let f be K-straight. There is a K-homomorphism $\tilde F\colon L_K(X,Y) \to M$ such that $f([a]) = \tilde F(\langle a\rangle_K)$, $a \in C(X,Y)$. Consider the homomorphism $F = \tilde F \circ e$:

$$\begin{tikzcd}
C(X,Y) \arrow[rr, "\langle ?\rangle_K"] \arrow[dr, "\langle ?\rangle"'] \arrow[dd, "{[?]}"'] & & L_K(X,Y) \arrow[dd, "\tilde F"] \\
& L(X,Y) \arrow[ur, "e"] \arrow[dr, "F"] & \\
{[X,Y]} \arrow[rr, "f"] & & M.
\end{tikzcd}$$

The diagram is commutative. We get $f([a]) = F(\langle a\rangle)$, $a \in C(X,Y)$. Therefore, f is straight.

Sufficiency. Let f be straight. There is a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a\rangle)$, $a \in C(X,Y)$. By Lemma 16.1, $e^{(K)}$ is a K-isomorphism. Consider the homomorphism $\tilde{F} = F^{(K)} \circ (e^{(K)})^{-1}$:

$$\begin{tikzcd}
C(X,Y) \arrow[rr, "\langle ?\rangle_K"] \arrow[dr, "\langle ?\rangle"'] \arrow[dd, "{[?]}"'] & & L_K(X,Y) \arrow[dd, bend left=60, "\tilde{F}"] \\
& L(X,Y) \arrow[ur, "e"] \arrow[r, "1\otimes ?"] \arrow[dr, "F"'] & K \otimes L(X,Y) \arrow[u, "e^{(K)}"'] \arrow[d, "F^{(K)}"] \\
{[X,Y]} \arrow[rr, "f"] & & M.
\end{tikzcd}$$

The diagram is commutative. We get $f([a]) = \tilde{F}(\langle a\rangle_K)$, $a \in C(X,Y)$. Therefore, f is K-straight. $\square$

The homomorphism $I\colon [SX,SY] \to [S_KX, S_KY]_K$. Let X and Y be spaces. A morphism $v\colon SX \to SY$ induces a K-morphism

$$S_KX = K \otimes SX \xrightarrow{\mathrm{id}\otimes v} K \otimes SY = S_KY.$$

Consider the homomorphism $I\colon [SX,SY] \to [S_KX, S_KY]_K$, $[v] \mapsto [\mathrm{id} \otimes v]_K$.

Lemma 16.2. *If the group $H_\bullet(X)$ is finitely generated, then the K-homomorphism*

$$I^{(K)}\colon K \otimes [SX,SY] \to [S_KX, S_KY]_K$$

is a K-split K-monomorphism, i. e. there exists a K-homomorphism $R\colon [S_KX, S_KY]_K \to K \otimes [SX,SY]$ such that $R \circ I^{(K)} = \mathrm{id}$.

Proof. This is a variant of the universal coefficient theorem, cf. [12, Theorems 5.2.8 and 5.5.10]. $\square$

Proof of Theorem 15.2. We have $\tilde{h} = I \circ h$, where $h\colon [X,Y] \to [SX,SY]$ is the main invariant. By Proposition 7.3, h is straight. Therefore, $\tilde{h}$ is straight. By Proposition 15.1, $\tilde{h}$ is K-straight.

This gives the sufficiency. Necessity. Let f be K-straight. By Proposition 15.1, f is straight. By Proposition 12.2, there is a homomorphism $d\colon [SX,SY] \to M$ such that $f = d \circ h$. By Lemma 16.2, there is a

K-homomorphism $\tilde{d}$ such that $\tilde{d} \circ I^{(K)} = d^{(K)}$:

$$\begin{tikzcd}
& {[SX, SY]} \arrow[d, "d"] \arrow[dr, "1\otimes ?"] \arrow[dd, bend left=90, looseness=2, "I"] & \\
{[X, Y]} \arrow[ur, "h"] \arrow[r, "f"] \arrow[dr, "\tilde{h}"'] & M & K \otimes [SX, SY] \arrow[l, "d^{(K)}"'] \arrow[dl, "I^{(K)}"'] \\
& {[S_K X, S_K Y]_K.} \arrow[u, "\tilde{d}"'] &
\end{tikzcd}$$

The diagram is commutative. In particular, $f = \tilde{d} \circ \tilde{h}$. □

References

[1] M. G. Barratt, J. Milnor, *An example of anomalous singular homology*, Proc. Amer. Math. Soc. **13** (1962), 293–297.

[2] L. Fuchs, *Infinite abelian groups*, vol. 2, Academic Press, 1973.

[3] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[4] G. Higman, *Unrestricted free products, and varieties of topological groups*, J. Lond. Math. Soc. **27** (1952), 73–81.

[5] P. Hill, *The additive group of commutative rings generated by idempotents*, Proc. Amer. Math. Soc. **38** (1973), 499–502.

[6] K. Kuratowski, *Topology*, vol. 2, Academic Press, PWN, 1968.

[7] S. S. Podkorytov, *An alternative proof of a weak form of Serre's theorem* (Russian), Zap. Nauchn. Sem. POMI **261** (1999), 210–221. English translation: J. Math. Sci. (N. Y.) **110** (2002), no. 4, 2875–2881.

[8] ———, *Mappings of the sphere to a simply connected space* (Russian), Zap. Nauchn. Sem. POMI **329** (2005), 159–194. English translation: J. Math. Sci. (N. Y.) **140** (2007), no. 4, 589–610.

[9] ———, *An iterated sum formula for a spheroid's homotopy class modulo 2-torsion*, arXiv:math/0606528 (2006).

[10] ———, *The order of a homotopy invariant in the stable case* (Russian), Mat. Sb. **202** (2011), no. 8, 95–116. English translation: Sb. Math. **202** (2011), no. 8, 1183–1206.

[11] ———, *On homotopy invariants of finite degree* (Russian), Zap. Nauchn. Sem. POMI **415** (2013), 109–136. English translation: J. Math. Sci. (N. Y.) **212** (2016), no. 5, 587–604.

[12] E. H. Spanier, *Algebraic topology*, McGraw-Hill, 1966.

[13] V. A. Zapol'skii, *Functional characterization of Vasil'ev invariants* (Russian), Zap. Nauchn. Sem. POMI **353** (2008), 39–53. English translation: J. Math. Sci. (N. Y.) **161** (2009), no. 3, 375–383.

Steklov Mathematical Institute at St. Petersburg; Fontanka 27, St. Petersburg, 191023, Russia

E-mail address: ssp@pdmi.ras.ru