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by

Ľubica Holá and Varun Jindal

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	Department of Mathematics & Statistics
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ON GRAPH AND FINE TOPOLOGIES

ĽUBICA HOLÁ AND VARUN JINDAL

Abstract. Let X be a Tychonoff space and (Y,d) be a metric space. Let $C(X,Y)$ be the space of continuous functions from X to Y and τ_Γ, τ_w be the graph and fine topologies on $C(X,Y)$, respectively. Let (Y,d) contain a nontrivial path. We prove nontrivial generalizations of some known results concerning τ_Γ and τ_w on $C(X)$. For example the following are equivalent (1) $(C(X,Y),\tau_\Gamma) = (C(X,Y),\tau_w)$; (2) X is a *cb*-space. Some topological properties of $(C(X,Y), \tau_\Gamma)$ and $(C(X,Y),\tau_w)$ are studied too.

Let X be a topological space and (Y,d) be a metric space. We will suppose that X and Y are infinite. As usual let $C(X,Y)$ be the space of continuous functions from X to Y and $C(X)$ be the space of continuous real-valued functions.

As in [10] denote by $\tau_p, \tau_u, \tau_w, \tau_\Gamma$ the topology of pointwise convergence, the topology of uniform convergence, the fine topology and the graph topology on $C(X,Y)$, respectively. Of course $\tau_p \subseteq \tau_u \subseteq \tau_w \subseteq \tau_\Gamma$ on $C(X,Y)$.

Given a function $\epsilon : X \to (0,\infty)$ and $f \in C(X,Y)$, define

$$B(f,\epsilon) = \{g \in C(X,Y) : d(f(x),g(x)) < \epsilon(x) \text{ for all } x \in X\}.$$

Denote by $C^+(X)$ ($LSC^+(X)$) the set of all strictly positive real-valued continuous (lower semicontinuous) functions defined on X.

The fine topology τ_w on $C(X,Y)$ (also called m-topology [2]) has as a base all sets of the form $B(f,\epsilon)$, where ϵ runs over all elements from

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$C^+(X)$. The fine topology on $C(X)$ was introduced by Hewitt [7] and it has been thoroughly investigated in the past [2, 1, 8, 9, 10]. It was proved in [2] and [10] that the graph topology τ_Γ on $C(X, Y)$ has as a base all sets of the form $B(f, \epsilon)$, where ϵ runs over all elements from $LSC^+(X)$.

Both $(C(X, Y), \tau_w)$ and $(C(X, Y), \tau_\Gamma)$ are Tychonoff topological spaces, in fact they are uniform spaces.

For general topological spaces X and Y the graph topology τ_Γ on $C(X, Y)$ was introduced by Naimpally in [15] as the topology which is generated by sets of the form

$$F_U = \{f \in C(X, Y) : \text{graph}(f) \subset U\},$$

where U runs over the family of open sets in $X \times Y$.

Notice that if X is T_1 and Y is T_2, then τ_Γ on $C(X, Y)$ is the relative Vietoris topology [14] inherited from the hyperspece of nonempty closed subsets of $X \times Y$ after identifying elements from $C(X, Y)$ with their graphs.

A topological space X is called a *cb*-space if it satisfies one of the following equivalent conditions [12, Theorem 1]:

(1) for every $f \in LSC^+(X)$ there is $\varphi \in C^+(X)$ such that $\varphi(x) \leq f(x)$ for every $x \in X$,

(2) for each decreasing sequence $(F_n)_n$ of closed sets with $\bigcap_n F_n = \emptyset$ there is a sequence $(Z_n)_n$ of zero sets with $\bigcap_n Z_n = \emptyset$ such that $F_n \subseteq Z_n$ for every n.

It was proved in [12, Corollary 2] that every *cb*-space is countably paracompact.

This note is motivated by a misprint in Proposition 1.2 in [10]. Proposition 1.2 in [10] states that for a Tychonoff space X and every metric space (Y, d), the coincidence $(C(X, Y), \tau_\Gamma) = (C(X, Y), \tau_w)$ is equivalent to the condition that X is a *cb*-space. Proposition 1.2 in [10] refers to [2] for the proof. In fact, if Y is the space of reals with the usual euclidean metric, the result was proved by van Dowen in [2].

We will present an example that the above mentioned equivalence does not work for any metric space (Y, d) and we will show that it holds for a metric space (Y, d), which contains a non-trivial path.

1. Main result

The following theorem was proved in [2].

Theorem 1.1. *Let X be a Tychonoff topological space. The following are equivalent:*

(1) $(C(X), \tau_\Gamma) = (C(X), \tau_w)$;

(2) X is a cb-space.

A question arises for which metric spaces Y the coincidence τ_Γ and τ_w on $C(X,Y)$ implies that X is a *cb*-space. Of course, if X is a *cb*-space, then $(C(X,Y),\tau_\Gamma) = (C(X,Y),\tau_w)$.

We have the following result.

Theorem 1.2. *Let X be a Tychonoff topological space and (Y,d) be a metric space which contains a nontrivial path. If $(C(X,Y),\tau_\Gamma) = (C(X,Y),\tau_w)$, then X is a cb-space.*

Proof. Let $\eta \in LSC^+(X)$, $h : [0,1] \to Y$ be a continuous function such that $h(z) \neq h(0)$ for every $z \neq 0$ and define $f(x) = h(0)$ for all $x \in X$. Then $f \in C(X,Y)$, and if $\eta^* = min\{\eta, d(h(0),h(1))/2\}$, then $\eta^* \in LSC^+(X)$.

Since $(C(X,Y),\tau_\Gamma) = (C(X,Y),\tau_w)$, there is $\varphi \in C^+(X)$ such that

$$B(f,\varphi) \subseteq B(f,\eta^*).$$

We claim that $\varphi(x) \leq \eta^*(x)$ for every $x \in X$, otherwise, $\eta^*(x_0) < \varphi(x_0)$ for some $x_0 \in X$. Let $O(x_0)$ be an open neighbourhood of x_0 such that $\eta^*(x_0) < \varphi(x)$ for every $x \in O(x_0)$. Since $\{z \in [0,1] : d(h(0),h(z)) \geq \eta^*(x_0)\}$ is a nonempty compact subset of $[0,1]$, it has a minimum $b > 0$. Note that $d(h(0),h(z)) < \eta^*(x_0)$ for all $z \in [0,b)$, and $d(h(0),h(b)) = \eta^*(x_0)$.

Since X is a Tychonoff space, there is a continuous function $H : X \to [0,b]$ such that $H(x_0) = b$ and $H(x) = 0$ for every $x \notin O(x_0)$. Define the function $G : X \to Y$ as follows: $G(z) = h(H(z))$ for every $z \in X$. Then G is a continuous function which is different from f and

$$G \in B(f,\varphi),$$

since for $x \in O(x_0)$, $d(f(x),G(x)) = d(h(0),h(H(x)) \leq \eta^*(x_0) < \varphi(x)$, and for $x \notin O(x_0)$, $d(f(x),G(x)) = d(h(0),h(0)) = 0 < \varphi(x)$. This implies $G \in B(f,\eta^*)$, which is a contradiction, since $d(f(x_0),G(x_0)) = d(h(0),h(b)) = \eta^*(x_0)$. In conclusion, we found $\varphi \in C^+(X)$ with $\varphi \leq \eta^* \leq \eta$, so X is a *cb*-space. □

The condition on Y to have a nontrivial path in Theorem 1.2 is essential.

Example 1.1. Let X be a connected topological space which is not countably paracompact (the Niemytzki plane is such a space). Let (Y, d) be a metric space with the $0-1$ metric. Then $(C(X,Y), \tau_p)$ is a discrete topological space. Thus we have $(C(X,Y), \tau_p) = (C(X,Y), \tau_u) = (C(X,Y), \tau_w) = (C(X,Y), \tau_\Gamma)$. Of course such a space X cannot be a *cb*-space.

Theorem 1.3. *Let X be a Tychonoff topological space and (Y, d) be a metric space which contains a nontrivial path. The following are equivalent:*

(1) $(C(X,Y), \tau_\Gamma) = (C(X,Y), \tau_w)$;
(2) X is a cb-space.

2. Other results

If X is a pseudocompact space and (Y, d) is a metric space, then $(C(X,Y), \tau_w) = (C(X,Y), \tau_u)$ [13, Proposition 2.1].

Theorem 2.1. *Let X be a Tychonoff space and (Y, d) be a metric space with a non isolated point. The following are equivalent:*
(1) $(C(X,Y), \tau_w) = (C(X,Y), \tau_u)$;
(2) X is pseudocompact.

Proof. It is sufficient to prove that (1) $\Rightarrow$ (2). Suppose that X is not pseudocompact. There is a continuous real-valued function h which is not bounded. Consider the function $g : X \to R$ defined as follows: $g(x) = \mid h(x) \mid +1$ for every $x \in X$.

Let $y \in Y$ be a non isolated point in Y and $\{y_n : n \in \omega\}$ be a sequence in Y which converges to y such that $y_n \neq y$ for every $n \in \omega$. Let $f : X \to Y$ be a function defined as follows: $f(x) = y$ for every $x \in X$. For every $n \in \omega$ define a function $f_n : X \to Y$ as $f_n(x) = y_n$ for every $x \in X$. It is easy to verify that $\{f_n : n \in \omega\}$ converges to f in $(C(X,Y), \tau_u)$. However $\{f_n : n \in \omega\}$ fails to converge to f in $(C(X,Y), \tau_w)$, since

$$f \in B(f, 1/g) = \{l \in C(X,Y) : d(f(x), l(x)) < 1/g(x) \text{ for all } x \in X\},$$

but no f_n is contained in $B(f, 1/g)$. □

The condition on a metric space (Y, d) to have a non isolated point in Theorem 2.1 is essential.

Example 2.1. Let X be a connected topological space which is not pseudocompact (the Niemytzki plane is such a space). Let (Y, d) be a metric space with the $0-1$ metric. Then $(C(X, Y), \tau_p)$ is a discrete topological space. Thus we have $(C(X, Y), \tau_p) = (C(X, Y), \tau_u) = (C(X, Y), \tau_w)$.

The following theorem was proved by Hansard in [6].

Theorem 2.2. *Let X be a Tychonoff space and (Y, d) be a metric space with a non isolated point. The following are equivalent:*
(1) $(C(X, Y), \tau_\Gamma) = (C(X, Y), \tau_u)$*;*
(2) X is countably compact.

Of course, if X is countably compact, then for every metric space (Y, d) we have $(C(X, Y), \tau_\Gamma) = (C(X, Y), \tau_u)$. However the condition on a metric space (Y, d) to have a non isolated point in Theorem 2.2 is essential.

Example 2.2. Let X be a connected topological space which is not countably compact (the Niemytzki plane is such a space). Let (Y, d) be a metric space with the $0-1$ metric. Then $(C(X, Y), \tau_p)$ is a discrete topological space. Thus we have $(C(X, Y), \tau_p) = (C(X, Y), \tau_u) = (C(X, Y), \tau_\Gamma)$.

3. Topological properties of τ_Γ and τ_w

Theorem 3.1. *Let X be a Tychonoff space and (Y, d) be a metric space which contains a nontrivial path. The following are equivalent:*
(1) $(C(X, Y), \tau_\Gamma)$ *is metrizable;*
(2) $(C(X, Y), \tau_\Gamma)$ *is a p-space;*
(3) $(C(X, Y), \tau_\Gamma)$ *is q-space;*
(4) $(C(X, Y), \tau_\Gamma)$ *is 1st countable;*
(5) $(C(X, Y), \tau_\Gamma)$ *is a Frechet space;*
(6) $(C(X, Y), \tau_\Gamma)$ *is sequential;*
(7) $(C(X, Y), \tau_\Gamma)$ *is a k-space;*
(8) $(C(X, Y), \tau_\Gamma)$ *is countably tight;*
(9) $(C(X, Y), \tau_\Gamma)$ *is a M-space;*
(10) $(C(X, Y), \tau_\Gamma)$ *is pointwise countable type;*
(11) $(C(X, Y), \tau_\Gamma)$ *is a r-space;*
(12) X is countably compact;
(13) $(C(X, Y), \tau_\Gamma)$ *is radial;*
(14) $(C(X, Y), \tau_\Gamma)$ *is pseudoradial.*

Proof. We prove $(8) \Rightarrow (12)$. Assume that X is not countably compact. There is a sequence $\{x_n : n \in \omega\}$ in X which has no cluster point in X. Let $\{U_n : n \in \omega\}$ be a pairwise disjoint sequence of open neighborhoods of the x_n's. There is a continuous function $H : [0, 1] \to Y$ such that

$H(x) \neq H(0)$ for every $x \in (0,1]$. Let f_0 be the function defined as $f_0(x) = H(0)$ for every $x \in X$. As in [10] define

$$L = \{g \in C(X,Y) : g(x_n) \neq H(0) \text{ for every } n \in \omega\}.$$

We claim that f_0 is in the closure of L in $(C(X,Y), \tau_\Gamma)$. Let G be an open set in $X \times Y$ such that $f_0 \in F_G$.

For every $n \in \omega$ there are V_n, an open neighborhood of x_n and $\epsilon_n > 0$ such that

$$\overline{V_n} \subseteq U_n \text{ and } V_n \times S(H(0), \epsilon_n) \subset G,$$

where $S(H(0), \epsilon_n) = \{y \in Y : d(y, H(0)) < \epsilon_n\}$.

For every $n \in \omega$ let $1 > \eta_n > 0$ be such that $d(H(0), H(z)) < \epsilon_n$ for every $z \in [0, \eta_n]$ and $\{\eta_n : n \in \omega\}$ converges to 0.

For every $n \in \omega$ define the continuous function $h_n : \overline{V_n} \to [0, \eta_n]$ such that $h_n(x_n) = \eta_n$ and $h_n(\overline{V_n} \setminus V_n) = 0$. Let $h : X \to [0,1]$ be a continuous function defined as follows:

$$h(x) = h_n(x) \text{ if } x \in \overline{V_n} \text{ for some } n \in \omega \text{ and } h(x) = 0 \text{ otherwise.}$$

It is easy to verify that the function $F : X \to Y$ defined as $F(x) = H(h(x))$ for every $x \in X$ is a continuous function and $F \in F_G$.

Since $(C(X,Y), \tau_\Gamma)$ is countably tight, there is a countable subset $L' = \{f_n : n \in \omega\}$ of L such that f_0 is in the closure of L' in $(C(X,Y), \tau_\Gamma)$.

Put $M = \{(x_n, f_n(x_n)) : n \in \omega\}$. Then M is a closed set in $X \times Y$ and $\text{graph}(f_0) \subset (X \times Y) \setminus M$, but no $\text{graph}(f_n)$ is contained in $(X \times Y) \setminus M$, a contradiction.

The implications (12) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are well known; for (3) $\Rightarrow$ (4) note that $(C(X,Y), \tau_\Gamma)$ is a Tychonoff space and it is a submetrizable space. The implication then follows from the fact that a regular q-space whose points are G_δ is 1st countable [4].

The implications (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are well known; as for (7) $\Rightarrow$ (8) see [10]. The implications (4) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (3) and (1) $\Rightarrow$ (9) $\Rightarrow$ (3) are well known. (12) $\Rightarrow$ (13) $\Rightarrow$ (14) are clear. To prove that (14) $\Rightarrow$ (6) we use the idea from the proof of Theorem 2.7 in [5]. □

Theorem 3.2. *Let X be a Tychonoff space and (Y,d) be a completely metrizable space which contains a nontrivial path. The following are equivalent:*

(1) $(C(X,Y), \tau_\Gamma)$ is completely metrizable;
(2) $(C(X,Y), \tau_\Gamma)$ is Čech complete;
(3) $(C(X,Y), \tau_\Gamma)$ is sieve complete;
(4) $(C(X,Y), \tau_\Gamma)$ is hereditarily Baire;
(5) X is countably compact.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are well known. We prove (4) $\Rightarrow$ (5). If X is not countably compact, there is a sequence $\{x_n : n \in \omega\}$ in X which has no cluster point in X. There is a continuous function $L : [0,1] \to Y$ such that $L(0) \neq L(1)$. By [3] there is a homeomorphism $H : [0,1] \to H([0,1]) \subseteq Y$ such that $H(0) = L(0)$ and $H(1) = L(1)$. For each $n \in \omega$ define, similarly as in [10],

$$S_n = \{f \in C(X,Y) : f(x_k) = H(0) \text{ for every } k \geq n\},$$

and put $S = \bigcup_{n \in \omega} S_n$.

It is easy to verify that S_n is closed in $(C(X,Y), \tau_\Gamma)$ for every $n \in \omega$, since S_n is closed in $(C(X,Y), \tau_p)$. Also, S is closed in $(C(X,Y), \tau_\Gamma)$. If $f \in C(X,Y) \setminus S$, then $f(x_n) \neq H(0)$ for infinitely many n (w.l.o.g., all n). Define

$$U = X \times Y \setminus (\{x_n : n \in \omega\} \times \{H(0)\}).$$

Then $\mathrm{graph}(f) \subset U$ and $F_U \subset C(X,Y) \setminus S$.

Put now $T = S \cap \{f \in C(X,Y) : f(X) \subseteq H([0,1])\}$ and for each $n \in \omega$, $T_n = S_n \cap \{f \in C(X,Y) : f(X) \subseteq H([0,1])\}$. $T = \bigcup_{n \in \omega} T_n$. Then T and T_n, $n \in \omega$ are closed in $(C(X,Y), \tau_\Gamma)$.

We will show that each T_n is nowhere dense in T. Assume, there is an open set V in $X \times Y$ such that $\emptyset \neq F_V \cap T \subseteq T_n$. Let $f \in F_V \cap T_n$. Then $f(x_n) = H(0)$. There is $\epsilon > 0$ and an open neighbourhood G of x_n such that

$$G \times S(H(0), \epsilon) \subseteq V \text{ and } G \cap \{x_j : j \in \omega\} = \{x_n\}.$$

Put

$$\eta = \min\{\epsilon/2, d(H(0), H(1))\}.$$

There must exist $a \in (0,1)$ such that for every $s \in (a,1)$, $d(H(0), H(s)) \geq \eta/2$. There is an open set $G_1 \subset G$, $x_n \in G_1$ with

$$d(f(x), H(0)) < \eta/2 \text{ for every } x \in G_1.$$

For every $x \in G_1$, $H^{-1}(f(x)) \in [0,a]$. Let $\alpha > 0$ be such that $a + \alpha < 1$ and $d(H(s), H(t)) < \eta/2$ for every $s, t \in [0,1]$ with $\mid s - t \mid \leq \alpha$.

Let $g : X \to [0, \alpha]$ be a continuous function such that $g(x_n) = \alpha$ and $g(x) = 0$ for every $x \notin G_1$. Then the function $l : X \to Y$ defined as $l(x) = H(g(x) + H^{-1}(f(x)))$ for every $x \in X$ belongs to the set $F_V \cap T \setminus T_n$, a contradiction.

The proof of (5) $\Rightarrow$ (1) follows from Theorem 3.1 and [10, Theorem 3.1]. □

Example 2.2 shows that the condition on Y to have a nontrivial path in Theorems 3.1 and 3.2 is essential.

Theorems 3.3 and 3.4 have been proved in [11] but these can also be proved using the ideas of the proofs of Theorems 3.1 and 3.2 above.

Theorem 3.3. *Let X be a Tychonoff space and (Y, d) be a metric space which contains a nontrivial path. The following are equivalent:*
(1) $(C(X, Y), \tau_w)$ is metrizable;
(2) $(C(X, Y), \tau_w)$ is a p-space;
(3) $(C(X, Y), \tau_w)$ is q-space;
(4) $(C(X, Y), \tau_w)$ is 1st countable;
(5) $(C(X, Y), \tau_w)$ is a Frechet space;
(6) $(C(X, Y), \tau_w)$ is sequential;
(7) $(C(X, Y), \tau_w)$ is a k-space;
(8) $(C(X, Y), \tau_w)$ is countably tight;
(9) $(C(X, Y), \tau_w)$ is a M-space;
(10) $(C(X, Y), \tau_w)$ is pointwise countable type;
(11) $(C(X, Y), \tau_w)$ is a r-space;
(12) X is pseudocompact;
(13) $(C(X, Y), \tau_w)$ is radial;
(14) $(C(X, Y), \tau_w)$ is pseudoradial.

Theorem 3.4. *Let X be a Tychonoff space and (Y, d) be a complete metric space which contains a nontrivial path. The following are equivalent:*
(1) $(C(X, Y), \tau_w)$ is completely metrizable;
(2) $(C(X, Y), \tau_w)$ is Čech complete;
(3) $(C(X, Y), \tau_w)$ is sieve complete;
(4) $(C(X, Y), \tau_w)$ is hereditarily Baire;
(5) X is pseudocompact.

Example 2.1 shows that the condition on Y to have a nontrivial path in Theorems 3.3 and 3.4 is essential.

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(Holá) Academy of Sciences, Institute of Mathematics
Štefánikova 49, 81473 Bratislava, Slovakia
E-mail address: `hola@mat.savba.sk`

(Jindal) Department of Mathematics,
Malaviya National Institute of Technology Jaipur
E-mail address: `vjindal.maths@mnit.ac.in`