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BOUNDEDNESS OF THE RELATIVES OF UNIFORMLY CONTINUOUS FUNCTIONS

MANISHA AGGARWAL AND S. KUNDU

ABSTRACT. A function f from a metric space (X, d) to another metric space (Y, ρ) is said to be Cauchy-continuous if $(f(x_n))$ is Cauchy in (Y, ρ) for every Cauchy sequence (x_n) in (X, d). Recently in [5], Beer and Garrido have characterized those metric spaces (X, d) on which each Cauchy-continuous function defined on X is bounded. Since in the literature, we have various other kinds of sequences that are weaker than Cauchy sequences, in this paper we have discussed a few properties of functions preserving different kinds of sequences and characterized those metric spaces on which each such function is bounded. It suffices in each case to consider real-valued functions. We observe that a uniformly continuous function preserves all those sequences, so those aforesaid functions are actually the relatives of uniformly continuous functions.

1. INTRODUCTION

The concepts of compactness and completeness play a vital role in the theory of metric spaces. Surely for discussing completeness of a metric space, one has to consider its corresponding Cauchy sequences. We recall that a sequence (x_n) in (X, d) is said to be Cauchy if for every $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for each $n, j \geq n_o$, we have $d(x_n, x_j) < \epsilon$. Some classes of metric spaces satisfying properties stronger than completeness

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but weaker than compactness have been recently studied by many authors. One of such spaces is an Atsuji space, also widely known as UC space. A metric space (X, d) is called an Atsuji space if every real-valued continuous function on (X, d) is uniformly continuous. The Atsuji spaces, first studied extensively by Atsuji [2], have been the subject of a number of articles over the years. One of the most useful characterization of Atsuji spaces is in terms of sequences (discovered by Toader [20]): a metric space is Atsuji if and only if each pseudo-Cauchy sequence with distinct terms in the space has a cluster point, where a sequence (x_n) is said to be pseudo-Cauchy if $\forall\ \epsilon > 0$ and $\forall\ n \in \mathbb{N}$, $\exists\ k, j \in \mathbb{N}$, $k \neq j$ such that $k, j > n$ and $d(x_k, x_j) < \epsilon$. Thus, Toader's pseudo-Cauchy sequences are those for which pairs of terms are arbitrarily close frequently. But there is a second natural way to generalize the definition of Cauchy sequence [11, 12], known as cofinally Cauchy, which are obtained by replacing residual by cofinal in the definition of Cauchy sequences. Precisely, a sequence (x_n) is called cofinally Cauchy if $\forall\ \epsilon > 0$, $\exists$ an infinite subset $\mathbb{N}_\epsilon$ of $\mathbb{N}$ such that for each $n, j \in \mathbb{N}_\epsilon$, we have $d(x_n, x_j) < \epsilon$. In 2008, Beer [3] cast new light on those metric spaces in which each cofinally Cauchy sequence has a cluster point which is widely known as cofinally complete spaces. Such spaces lie strictly in between Atsuji and complete metric spaces.

Recently, two new kinds of metric spaces are introduced by Garrido and Meroño in [9], namely Bourbaki-complete and cofinally Bourbaki-complete metric spaces, which are stronger than complete metric spaces and weaker than compact metric spaces. Since a way to achieve a property stronger than completeness for a metric space consists of asking for the clustering of all the sequences belonging to some class bigger than the class of Cauchy sequences, Garrido and Meroño defined the class of Bourbaki-Cauchy sequences and the class of cofinally Bourbaki-Cauchy sequences (see definitions 2.18, 2.22). These sequences appeared when they considered the so-called Bourbaki-bounded sets. This notion of boundedness was introduced by Atsuji in [2], under the name of finitely chainable, in order to exhibit metric spaces where every real-valued uniformly continuous function is bounded. Garrido and Meroño characterized Bourbaki-bounded sets in terms of sequences in the same way that Cauchy sequences characterize total boundedness. In this way, Bourbaki-Cauchy sequences came into picture, whereas the concept of cofinally Bourbaki-Cauchy sequences is similar to that of cofinally Cauchy sequences. This parallelism gave us an idea of defining pseudo Bourbaki-Cauchy sequences, which act as a generalization of pseudo-Cauchy sequences (see Definition 2.27). We observe that the clustering of pseudo Bourbaki-Cauchy sequences of distinct terms also characterizes Atsuji spaces.

The role of functions is inevitable in the theory of metric spaces. Two important classes of functions, namely the class of continuous functions and that of uniformly continuous functions, are well known to all of us. In [19], R. F. Snipes has studied the functions, that lie strictly in between the two aforesaid important classes of functions, which he called Cauchy sequentially-regular, widely known as Cauchy-continuous functions. A function $f : (X, d) \to (Y, \rho)$ between two metric spaces is called Cauchy-continuous if for any Cauchy sequence (x_n) in (X, d), $(f(x_n))$ is a Cauchy sequence in (Y, ρ). Cauchy-continuous functions are important because: (1) some of the most useful theorems about uniformly continuous functions hold also for Cauchy-continuous functions; and (2) many of the important functions which occur in analysis are Cauchy-continuous but not uniformly continuous. This motivated us to look for the functions that preserve the aforesaid sequences which are generalizations of Cauchy sequences. Certainly, uniformly continuous functions preserve all those sequences. Though these newly defined classes of functions are not contained in the class of continuous functions, they help in giving characterizations of spaces like Atsuji, cofinally complete, Bourbaki-complete and cofinally Bourbaki-complete. We discuss a few of them in this paper. Moreover, we discuss a few properties of these functions and give various examples in the context of their relation with each other.

Recall that a function $f : (X, d) \to (Y, \rho)$ between two metric spaces is bounded if the image set $f(X)$ is bounded in (Y, ρ). It is known that a metric space (X, d) is compact if and only if every real-valued continuous function on X is bounded (usually termed as pseudocompact space). This motivated several authors, [2, 8, 15, 17], to find necessary and sufficient conditions for a metric space (X, d) such that each uniformly continuous function from X into any other metric space is bounded. Recently in 2014, Beer and Garrido have characterized the metric spaces on which every real-valued Cauchy-continuous function is bounded and shown them to be totally bounded metric spaces in [5]. This has inspired us to look for several interesting classes of bounded real-valued functions on metric spaces and subsequent characterization of such metric spaces. In fact, this serves as the main goal of this paper. Meanwhile, we obtain new characterizations of totally bounded and Bourbaki-bounded spaces.

The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}$ and $\mathbb{Q}$ denote the sets of real numbers, complex numbers, natural numbers and rational numbers respectively. Unless mentioned otherwise, $\mathbb{R}$ and its subsets carry the usual distance metric. If (X, d) is a metric space, $x \in X$ and $\delta > 0$, then $B(x, \delta)$ (or $B_\delta(x)$) denotes the open ball in (X, d), centered at x with radius δ. Also, $(\widehat{X}, d)$ denotes the completion of the metric space (X, d).

2. Results

It is known that the class of Cauchy-continuous functions is bigger than the class of uniformly continuous functions, where Cauchy-continuous functions are defined as follows:

Definition 2.1. A function $f : (X, d) \to (Y, \rho)$ between two metric spaces is said to be *Cauchy-continuous* if $(f(x_n))$ is Cauchy in (Y, ρ) for every Cauchy sequence (x_n) in (X, d).

Clearly, every Cauchy-continuous function between two metric spaces is continuous. The significance of Cauchy sequences in the theory of metric spaces makes this class of functions equally significant. Similar to Cauchy-continuous functions, we define another class of functions that preserve cofinally Cauchy sequences. But before that, we recall a few relevant definitions.

Definition 2.2. A sequence (x_n) in a metric space (X, d) is called *cofinally Cauchy* if $\forall\ \epsilon > 0$, $\exists$ an infinite subset $\mathbb{N}_\epsilon$ of $\mathbb{N}$ such that for each $n, j \in \mathbb{N}_\epsilon$, we have $d(x_n, x_j) < \epsilon$.

Definition 2.3. A metric space (X, d) is said to be *cofinally complete* if every cofinally Cauchy sequence in X clusters.

Definition 2.4. A function $f : (X, d) \to (Y, \rho)$ between two metric spaces is called *cofinally Cauchy regular* (or *CC-regular* for short) if $(f(x_n))$ is cofinally Cauchy in (Y, ρ) for every cofinally Cauchy sequence (x_n) in (X, d).

Clearly, every uniformly continuous function between two metric spaces is CC-regular but a CC-regular function need not be even continuous. For example, let $X = \{0, \frac{1}{n+1} : n \in \mathbb{N}\}$ and $Y = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual metric and let $f : X \to Y$ be defined as:

$$f(x) = \left\{ \begin{array}{ccc} 1 & : & x = 0 \\ x & : & else \end{array} \right\}.$$

Remark 2.5. Observe that a Cauchy-continuous function need not be CC-regular: as a metric subspace of the Hilbert space l_2 of square summable sequences, let $X = \{e_n + \frac{1}{n} e_k : n, k \in \mathbb{N}\}$, where e_n is a sequence with 1 at n^{th} place and 0 at all other places. Note X itself is countable. Let (x_n) be an enumeration. Then (x_n) is a cofinally Cauchy sequence of distinct terms with no Cauchy subsequence. Hence, the function defined as: $f : X \to \mathbb{N}$, $f(x_n) = n$, is Cauchy-continuous which is not CC-regular.

However, every continuous function on a cofinally complete space with values in a metric space is CC-regular. Thus, the function $f : \mathbb{R} \to \mathbb{R}$ defined as : $f(x) = x^2$ is CC-regular.

Since Cauchy-continuous functions preserve Cauchy sequences, they preserve totally bounded sets as well. Our next proposition says that CC-regular functions also preserve totally bounded sets. It can be proved using the following result given by Beer in [3].

Lemma 2.6. *Let (X, d) be a metric space. Then the following are equivalent:*
(a) *(X, d) is totally bounded.*
(b) *Each sequence in X is cofinally Cauchy.*
(c) *Each sequence in X is pseudo-Cauchy (see definition 2.9).*

Proposition 2.7. *Let $f : (X, d) \to (Y, \rho)$ be a CC-regular function from a metric space (X, d) to another metric space (Y, ρ) and A be a totally bounded subset of X. Then $f(A)$ is a totally bounded subset of Y.*

The converse of the previous proposition need not hold as we know that there exists a Cauchy-continuous function which is not CC-regular and every Cauchy-continuous function preserves totally bounded sets. Now we give one of our main results.

Theorem 2.8. *Let (X, d) be a metric space. Then the following are equivalent:*

- (a) *(X, d) is totally bounded.*
- (b) *Whenever (Y, ρ) is a metric space and $f : (X, d) \to (Y, \rho)$ is CC-regular, then f is bounded.*
- (c) *Whenever $f : (X, d) \to (Z, \sigma)$ is CC-regular where (Z, σ) is an unbounded metric space, then f is bounded.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Proposition 2.7, whereas $(b) \Rightarrow (c)$ is immediate.

$(c) \Rightarrow (a)$: Let $z_o \in Z$. Since (Z, σ) is unbounded, for every $n \in \mathbb{N}$, there exists $z_n \in Z$ such that $\sigma(z_n, z_o) > n$. Suppose that (X, d) is not totally bounded. Then there exists a sequence, say (x_n), in X satisfying the following condition: there exists $\delta > 0$ such that $d(x_n, x_m) > \delta$ for all $n \neq m$. Now define a function $f : X \to Z$ as:

$$f(x) = \left\{ \begin{array}{lcl} z_n & : & x = x_n \text{ for some n} \\ z_o & : & else \end{array} \right\}.$$

Then f is an unbounded CC-regular function on X because if (y_n) is a cofinally Cauchy sequence in X, then the sequence $(f(y_n))$ has a constant subsequence. Hence we get a contradiction. $\square$

In the literature one can find another important generalization of Cauchy sequences, known as pseudo-Cauchy sequences. These sequences play an important role in characterizing Atsuji spaces: a metric space is Atsuji if and only if every pseudo-Cauchy sequence of distinct points in the space clusters. The significance of pseudo-Cauchy sequences motivated us to talk about the functions that preserve such sequences. Before we define a new type of functions, let us recall a few relevant definitions.

Definition 2.9. A sequence (x_n) in (X, d) is said to be *pseudo-Cauchy* if $\forall\ \epsilon > 0$ and $\forall\ n \in \mathbb{N}$, there exist $k, j \in \mathbb{N}$, $k \neq j$ such that $k, j > n$ and $d(x_k, x_j) < \epsilon$.

Definition 2.10. A metric space (X, d) is called an *Atsuji space* if every real-valued continuous function on X is uniformly continuous.

Note that Atsuji spaces are also widely known as *UC spaces* in the literature. Now we are ready to define another kind of functions.

Definition 2.11. A function $f : (X, d) \to (Y, \rho)$ between two metric spaces is said to be *pseudo-Cauchy regular* (or *PC-regular* for short) if given any pseudo-Cauchy sequence (x_n) in (X, d), the sequence $(f(x_n))$ is pseudo-Cauchy in (Y, ρ).

Note that every uniformly continuous function between two metric spaces is PC-regular but a PC-regular function need not be continuous. Let us look at the relation between PC-regular and CC-regular functions.

Proposition 2.12. *Every PC-regular function from a metric space (X, d) to a metric space (Y, ρ) is CC-regular.*

Proof. Let (x_n) be a cofinally Cauchy sequence in X. Suppose that $(f(x_n))$ is not cofinally Cauchy. Then there exists $\epsilon_o > 0$, such that for all $n \in \mathbb{N}$, $f(x_m) \in B(f(x_n), \epsilon_o)$ for at most finitely many $m \in \mathbb{N}$. For $k = 1$, choose $m_1,\ n_1 \in \mathbb{N},\ m_1 < n_1$ such that $d(x_{m_1}, x_{n_1}) < 1$ and $\rho(f(x_{m_1}), f(x_{n_1})) \geq \epsilon_o$. Similarly, for $k = 2$, choose $m_2,\ n_2 \in \mathbb{N},\ n_1 < m_2 < n_2$ such that $d(x_{m_2}, x_{n_2}) < \frac{1}{2}$ and $\rho(f(x_{v_p}), f(x_{w_q})) \geq \epsilon_o$ for $v_p \neq w_q,\ p,\ q \in \{1, 2\}$ and $v,\ w \in \{m, n\}$. By induction, we can choose $m_k,\ n_k \in \mathbb{N},\ n_{k-1} < m_k < n_k$ such that $d(x_{m_k}, x_{n_k}) < \frac{1}{k}$ and $\rho(f(x_{v_p}), f(x_{w_q})) \geq \epsilon_o$ for $v_p \neq w_q$, $p,\ q \in \{1, ..., k\}$ and $v,\ w \in \{m, n\}$. Then the sequence $(x_{m_1}, x_{n_1}, x_{m_2}, x_{n_2}, ...)$ is pseudo-Cauchy in X but $(f(x_{m_1}), f(x_{n_1}), f(x_{m_2}), f(x_{n_2}), ...)$ is not pseudo-Cauchy in Y, which is a contradiction. $\square$

Note that a CC-regular function need not be PC-regular. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as : $f(x) = x^2$. Then f is CC-regular, but f is not PC-regular as the image of the pseudo-Cauchy sequence $(1, 1+1, 2, 2+\frac{1}{2}, 3, 3+\frac{1}{3}, ...)$ is not pseudo-Cauchy under f. One can find various properties of CC-regular and PC-regular functions in [1].

The proof of our next result is similar to that of Theorem 2.8.

Theorem 2.13. *Let (X,d) be a metric space. Then the following are equivalent:*

(a) *(X,d) is totally bounded.*
(b) *Whenever $f:(X,d)\to(Y,\rho)$ is a metric space and $f:X\to Y$ is PC-regular, then f is bounded.*
(c) *Whenever $f:(X,d)\to(Z,\sigma)$ is PC-regular where (Z,σ) is an unbounded metric space, then f is bounded.*

Before looking at the boundedness of our next class of functions, which is yet another relative of uniformly continuous functions, we need to give the following definition.

Definition 2.14. Let f be an arbitrary function defined on a metric space (X,d) with values in a metric space (Y,ρ). Then f is said to be *uniformly locally bounded* if $\exists\ \delta>0$ such that $\forall\ x\in X, f(B(x,\delta))$ is a bounded subset of Y.

The significance of these functions follows from their role in characterizing cofinally complete spaces via functions ([3]): a metric space (X,d) is cofinally complete if and only if every real-valued continuous function on X is uniformly locally bounded.

Theorem 2.15. *Let (X,d) be a metric space. Then the following are equivalent:*

(a) *(X,d) is totally bounded.*
(b) *Whenever (Y,ρ) is a metric space and $f:(X,d)\to(Y,\rho)$ is uniformly locally bounded, then f is bounded.*
(c) *Whenever $f:(X,d)\to(Z,\sigma)$ is uniformly locally bounded where (Z,σ) is an unbounded metric space, then f is bounded.*

Proof. The implication $(b)\Rightarrow(c)$ is immediate. The proof of $(c)\Rightarrow(a)$ is similar to that of Theorem 2.8.

$(a)\Rightarrow(b)$: Let f be a uniformly locally bounded function from X to Y. Thus, $\exists\ \epsilon>0$ such that $f(B(x,\epsilon))$ is bounded for all $x\in X$. Since X is totally bounded, $\exists\ x_1,...,x_n$ in X such that $X=\bigcup_{i=1}^n B(x_i,\epsilon)$. Hence, $f(X)=\bigcup_{i=1}^n f(B(x_i,\epsilon))$, which is bounded. □

In 1958, Atsuji introduced in [2] a notion, namely finitely chainable metric space, which was weaker than totally bounded metric space but stronger than bounded metric space. Recently in 2014, Garrido and Meroño called such spaces in [9] to be Bourbaki-bounded and gave nice characterization of such spaces in terms of sequences, which they called Bourbaki-Cauchy.

Proposition 2.16. ([9]) *A subset B of (X, d) is Bourbaki-bounded in X if and only if every sequence in B has a Bourbaki-Cauchy subsequence in X.*

This motivated us to look for the functions that preserve Bourbaki-Cauchy sequences. Now before giving our next result, let us recall a few relevant definitions.

Definitions 2.17. Let (X, d) be a metric space and ϵ be a positive number, then an ordered set of points $\{x_0, x_1, ..., x_m\}$ in X satisfying $d(x_{i-1}, x_i) < \epsilon$, where $i = 1, 2, ..., m$, is said to be an ϵ-chain of length m from x_o to x_m. We call X *$\epsilon-$chainable* if each two points in X can be joined by an $\epsilon-$chain, and X is called *chainable* if X is $\epsilon-$chainable for every $\epsilon > 0$.

Clearly, $\mathbb{R}$ is chainable. In fact, every connected metric space is chainable.

Definitions 2.18. Let (X, d) be a metric space. A sequence (x_n) is said to be *Bourbaki-Cauchy* in X if for every $\epsilon > 0$, there exist $m \in \mathbb{N}$ and $n_o \in \mathbb{N}$ such that whenever $n > j \geq n_o$, the points x_j and x_n can be joined by an $\epsilon-$chain of length m. Moreover, a subset B of (X, d) is said to be *Bourbaki-bounded* subset of X if for every $\epsilon > 0$, $\exists\ m \in \mathbb{N}$ and a finite collection of points $p_1, p_2, ..., p_k \in X$ such that every point of B can be joined with some element of the finite collection by an $\epsilon-$chain of length m.

Note that a subset of a metric space (X, d) which is Bourbaki-bounded in X, need not be Bourbaki-bounded in itself. For example, consider any infinite bounded uniformly discrete set in a normed linear space, where a subset A of a metric space (X, d) is said to be uniformly discrete if there exists $\delta > 0$ such that $d(x, y) > \delta$ for all $x, y \in A$, $x \neq y$. Now, we give the following terminology.

Definition 2.19. A function f from a metric space (X, d) to another metric space (Y, ρ) is said to be *Bourbaki-Cauchy regular* (or *BC-regular* for short) if $(f(x_n))$ is Bourbaki-Cauchy in (Y, ρ) for every Bourbaki-Cauchy sequence (x_n) in (X, d).

Note that every uniformly continuous function from a metric space (X, d) to another metric space (Y, ρ) is BC-regular, but a BC-regular function need not be continuous. For example, let $X = \{0, \frac{1}{n+1} : n \in \mathbb{N}\}$ and $Y = \mathbb{R}$ with the usual metric and let $f : X \to Y$ be defined as:

$$f(x) = \left\{ \begin{array}{ccc} 1 & : & x = 0 \\ x & : & else \end{array} \right\}.$$

A Cauchy-continuous function need not be BC-regular. For example, consider the l_2-space. Let $A = \{e_n : n \in \mathbb{N}\}$. Then (e_n) is a Bourbaki-Cauchy sequence in l_2 with no Cauchy subsequence. Consider the function $f : A \to \mathbb{R}$: $f(e_n) = n$. Then f is Cauchy-continuous. Hence, by Proposition 2.15 of [13], f can be extended to a real-valued Cauchy-continuous function f' on l_2 which will not be BC-regular as (e_n) is Bourbaki-Cauchy in l_2 but $(f'(e_n)) = (f(e_n)) = (n)$ is not Bourbaki-Cauchy in $\mathbb{R}$.

Using Proposition 2.16, we can easily prove our next proposition.

Proposition 2.20. *If $f : (X, d) \to (Y, \rho)$ is a BC-regular function between two metric spaces and A is a Bourbaki-bounded subset of X, then $f(A)$ is a Bourbaki-bounded subset of Y.*

Our next result characterizes the metric spaces on which every real-valued BC-regular function is bounded.

Theorem 2.21. *Let (X, d) be a metric space. Then the following are equivalent:*

- (a) *(X, d) is Bourbaki-bounded.*
- (b) *Whenever (Y, ρ) is a metric space and $f : (X, d) \to (Y, \rho)$ is BC-regular, then f is bounded.*
- (c) *Whenever $f : (X, d) \to (Z, \sigma)$ is BC-regular where (Z, σ) is an unbounded chainable metric space, then f is bounded.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Proposition 2.20 and $(b) \Rightarrow (c)$ is immediate.

$(c) \Rightarrow (a)$: Let $z_o \in Z$. Since (Z, σ) is unbounded, for every $n \in \mathbb{N}$, there exists $z_n \in Z$ such that $\sigma(z_n, z_o) > n$. Suppose that (X, d) is not Bourbaki-bounded. Then there exists a sequence (a_n) in X with distinct terms which has no Bourbaki-Cauchy subsequence in X. Now define a function $f : X \to Z$ as:

$$f(x) = \left\{ \begin{array}{lll} z_n & : & x = a_n \text{ for some n} \\ z_o & : & else \end{array} \right\}.$$

We claim that f is BC-regular. Let (x_n) be a Bourbaki-Cauchy sequence in X, then $a_m \in \{x_n : n \in \mathbb{N}\}$ for at most finitely many m otherwise (a_n) will have a Bourbaki-Cauchy subsequence. Hence, $\{f(x_n) : n \in \mathbb{N}\}$ is a finite set. Since (Z, σ) is chainable, $(f(x_n))$ is Bourbaki-Cauchy. Hence f is an unbounded BC-regular function, which is a contradiction. □

Note that chainability of (Z, σ) in the last theorem is necessary for (X, d) to be Bourbaki-bounded. For example, consider (X, d) to be $\mathbb{R}$ and (Z, σ) to be $\mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{N}$ be any BC-regular function then f must be a constant function otherwise there exists $x, y \in \mathbb{R}$, $x \neq y$ such that $f(x) = n_1$ and $f(y) = n_2$ where $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$.

Then the sequence $(x, y, x, y, ...)$ is Bourbaki-Cauchy in $\mathbb{R}$ but its image is not Bourbaki-Cauchy in $\mathbb{N}$, which is a contradiction.

The way Bourbaki-Cauchy sequences generalize the concept of Cauchy sequences, one can naturally generalize the concept of cofinally Cauchy sequences as well. In [9], Garrido and Meroño introduced this notion and called such sequences to be cofinally Bourbaki-Cauchy. Let us give the precise definition.

Definitions 2.22. Let (X, d) be a metric space. A sequence (x_n) is said to be *cofinally Bourbaki-Cauchy* in X if for every $\epsilon > 0$, there exist $m \in \mathbb{N}$ and an infinite subset $\mathbb{N}_\epsilon$ of $\mathbb{N}$ such that the points x_j and x_n can be joined by an $\epsilon-$chain of length m for every $j, n \in \mathbb{N}_\epsilon$. Moreover, a metric space is said to be *cofinally Bourbaki-complete* if every cofinally Bourbaki-Cauchy sequence in the space clusters.

Since the collection of cofinally Cauchy sequences in a metric space is contained in a bigger collection of cofinally Bourbaki-Cauchy sequences, every cofinally Bourbaki-complete space is cofinally complete. Now, we introduce another relative of uniformly continuous function.

Definition 2.23. A function f from a metric space (X, d) to another metric space (Y, ρ) is said to be *cofinally Bourbaki-Cauchy regular* (or *CBC-regular* for short) if $(f(x_n))$ is cofinally Bourbaki-Cauchy in (Y, ρ) for every cofinally Bourbaki-Cauchy sequence (x_n) in (X, d).

A CBC-regular function need not be BC-regular. For example, let $f : \mathbb{R} \to \{0, 1\}$ be defined as:

$$f(x) = \left\{ \begin{array}{ccc} 1 & : & x \in \mathbb{Q} \\ 0 & : & else \end{array} \right\}.$$

Here note that if we would have taken the range of the function to be $\mathbb{R}$ instead of $\{0, 1\}$, then this example won't have worked. Moreover, a BC-regular function need not be CBC-regular: consider the example given in remark 2.5, the function f is BC-regular as every Bourbaki-Cauchy sequence in X is eventually constant.

A CBC-regular function need not be CC-regular. For example, let $f : \{\frac{1}{n} : n \in \mathbb{N}\} \to l_2$ be defined as: $f(\frac{1}{n}) = e_n$. In fact, the converse is also not true. For example, let $f : l_2 \to \mathbb{N}$ be defined as,

$$f(x) = \left\{ \begin{array}{ccl} n & : & x = e_n \text{ for some n} \in \mathbb{N} \\ 1 & : & else \end{array} \right\}.$$

Our next result characterizes cofinally Boubaki-complete spaces in terms of functions.

Theorem 2.24. *Let (X, d) be a metric space. Then the following statements are equivalent:*

(a) (X,d) *is cofinally Bourbaki-complete.*
(b) *Each continuous function on* X *with values in a metric space* (Y,ρ) *is CBC-regular.*
(c) *Each real-valued continuous function on* X *is CBC-regular.*

Proof. Only $(c) \Rightarrow (a)$ needs some explanation, the rest is immediate.

$(c) \Rightarrow (a)$: Let (y_n) be a cofinally Bourbaki-Cauchy sequence in X. If (y_n) has a constant subsequence then we are done, else it has a cofinally Bourbaki-Cauchy subsequence of distinct points. Thus it is enough to prove that every cofinally Bourbaki-Cauchy sequence in X with distinct terms clusters. Suppose that there exists a cofinally Bourbaki-Cauchy sequence (x_n) of distinct points in X which does not cluster. Let $A = \{x_n : n \in \mathbb{N}\}$. Consider the function $f : A \to \mathbb{R}$: $f(x_n) = 2^n$. Then f is a real-valued continuous function on the closed set A. Hence by Tietze's extension theorem, f can be extended to a real-valued continuous function f' on X, which will not be CBC-regular. Hence a contradiction. □

Since a cofinally Boubaki-complete space is complete, it is very natural to consider metric spaces whose completions are cofinally Bourbaki-complete. In our next result, we present some equivalent conditions for such metric spaces.

Theorem 2.25. *Let* (X,d) *be a metric space. Then the following statements are equivalent:*

(a) *The completion* $(\widehat{X},d)$ *of* (X,d) *is cofinally Bourbaki-complete.*
(b) *Each Cauchy-continuous function on* X *with values in a metric space* (Y,ρ) *is CBC-regular.*
(c) *Each real-valued Cauchy-continuous function on* X *is CBC-regular.*
(d) *Every cofinally Bourbaki-Cauchy sequence in* X *has a Cauchy subsequence.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$: Easy to prove.

$(c) \Rightarrow (d)$: Let (x_n) be a cofinally Bourbaki-Cauchy sequence of distinct terms in X with no Cauchy subsequence. Let $A = \{x_n : n \in \mathbb{N}\}$. Consider the function $f : A \to \mathbb{R}$: $f(x_n) = 2^n$. Then f is a Cauchy-continuous function on A. Hence by Proposition 2.15 of [13], f can be extended to a real-valued Cauchy-continuous function f' on X, which is not CBC-regular. Hence a contradiction.

$(d) \Rightarrow (a)$: Let $(\widehat{x}_n)$ be a cofinally Bourbaki-Cauchy sequence in $\widehat{X}$ with no Cauchy subsequence. Let $(x_n) \subseteq X$ such that $d(x_n, \widehat{x}_n) < \frac{1}{n}$. Therefore, (x_n) is a cofinally Bourbaki-Cauchy sequence in X with no Cauchy subsequence, which is a contradiction. □

By the previous theorem, one can conclude that every real-valued Cauchy-continuous function need not be CBC-regular. Now we would like to talk about the characterization of the metric spaces on which every real-valued CBC-regular function is bounded. Note that the proof of our next theorem is similar to that of Theorem 2.21 using the following result given by Garrido and Meroño in [9]: a subset B of (X, d) is Bourbaki-bounded in X if and only if every sequence in B is cofinally Bourbaki-Cauchy in X.

Theorem 2.26. *Let (X, d) be a metric space. Then the following are equivalent:*

- (a) *(X, d) is Bourbaki-bounded.*
- (b) *Whenever $f : (X, d) \to (Y, \rho)$ is a metric space and $f : X \to Y$ is CBC-regular, then f is bounded.*
- (c) *Whenever $f : (X, d) \to (Z, \sigma)$ is CBC-regular where (Z, σ) is an unbounded metric space, then f is bounded.*

So far we have seen generalizations of Cauchy and cofinally Cauchy sequences. Now we would like to define sequences which are weaker than pseudo-Cauchy sequences and we call them pseudo Bourbaki-Cauchy.

Definition 2.27. Let (X, d) be a metric space. A sequence (x_n) is said to be *pseudo Bourbaki-Cauchy* in X if for every $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}$, $\exists\ j, k > n$, $j \neq k$ such that the points x_j and x_k can be joined by an $\epsilon-$chain of length m.

Clearly, every cofinally Bourbaki-Cauchy sequence is pseudo Bourbaki-Cauchy, but a pseudo Bourbaki-Cauchy sequence need not be cofinally Bourbaki-Cauchy. The sequence $(n)_{n\in\mathbb{N}}$ is pseudo Bourbaki-Cauchy in $\mathbb{R}$ but not cofinally Bourbaki-Cauchy.

Our next proposition is another sequential characterization of Bourbaki-bounded sets.

Proposition 2.28. *A subset B of (X, d) is Bourbaki-bounded in X if and only if every sequence in B is pseudo Bourbaki-Cauchy in X.*

Proof. Let B be a Bourbaki-bounded subset of X, then every sequence in B is pseudo Bourbaki-Cauchy in X by Proposition 2.16. For the converse, suppose that B is not Bourbaki-bounded in X. Then there exist $\epsilon_o > 0$ and a sequence $(a_n) \subseteq B$ such that for every $n \in \mathbb{N}$, a_{n+1} can't be joined with any a_i, $i \in \{1, 2, ..., n\}$, by an ϵ_o-chain of length n. Thus, (a_n) is not pseudo Bourbaki-Cauchy in X, which is a contradiction. $\square$

Now, let us introduce a terminology for functions preserving pseudo Bourbaki-Cauchy sequences.

Definition 2.29. A function $f : (X, d) \to (Y, \rho)$ between two metric spaces is said to be *pseudo Bourbaki-Cauchy regular* (or *PBC-regular* for short) if $(f(x_n))$ is pseudo Bourbaki-Cauchy in (Y, ρ) for every pseudo Bourbaki-Cauchy sequence (x_n) in (X, d).

A PC-regular function need not be PBC-regular. For example, let $f : \mathbb{R} \to \{e_n : n \in \mathbb{N}\}$ be defined as follows:

$$f(x) = \left\{ \begin{array}{lll} e_n & : & x = n \text{ for some n} \in \mathbb{N} \\ e_1 & : & else \end{array} \right\}.$$

If (x_n) is a pseudo-Cauchy sequence of distinct points in $\mathbb{R}$, then $(f(x_n))$ must have a constant subsequence and hence $(f(x_n))$ is pseudo-Cauchy. Moreover, a PBC-regular function need not be PC-regular.

Now one might be thinking why we didn't introduce a terminology for the metric spaces in which every pseudo Bourbaki-Cauchy sequence of distinct terms clusters. Our next result says that such a metric space is actually an Atsuji space.

Theorem 2.30. *Let (X, d) be a metric space. Then the following statements are equivalent:*

- (a) *(X, d) is an Atsuji space.*
- (b) *Each continuous function on X with values in a metric space (Y, ρ) is PBC-regular.*
- (c) *Each real-valued continuous function on X is PBC-regular.*
- (d) *Every pseudo Bourbaki-Cauchy sequence in X with distinct terms clusters.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c), (d) \Rightarrow (a)$ are immediate. The proof of $(c) \Rightarrow (d)$ is similar to that of Theorem 2.24. □

By Theorems 2.24 and 2.30, one can conclude that there exists a CBC-regular function which is not PBC-regular, otherwise every cofinally Bourbaki-complete space would have been an Atsuji space which is not true, for example, $\mathbb{R}$ with the usual metric is not an Atsuji space.

Theorem 2.31. *Let (X, d) be a metric space. Then the following statements are equivalent:*

- (a) *The completion $(\widehat{X}, d)$ of (X, d) is an Atsuji space.*
- (b) *Each Cauchy-continuous function on X with values in a metric space (Y, ρ) is PBC-regular.*
- (c) *Each real-valued Cauchy-continuous function on X is PBC-regular.*
- (d) *Every pseudo Bourbaki-Cauchy sequence of distinct terms in X has a Cauchy subsequence.*

Proof. Similar to the proofs of Theorems 2.25 and 2.30. □

Thus, every Cauchy-continuous function need not be PBC-regular. Our next theorem talks about the boundedness of PBC-regular functions.

Theorem 2.32. *Let (X, d) be a metric space. Then the following are equivalent:*

(a) *(X, d) is Bourbaki-bounded.*
(b) *Whenever (Y, ρ) is a metric space and $f : (X, d) \to (Y, \rho)$ is PBC-regular, then f is bounded.*
(c) *Whenever $f : (X, d) \to (Z, \sigma)$ is PBC-regular where (Z, σ) is an unbounded metric space, then f is bounded.*

Proof. The proof is similar to that of Theorem 2.21. □

It is well known that a metric space (X, d) is compact if and only if every real-valued continuous function on X is bounded. Using this, we get the following corollary.

Corollary 2.33. *The following statements are equivalent for a metric space (X, d):*

(a) *(X, d) is compact.*
(b) *(X, d) is Bourbaki-bounded and an Atsuji space.*

Proof. The implication $(b) \Rightarrow (a)$ follows from Theorems 2.30 and 2.32. □

Corollary 2.34. *The following statements are equivalent for a metric space (X, d):*

(a) *(X, d) is totally bounded.*
(b) *(X, d) is Bourbaki-bounded and its completion is an Atsuji space.*
(c) *(X, d) is Bourbaki-bounded and its completion is cofinally Bourbaki-complete.*

Proof. The implication $(a) \Rightarrow (b)$ is immediate, while $(b) \Rightarrow (c)$ follows from Proposition 17 of [9].

$(c) \Rightarrow (a)$: It follows from Theorem 2.25 and Theorem 2.26. □

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