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by

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EXTENDING T_1 TOPOLOGIES TO HAUSDORFF WITH THE SAME SETS OF LIMIT POINTS

KYRIAKOS KEREMEDIS

ABSTRACT. Within the framework of **ZF** set theory we show that the statements: "Every infinite T_1 topological space (X, Q) with a finite set of limit points can be extended to a T_2 space with the same set of limit points" and "there exist no free ultrafilters" are equivalent.

1. NOTATION AND TERMINOLOGY

Given a set X, a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X) \backslash \{\varnothing\}$ is called a *filter* iff it is closed under finite intersections and for every $F \in \mathcal{F}$ and $O \subseteq X$ if $F \subseteq O$ then $O \in \mathcal{F}$.

A non-empty collection $\mathcal{H} \subseteq \mathcal{P}(X) \backslash \{\varnothing\}$ is a *filterbase* iff it is closed under finite intersections.

A filterbase $\mathcal{F}$ of X is called *free* if $\bigcap \mathcal{F} = \varnothing$. A maximal with respect to inclusion filter of X is called *ultrafilter*. $cof(X)$ will denote the filter of all cofinite subsets of X. i.e., $A \in cof(X)$ iff $|X \backslash A| < \aleph_0$.

Let $\mathbf{X} = (X, T)$ be a topological space and $A \subseteq X$. An element $x \in X$ is said to be a *limit point* of A iff for every neighborhood V_x of x, $V_x \cap A \backslash \{x\} \neq \varnothing$. A non-limit point of X is called *isolated*. $Lim_T(X)$ denotes the set of all limit points of $\mathbf{X}$ and $Iso_T(X)$ denotes the set of all isolated points of $\mathbf{X}$. If no confusion is likely to arise we shall omit the subscript T from $Iso_T(X)$ and $Lim_T(X)$. If $A \subseteq X$ then T_A will denote the topology A inherits as a subspace of $\mathbf{X}$.

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Given an infinite set A, a free filter $\mathcal{H}$ of A and a non-empty set X disjoint from A, $T_X^{A,\mathcal{H}}$ (resp. T_X^A) will denote the topology on $Y = X \cup A$ generated by the collection:

$$\begin{aligned} C_X^{A,\mathcal{H}} &= \{\{a\} : a \in A\} \cup \{\{x\} \cup H : x \in X, H \in \mathcal{H}\} \\ (\text{resp. } C_X^A &= \{\{a\} : a \in A\} \cup cof(Y)). \end{aligned}$$

Clearly, $(Y, T_X^{A,\mathcal{H}})$ is a T_1 space such that $Iso(Y) = A$ and $Lim(Y) = X$. Likewise, (Y, T_X^A) is a compact T_1 space with $Iso(Y) = A$ and $Lim(Y) = X$. Note that if $A = \varnothing$ then $T_X^{\varnothing}$ is just the *cofinite topology* on X.

The following properties of T_X^A and $T_X^{A,cof(A)}$ are simple observations:

- $T_X^A \subseteq T_X^{A,cof(A)}$,
- $T_X^{A,cof(A)}$ is generated by the family $T_X^A \cup \{\{x\} \cup F : x \in X, F \in cof(A)\}$,
- if X is finite then $T_X^{A,cof(A)} = T_X^A$,
- if X is infinite then $T_X^{A,cof(A)} \neq T_X^A$.

An infinite set X is said to be:

- *amorphous* iff X cannot be partitioned into two infinite sets;
- *Dedekind-infinite,* denoted by $\mathbf{DI}(X)$, iff it contains a countably infinite set. Otherwise is said to be *Dedekind-finite;*
- *weakly Dedekind-infinite*, denoted by $\mathbf{WDI}(X)$, iff $\mathcal{P}(X)$ contains a countably infinite set. Otherwise is said to be *weakly Dedekind-finite*;
- *filterbase infinite*, denoted by $\mathbf{FBI}(X)$, iff there exists a family $\mathcal{V} = \{\mathcal{V}_i : i \in \omega\}$ of free filterbases of X such that for every $i, j \in \omega$ with $i \neq j$ there exists $V \in \mathcal{V}_i$, $U \in \mathcal{V}_j$ with $V \cap U = \varnothing$. Otherwise X is said to be *filterbase-finite*;
- (T_1, T_2)*-infinite*, denoted by $\mathbf{EI}(X, T_1, T_2)$, iff every T_1 topology Q on X with $|Lim_Q(X)| < \aleph_0$ can be extended to a T_2 topology T on X such that $Lim_Q(X) = Lim_T(X)$. Otherwise is said to be (T_1, T_2)*-finite*.

By universal quantifying over X, each of these notions give rise to a choice principle. For example, **IDI** is the statement

$$\forall X (X \text{ infinite } \rightarrow \mathbf{DI}(X))$$

that is "every infinite set is Dedekind-infinite" (Form 9 of [1]). Similarly one defines **IWDI** (Form 82 of [1]), **IFBI** and $\mathbf{IEI}(T_1, T_2)$.

There are three more weak forms of choice that we will use in this paper:

- **NAS** : There are no amorphous sets (Form 64 of [1]) and,
- **EFU** : There exists an infinite set X and a free ultrafilter $\mathcal{F}$ on X (Form 206 of [1]),
- **SPI** : Every infinite set X has a free ultrafilter (Form 63 of [1]).

2. Introduction and some preliminary results

Clearly, for every infinite set X, $cof(X)$ is a free filter of X. However, one cannot prove in **ZF** that given an infinite set X the statement:

- $\mathbf{FBI}_2(X)$: There exist two free filterbases $\mathcal{V}$, $\mathcal{U}$ of X such that $V \cap U = \varnothing$ for some $V \in \mathcal{V}$, $U \in \mathcal{U}$

holds true. Indeed, $\mathbf{FBI}_2(X)$ clearly implies "X is not amorphous" and it is known, see e.g., [1], that there exist **ZF** models including amorphous sets. On the other hand, the statement "X is not amorphous" implies $\mathbf{FBI}_2(X)$. Indeed, if $\{A, A^c\}$ is a partition of X into infinite sets then $\mathcal{V} = cof(A)$, $\mathcal{U} = cof(A^c)$ satisfy the conclusion of $\mathbf{FBI}_2(X)$. Thus, part (i) of Proposition 2.1 is proved.

Proposition 2.1. *Let X be an infinite set.*
(i) X is not amorphous iff $\mathbf{FBI}_2(X)$. *In particular,* **NAS** *iff* $\mathbf{IFBI}_2$ $(= \forall Y(Y \text{ infinite} \to \mathbf{FBI}_2(Y)))$.
(ii) $\rceil$**EFU** *implies* $\mathbf{IFBI}_2$. *The converse fails in* **ZF**.
(iii) For all $m, n \in \omega\backslash 2$, $m < n$, $\mathbf{FBI}_n(X)$ *(: There exist free filterbases $\mathcal{V}_i$, $i \leq n$ of X such that for every i, $j \leq n$, $i \neq j$, there exists $V_i \in \mathcal{V}_i$, $V_j \in \mathcal{V}_j$ with $V_i \cap V_j = \varnothing$) implies* $\mathbf{FBI}_m(X)$. *The converse fails in* **ZF**.
(iv) $\mathbf{IFBI}_2$ *iff for all $n \in \omega\backslash 2$,* $\mathbf{IFBI}_n$ $(= \forall X(X \text{ infinite} \to \mathbf{FBI}_n(X)))$.

Proof. (ii) By part (i), it suffices to show that $\rceil$**EFU** implies **NAS**. Fix X an infinite set. Since $cof(X)$ is a free filter of X it follows, by our hypothesis, that $cof(X)$ is not an ultrafilter. Thus, there exists a subset A of X such that $A \notin cof(X)$ and $A^c \notin cof(X)$. Hence, $\{A, A^c\}$ is a partition of X into infinite sets and X is not amorphous as required.

For the second assertion, we note that in the Second Fraenkel model $\mathcal{N}2$ in [1], **NAS** and **EFU** are both true and transferable to **ZF**.

(iii) The first part is obvious. For the second part we note that if $\mathcal{M}$ is a **ZF** model including an amorphous set Y, and $X = Y \times \{0\} \cup Y \times \{1\}$ then $\mathbf{FBI}_2(X)$ holds true but $\mathbf{FBI}_3(X)$ fails in $\mathcal{M}$.

(iv) This is straightforward. □

Let X be an infinite set. Clearly, $\mathbf{FBI}(X)$ implies $\mathbf{FBI}_2(X)$. Hence, **IFBI** implies $\mathbf{IFBI}_2$. In view of Proposition 2.1 and the latter implication one may ask whether any of the following implications $\mathbf{IFBI}_2 \to \mathbf{IFBI}$, $\mathbf{IFBI} \to \rceil\mathbf{EFU}$ holds true in **ZF**. In Theorem 3.5 we show that none of these implications holds true and, in addition, in the Weglorz/Brunner Model $\mathcal{N}51$ in [1] there exists a set A having neither a free ultrafilter nor a countable family $\mathcal{V} = \{\mathcal{V}_i : i \in \omega\}$ of free filterbases satisfying the conclusion of $\mathbf{FBI}(A)$.

In Theorem 3.1 we show that $\mathbf{FBI}(X)$ is equivalent to each one of the following topological statements:

- $\mathbf{EI}_\omega(X)$: T_ω^X extends to a T_2 topology T such that $Lim_T(X \cup \omega) = \omega$,
- $\mathbf{EI}_\omega^{cof(X)}(X)$: $T_\omega^{X,cof(X)}$ extends to a T_2 topology T such that $Lim_T(X \cup \omega) = \omega$.

We would like to remark here that in $\mathbf{EI}_\omega(X)$ and $\mathbf{EI}_\omega^{cof(X)}(X)$ we need the set X to be disjoint from ω. Since X and $X \times \{0\}$ share the same finiteness properties and $X \times \{0\}$, ω are disjoint we shall assume in the sequel that whenever X and ω are not disjoint then X is replaced by another set of equal cardinality disjoint from ω such as $X \times \{0\}$.

In [3] it has been shown that

Lemma 2.2. [3] *Let X be an infinite set. Then, the following are equivalent:*
(i) $\mathbf{WDI}(X)$.
(ii) $(\mathcal{P}(X), \subseteq)$ has infinite towers (subsets T of $\mathcal{P}(X)$ which are well-ordered by $\supseteq$).
(iii) X has a countable partition.
(iv) For every infinite set X there exists a metric d on X such that (X, d) has at least one limit point.

From Lemma 2.2 it follows immediately that:

Proposition 2.3. $\mathbf{WDI}(X)$ *implies* $\mathbf{FBI}(X)$.

Proof. Fix an infinite set X and let $\mathcal{P} = \{X_n : n \in \omega\}$ be a partition of X. By partitioning ω into countably many infinite sets we can easily pass to a countable partition of X into infinite sets. So, we assume that each member of $\mathcal{P}$ is an infinite set. Then, $\{cof(X_n) : n \in \omega\}$ is the required family of filterbases satisfying the conclusion of $\mathbf{FBI}(X)$. □

On the basis of Proposition 2.3 one may ask the following question.

Question 1. Does $\mathbf{FBI}(X)$ imply "X has a countable partition"?

It is known there exist compact T_1 spaces (X, Q) such that the topology Q cannot be enlarged to a compact T_2 topology on X. As an example consider the following:

Example 1. Take $X = \{x \in \mathbb{R} : x \geq 0\}$ with the topology it inherits as a subspace of $\mathbb{R}$ with the usual topology. Let O (resp. E) be the set of odd (resp. even) integers. Let S be the subspace topology X inherits from $\mathbb{R}$ and embed X as an open subspace into the space (Y, W) where, $Y = X \cup \{a, b\}$ and W is the topology generated by S together with all sets of the form

$$\begin{aligned} U(r) &= \{a\} \cup (\{x \in X : r < x\} - O),\ r \in X \\ V(r) &= \{b\} \cup (\{x \in X : r < x\} - E),\ r \in X. \end{aligned}$$

Then Y is a compact T_1 space but W cannot be enlarged to a compact T_2 topology T. Indeed, let T be a compact topology on Y that enlarges W. Because of the local compactness of (X, S), the subspace topology T_X which X inherits from T, coincides with S. The inclusion, $S \subseteq T_X$ is clear. If $S \neq T_X$ then there exists a set $A \subset X$ which is T_X-closed but not S-closed. Hence, A has a limit point $x \in X \backslash A$. Fix B a compact neighborhood of x in (X, S). Clearly, B is closed in (Y, T) and x is a limit point of the T-closed set $C = A \cap B$ in (X, S). Let $\mathcal{U}$ be the trace of the neigborhood filter of x in (X, S) on the set C. Since (X, S) is T_2, the filter $\mathcal{U}$ has no accumulation point in C. Hence, C is not T-compact and consequently (Y, T) is not compact which is a contradiction. Thus, $S = T_X$ as required.

We show that T is not T_2. Assume on the contrary and fix two disjoint open neighborhoods U_a, V_b of a and b respectively. Clearly,

$$\mathcal{U} = \{U_a, V_b\} \cup \{[0, r) : r \in X\}$$

is a T-cover or Y. Hence, by the compactness of (R, T), $\mathcal{U}$ has a finite subcover. Hence, there exists $x \in (0, \infty)$ such that

$$U_a \cup V_b \cup [0, x) = Y.$$

Since (x, ∞) is a connected subset of (X, S), it follows that $(x, \infty) \subseteq U_a$ or $(x, \infty) \subseteq V_b$. Contradiction! Thus, T fails to be T_2 contradicting our hypothesis.

In view of Example 1, one may ask:

Question 2. Given a set X, does every T_1 topology Q on X extend to a T_2 topology T such that $Lim_Q(X) = Lim_T(X)$?

The answer to Question 2 is in the negative as the following Example 2 demonstrates:

Example 2. Let $Y = \mathcal{P}(\mathcal{P}(\mathcal{P}(\omega)))$ and $X = Y \cup \omega$. We claim that the T_1 topology $Q = T_Y^{\omega, cof(\omega)}$ on X does not extend to a T_2 topology

T on X with $Lim_Q(X) = Lim_T(X) = Y$. Assume the contrary and fix a T_2 topology T on X satisfying:

$$Q \subset T \text{ and } Lim_Q(X) = Lim_T(X). \tag{2.1}$$

In view of (2.1) we may assume that for every $y \in Y$,

$$\mathcal{V}_y = \{V \in T : y \in V \text{ and } V \cap Y = \{y\}\} \tag{2.2}$$

is a T-neighborhood base of y.

Since T is T_2, it follows from (2.1) and (2.2), that the function $F : Y \to \mathcal{P}(\mathcal{P}(\omega))$ given by

$$F(y) = \{H : H \in [\omega]^\omega \text{ and } \{y\} \cup H \ \in \mathcal{V}_y\}$$

is one-to-one. Hence, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\omega)))| \leq |\mathcal{P}(\mathcal{P}(\omega))|$. Contradiction!

In view of Example 2, it follows that if we want to extend T_1 topologies to T_2 while retaining the same set L of limit points then some bound on the size of L must be imposed.

Even in case $|Lim_Q(X)| < \aleph_0$, $\mathbf{EI}(X, T_1, T_2)$ fails in case X has free ultrafilters as the next example shows:

Example 3. Let $\mathcal{F}$ be a free ultrafilter of $\omega\backslash 2$. Clearly, $(\omega, T_2^{\omega\backslash 2,\mathcal{F}})$ is a T_1 space such that $Lim_{T_2^{\omega\backslash 2,\mathcal{F}}}(\omega) = 2 = \{0, 1\}$ and the points $0, 1$ have no disjoint $T_2^{\omega\backslash 2,\mathcal{F}}$ - neighborhoods. We claim that there is no T_2 topology T on ω extending $T_2^{\omega\backslash 2,\mathcal{F}}$ with $Lim_T(\omega) = 2$. Indeed, if T is such a topology and V_0, V_1 are disjoint T - neighborhoods of $0, 1$ respectively, then both $\{V_0\backslash\{0\}\}\cup\mathcal{F}$ and $\{V_1\backslash\{1\}\}\cup\mathcal{F}$ have the fip (finite intersection property). So, $V_0\backslash\{0\}, V_1\backslash\{1\} \in \mathcal{F}$ contradicting the fact that $\mathcal{F}$ is a filter.

Example 3 shows that if we want $\mathbf{EI}(\omega, T_1, T_2)$ to be true then ω must have no free ultrafilters. Based on this observation we show in Theorem 3.3 that $\mathbf{IEI}(T_1, T_2)$ is equivalent to the negation of $\mathbf{EFU}$.

3. Main results

Theorem 3.1. *Let X be an infinite set. The following are equivalent:*
(i) $\mathbf{EI}_\omega(X)$.
(ii) $\mathbf{EI}_\omega^{cof(X)}(X)$.
(iii) $\mathbf{FBI}(X)$.
In particular, $\mathbf{IEI}_\omega$ *:* $\forall X(X$ *infinite* $\to \mathbf{EI}_\omega(X))$, $\mathbf{IEI}_\omega^{cof}$ *:* $\forall X(X$ *infinite* $\to \mathbf{E}_\omega^{cof(X)})$ *and* $\mathbf{IFBI}$ *are equivalent and none is provable in* $\mathbf{ZF}$.

Proof. Fix an infinite set X.

(i) $\to$ (ii) Let, by $\mathbf{EI}_\omega(X)$, W be a T_2 topology on $X \cup \omega$ extending T_ω^X such that $Lim_W(X) = \omega$. Clearly, the topology T on $X \cup \omega$ generated by

$$W \cup \{\{n\} \cup F : F \in cof(X), n \in \omega\}$$

is T_2 such that $T_\omega^{X,cof(X)} \subseteq T$. To complete the proof of (i) $\to$ (ii) it suffices to show that $Lim_T(X) = \omega$. Assume on the contrary that $n \in \omega$ is not a T limit point of X. Fix a T neighborhood V_n of n such that $V_n \cap X = \varnothing$. Clearly, $V_n = O \cap (\{n\} \cup F)$ for some $O \in W$ and $F \in cof(X)$. It follows that $O \cap F = \varnothing$ and consequently $O \cap X$ is finite. Thus, $n \notin Lim_W(X)$. Contradiction! Hence, $Lim_T(X) = \omega$ and $\mathbf{EI}_\omega^{cof(X)}(X)$ holds true.

(ii) $\to$ (iii) Let, by $\mathbf{EI}_\omega^{cof(X)}(X)$, T be a T_2 topology on $X \cup \omega$ extending $T_\omega^{X,cof(X)}$ such that $Lim_T(X) = \omega$. Clearly, $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ where for every $n \in \omega$, $\mathcal{V}_n = \{V \in T : n \in V\}$ satisfies: For every $i, j \in \omega$ with $i \neq j$ there exists $V \in \mathcal{V}_i$, $U \in \mathcal{V}_j$ with $V \cap U = \varnothing$.

(iii) $\to$ (i) Let, by our hypothesis, $\mathcal{F} = \{\mathcal{F}_i : i \in \omega\}$ be a family of free filterbases of X such that for every $i, j \in \omega$ with $i \neq j$ there exists $V \in \mathcal{F}_i$, $U \in \mathcal{F}_j$ with $V \cap U = \varnothing$. It is straightforward to verify that the topology T on $X \cup \omega$ generated by the family $\{\{x\} : x \in X\} \cup \{\{i\} \cup V : i \in \omega, V \in \mathcal{F}_i\}$ satisfies the conclusion of $\mathbf{EI}_\omega(X)$.

The second assertion, in view of (i)-(iii) and Proposition 2.1, is straightforward. $\square$

Theorem 3.2. *Let X be an infinite set.*
(i) $\mathbf{IWDI}(X)$ *implies* $\mathbf{EI}_\omega(X)$. *In particular,* $\mathbf{IEI}_\omega$ *lies in the hierarchy of choice principles between the statements* **NAS** *and* **IWDI**.
(ii) $\mathbf{EI}(X, T_1, T_2)$ *implies "for all $n \in \omega$, X has a partition $\{X_i : i \leq n\}$ into infinite sets".*

Proof. (i) This follows at once from Proposition 2.1 and Theorem 3.1.

(ii) Fix an infinite set X and $n \in \omega$, $n > 1$. By our hypothesis, there exists a T_2 topology T on $Y = X \cup (n+1)$ extending $T_{n+1}^{X,cof(X)}$ such that $Lim_Q(X) = Lim_T(X)$. Fix for every $i \leq n$ an open neighborhood V_i of i such that for all $i, j \leq n, i \neq j, V_i \cap V_j = \varnothing$. It follows that $\{V_i : i < n\} \cup \{X \setminus \bigcup\{V_i : i < n\}\}$ is a partition of X into n infinite sets. $\square$

Our next result shows that $\mathbf{IEI}(T_1, T_2)$ is equivalent to $\rceil\mathbf{EFU}$.

Theorem 3.3. *Let X be an infinite set. The following are equivalent:*
(i) X has no free ultrafilter.
(ii) $\mathbf{EI}(X, T_1, T_2)$.
(iii) For every T_1 topology Q on X with $|Lim_Q(X)| = 2$ there exists a T_2 extension T of Q with $Lim_T(X) = Lim_Q(X)$.
In particular, $\mathbf{IEI}(T_1, T_2)$ *is equivalent to* $\rceil\mathbf{EFU}$.

Proof. (i) $\rightarrow$ (ii) Fix (Y, Q) a T_1 space and let $\{x_i : i \in n\}$ be an enumeration of the set $X = Lim_Q(Y)$. We prove, via a straightforward induction, that Q extends to a T_2 topology T on Y such that $Lim_Q(Y) = Lim_T(Y)$. If $n = 1$ then the conclusion is straightforward. So assume that the conclusion holds true whenever $X = \{x_i : i \leq k\}$ and show that it remains true in case $X = \{x_i : i \leq k+1\}$.

Let, by our hypothesis, T_k be a T_2 topology on the open set $Y_k = Y \backslash \{x_{k+1}\}$ of $\mathbf{Y}$ extending Q_{Y_k} such that $Lim_{Q_{Y_k}}(Y_k) = Lim_{T_k}(Y_k) = \{x_i : i \leq k\}$. Let S be the topology on Y generated by $T_k \cup Q$. Clearly, $Q \subseteq S$ and (Y, S) is a T_1 space such that for all $i, j \leq k, i \neq j, x_i$ and x_j have disjoint open neighborhoods.

Claim. For every topology $K \supseteq S$ on Y with $Lim_K(Y) = \{x_j : j \leq k+1\}$, for every $i \leq k$ there exists a topology S_i on Y such that: $S_i \supseteq K$, $Lim_{S_i}(Y) = \{x_j : j \leq k+1\}$ and x_i, x_{k+1} have disjoint S_i - neighborhoods.

Proof of the claim. Let $\mathcal{V}_i, \mathcal{V}_{k+1}$ denote the neighborhood bases of all K - open sets of Y including x_i and x_{k+1} respectively. Let $A = Y \backslash \{x_j : j \leq k+1\}$. Since (Y, K) is T_1, it follows that

$$\mathcal{H}_i = \{V \cap A : V \in \mathcal{V}_i\} \text{ and } \mathcal{H}_{k+1} = \{V \cap A : V \in \mathcal{V}_{k+1}\}$$

are free filters of A. We consider the following two cases:

(a) $\mathcal{U} = \mathcal{H}_i \cup \mathcal{H}_{k+1}$ does not have the fip. In this case there exists a finite subset $\mathcal{V} = \{V_1, V_2 ... V_s\}$ of $\mathcal{U}$ such that $V_1 \cap V_2 \cap ... \cap V_s \cap A = \varnothing$. Clearly, $O_i = \{x_i\} \cup \bigcap(\mathcal{H}_i \cap \mathcal{V})$ and $O_{k+1} = \{x_{k+1}\} \cup \bigcap(\mathcal{H}_{k+1} \cap \mathcal{V})$ are disjoint K - neighborhoods of x_i and x_{k+1} respectively. Hence, $S_i = K$ is the required extension of K.

(b) $\mathcal{U} = \mathcal{H}_i \cup \mathcal{H}_{k+1}$ has the fip. Since, by our hypothesis, A has no free ultrafilters it follows that the free filter $\mathcal{F}$ of A generated by $\mathcal{U}$ is not an ultrafilter of A. Hence, there exists a subset D of A such that $\{D\} \cup \mathcal{F}$ has the fip and $\{D^c\} \cup \mathcal{F}$ has the fip. Let S_i be the topology generated by the collection:

$$C_{i,k+1} = K \cup \{\{x_i\} \cup D \backslash L : L \in [D]^{<\omega}\} \cup \{\{x_{k+1}\} \cup D^c \backslash L : L \in [D^c]^{<\omega}\}.$$

Clearly, $K \subseteq S_i$ and $V_i = \{x_i\} \cup D$, $V_{k+1} = \{x_{k+1}\} \cup D^c$ are disjoint S_i - neighborhoods of x_i and x_{k+1} respectively.

We show next that $Lim_{S_i}(Y) = \{x_j : j \leq k+1\}$. Fix $j \leq k+1$ and consider the following two cases:

(c) $j \in \{i, k+1\}$. Assume $j = i$ and fix

$$V = U \cap W, U \in K, W = \{x_i\} \cup D \backslash L, L \in [D]^{<\omega}$$

an S_i - neighborhood of x_i. Since $\{D\} \cup \mathcal{F}$ has the fip, $U \cap A \in \mathcal{F}$ and $\mathcal{F}$ is free it follows that $V \cap A = U \cap A \cap D \backslash L$ is infinite. Hence, $x_i \in Lim_{S_i}(Y)$.

Similarly we can show that $x_{k+1} \in Lim_{S_i}(Y)$.

(d) $j \notin \{i, k+1\}$. Since S_i adds no new neighborhoods of x_j it follows that $x_j \in Lim_{S_i}(Y)$.

From cases (c) and (d) it follows that $Lim_{S_i}(Y) = \{x_j : j \leq k+1\}$ as required finishing the proof of the claim.

Using the claim, we construct iteratively extensions

$$S_0 \subseteq S_1 \subseteq ... \subseteq S_k$$

of S such that S_0 is a topology on Y satisfying: $Lim_{S_0}(Y) = \{x_j : j \leq k+1\}$ and the points of x_0, x_{k+1} have disjoint neighborhoods in (Y, S_0).

In general, for every $0 < j \leq k, S_j$ is an extension of S_{j-1} satisfying: $Lim_{S_j}(Y) = \{x_i : i \leq k+1\}$ and the points of x_j, x_{k+1} have disjoint neighborhoods in (Y, S_j). Evidently, $T = S_k$ is the required T_2 extension of Q.

(ii) $\rightarrow$ (iii) This is straightforward.

(iii) $\rightarrow$ (i) Fix an infinite set X and let $Y = \{a, b\}$ be a subset of X. Assume, aiming for a contradiction, that there exists a free ultrafilter $\mathcal{H}$ on $A = X \backslash Y$. Arguing as in Example 3 we end up in a contradiction. Thus, X has no free ultrafilter as required. $\square$

As a corollary to Theorem 3.3 we get the following characterizations of **EFU** and **SPI** whose proof is left as an easy exercise for the reader.

Corollary 3.4. *(i)* **EFU** *iff there exists a T_1 space (X, Q) with $|Lim_Q(X)| < \aleph_0$ such that for every T_2 topology T on X extending Q, $Lim_T(X) \neq Lim_Q(X)$.*
(ii) **SPI** *iff for every infinite set X there exists a T_1 topology Q on X with $|Lim_Q(X)| < \aleph_0$ such that for every T_2 topology T on X extending Q, $Lim_T(X) \neq Lim_Q(X)$.*

Remark. A natural strengthening of $\mathbf{EI}(X, T_1, T_2)$ is the proposition:

- $\mathbf{EI}_{\aleph_0}(X, T_1, T_2)$: Every T_1 topology Q on X with $|Lim_Q(X)| \leq \aleph_0$ can be extended to a T_2 topology T on X such that $Lim_Q(X) = Lim_T(X)$.

One might think that working as in Theorem 3.3 can prove that the conjunction of $\rceil$**EFU** and some weak form of the axiom of choice such as the axiom of dependent choice **DC**, implies the statement: $\mathbf{IEI}_{\aleph_0}(T_1, T_2)$: $\forall X(X$ infinite $\rightarrow \mathbf{EI}_{\aleph_0}(X, T_1, T_2)$ is relatively consistent with **ZF**. However, there exist countable T_1 spaces (X, Q) without isolated points and the proof of Theorem 3.3 does not apply to these cases. We do not know whether the statement: *Every dense-in-itself T_1 topology Q on ω extends to a dense-in-itself T_2 topology on ω* is consistent with **ZF**. This explains why we preferred the condition $|Lim_Q(X)| < \aleph_0$ over $|Lim_Q(X)| \leq \aleph_0$.

Theorem 3.5. *(i)* $\mathbf{IEI}_\omega \nrightarrow \mathbf{IEI}(T_1,T_2)$, $\mathbf{IEI}_\omega \nrightarrow \mathbf{IDI}$, $\mathbf{IDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$ *and* $\mathbf{IWDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$ *in* **ZF**.
(ii) $\mathbf{EI}(X,T_1,T_2) \nrightarrow \mathbf{EI}_\omega(X)$ *in* **ZFA** *(=* **ZF** *plus the existence of a set of atoms).*
(iii) $\mathbf{IFBI}_2 \nrightarrow \mathbf{IFBI}$ *in* **ZF**.

Proof. (i) $\mathbf{IEI}_\omega \nrightarrow \mathbf{IEI}(T_1,T_2)$, $\mathbf{IWDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$, $\mathbf{IEI}_\omega \nrightarrow \mathbf{IDI}$ in **ZF**. It is known that in Cohen's basic model $\mathcal{M}1$ in [1] **SPI** and **IWDI** hold true but **IDI** fails. Hence, by Corollary 3.4, $\mathbf{IEI}(T_1,T_2)$ fails in $\mathcal{M}1$ and by Theorem 3.2 $\mathbf{IEI}_\omega$ holds true in $\mathcal{M}1$. Thus, $\mathbf{IEI}_\omega \nrightarrow \mathbf{IEI}(T_1,T_2)$, $\mathbf{IWDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$ and $\mathbf{IEI}_\omega \nrightarrow \mathbf{IDI}$ in **ZF**.

$\mathbf{IDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$. It is known that in the Pincus' Model IX, Model $\mathcal{M}47(n,M)$ in [1], **IDI** holds true and ω has a free ultrafilter. Hence, by Example 3, there exists a T_1 topology Q on ω with just two limit points which does not extent to a T_2 topology with the same set of limit points. Thus, $\mathbf{EI}(\omega,T_1,T_2)$ fails in $\mathcal{M}47(n,M)$. Hence, $\mathbf{IEI}(T_1,T_2)$ fails also and $\mathbf{IDI} \nrightarrow \mathbf{IEI}(T_1,T_2)$ in **ZF**.

(ii) We note that in the Weglorz/Brunner Model $\mathcal{N}51$ in [1] (the set of atoms A is countably infinite, the group of permutations G is the set of all permutations ϕ of A and the normal filter $\mathcal{H}$ is generated by the set of all subgroups G_B, where $B \subseteq A$ and $G_B = \{\pi \in G : \pi(B) = B\}$) the set of atoms A has no free ultrafilters. So, $\rceil\mathbf{UF}(A)$ and consequently by Theorem 3.3, $\mathbf{EI}(A,T_1,T_2)$ holds true. We show that $\mathbf{EI}_\omega(A)$ fails in $\mathcal{N}51$. To this end it suffices by Theorem 3.1 to show that $\mathbf{FBI}(A)$ fails. Assume, aiming for a contradiction, that $\mathbf{FBI}(A)$ holds true and let $\mathfrak{F} = \{\mathcal{F}_n : n \in \omega\}$ be a family of free filters of X such that:

(3.1) For every $n,m \in \omega, n \neq m$ there exists $U \in \mathcal{F}_n, V \in \mathcal{F}_m$ with $U \cap V = \varnothing$.

Let $H \in \mathcal{H}$ satisfy $H \subseteq Sym_G(\mathfrak{F})$. Assume

$$H = \bigcap_{i \leq n} G_{B_i}, B_i \in \mathcal{P}(A), i \leq n, n \in \mathbb{N}.$$

Clearly, the relation $\sim$ on $X = \bigcup_{i \leq n} B_i$ given by:

$$x \sim y \text{ iff for all } i \leq n, x \in B_i \leftrightarrow y \in B_i$$

is an equivalence. Clearly, for every $Y \in X/\sim$, if $x \in Y$ and $K_x = \{i \leq n : x \in B_i\}$ then $Y = \bigcap_{i \in K_x} B_i \cap \bigcap_{i \in K_x^c} B_i^c$. Thus,

$$(\forall Y \in X/\sim)(\exists \varnothing \neq K_Y \subseteq n+1)(Y = \bigcap_{i \in K_Y} B_i \cap \bigcap_{i \in K_Y^c} B_i^c). \tag{3.2}$$

We claim that:

$$\pi \in H \text{ iff for all } Y \in X/\sim,\ \pi(Y) = Y. \tag{3.3}$$

Indeed, if $\pi \in H$ and $Y \in X/\sim$ then, in view of (3.2), $Y = \bigcap_{i \in K_Y} B_i \cap \bigcap_{i \in K_Y^c} B_i^c$ for some $\varnothing \neq K_Y \subseteq n+1$. Hence, $\pi(Y) = \pi(\bigcap_{i \in K_Y} B_i \cap \bigcap_{i \in K_Y^c} B_i^c) = \pi(\bigcap_{i \in K_Y} B_i) \cap \pi(\bigcap_{i \in K_Y^c} B_i^c) = \bigcap_{i \in K_Y} \pi(B_{i_j}) \cap \bigcap_{i \in K_Y^c} \pi(B_i^c) = \bigcap_{i \in K_Y} B_{i_j} \cap \bigcap_{i \in K_Y^c} B_i^c = Y$.

Conversely, assume that for all $Y \in X/\sim$, $\pi(Y) = Y$ and show that $\pi \in H$. To this end, it suffices to show that for all $i \leq n, \pi \in G_{B_i}$. Fix $i \leq n$ and let $y \in B_i$. Since $X = \bigcup X/\sim$, it follows that $y \in Y$ for some $Y \in X/\sim$ with $Y \subseteq B_i$. By our hypothesis, $\pi(Y) = Y$ and consequently $\pi(y) \in B_i$ and $\pi(z) = y$ for some $z \in Y(\subseteq B_i)$. Thus, $\pi(B_i) \subseteq B_i$ and $B_i \subseteq \pi(B_i)$ meaning that $\pi(B_i) = B_i$. Hence, $\pi \in G_{B_i}$ as required.

We will be needing the following claim.

Claim. Let $Y \in X\backslash\sim$, $L = \{k \in \omega : \{Y\} \cup \mathcal{F}_k \text{ has the fip}\}$ and for all $k \in L$ let $\mathcal{F}_k^{'}$ denote the filter of A generated by $\{Y\} \cup \mathcal{F}_k$. Then, $H \subseteq Sym_G(\mathfrak{F}')$ where, $\mathfrak{F}' = \{\mathcal{F}_n : n \in \omega\backslash L\} \cup \{\mathcal{F}_k' : k \in L\}$.

Proof of the claim. Fix $\pi \in H$. It suffices to show that $\pi(\{\mathcal{F}_k' : k \in L\}) = \{\mathcal{F}_k' : k \in L\}$ and $\pi(\{\mathcal{F}_n : n \in \omega\backslash L\}) = \{\mathcal{F}_n : n \in \omega\backslash L\}$.

To see that $\pi(\{\mathcal{F}_k' : k \in L\}) \subseteq \{\mathcal{F}_k' : k \in L\}$ fix $k \in L$ and let $\pi(\mathcal{F}_k) = \{\pi(F) : F \in \mathcal{F}_k\} = \mathcal{F}_m$ for some $m \in L$. Since $\mathcal{F}_k \cup \{Y\}$ has the fip and, by (3.3), $\pi(Y) = Y$ we see that $\mathcal{F}_m \cup \{Y\}$ has the fip. Indeed, if $\pi(F_j) \in \mathcal{F}_m, j = 1, 2, ..., v$ then $\bigcap_{j \leq v} F_j \cap Y \neq \varnothing$ and consequently

$$\varnothing \neq \pi(\bigcap_{j \leq v} F_j \cap Y) = \bigcap_{j \leq v} \pi(F_j \cap Y) = \bigcap_{j \leq v} \pi(F_j) \cap \pi(Y) = \bigcap_{j \leq v} \pi(F_j) \cap Y \tag{3.4}$$

meaning that $\mathcal{F}_m \cup \{Y\}$ has the fip. Hence, $m \in L$ and (3.4) shows that $\pi(\mathcal{F}_k') = \mathcal{F}_m'$. Thus,

$$\pi(\{\mathcal{F}_k' : k \in L\}) \subseteq \{\mathcal{F}_k' : k \in L\}.$$

For the reverse inclusion, fix $k \in L$ and let $m \in \omega$ be such that $\pi(\mathcal{F}_m) = \mathcal{F}_k$. It is easy to see that, $m \in L$ and $\pi(\mathcal{F}_m') = \mathcal{F}_k'$. Thus, $\{\mathcal{F}_k' : k \in L\} \subseteq \pi(\{\mathcal{F}_k' : k \in L\})$ and $\pi(\{\mathcal{F}_k' : k \in L\}) = \{\mathcal{F}_k' : k \in L\}$.

To see that $\pi(\{\mathcal{F}_n : n \in \omega\backslash L\}) \subseteq \{\mathcal{F}_n : n \in \omega\backslash L\}$ we note that if $n \in \omega\backslash L$ then $\mathcal{F}_n \cup \{Y\}$ does not have the fip. Thus, there exists $F_i \in \mathcal{F}_n, i = 1, 2, ..., k$ such that $F_1 \cap F_2 \cap ... \cap F_k \cap Y = \varnothing$. Hence, $\pi(F_1) \cap \pi(F_2) \cap ... \cap \pi(F_k) \cap Y = \varnothing$ meaning that $\pi(\mathcal{F}_n) \cup Y$ does not have the fip. Thus, $\pi(\mathcal{F}_n) \in \{\mathcal{F}_v : v \in \omega\backslash L\}$ and consequently $\pi(\{\mathcal{F}_n : n \in \omega\backslash L\}) \subseteq \{\mathcal{F}_n : n \in \omega\backslash L\}$.

For the reverse inclusion, fix $n \in \omega\backslash L$. Since $\pi(\mathfrak{F}) = \mathfrak{F}$ we see that there exists $m \in \omega$ such that $\pi(\mathcal{F}_m) = \mathcal{F}_n$. Clearly, $m \in \omega\backslash L$ (if $m \in L$ then $n \in L$) and $\mathcal{F}_n \in \pi(\{\mathcal{F}_n : n \in \omega\backslash L\})$. Thus, $\{\mathcal{F}_n : n \in \omega\backslash L\} \subseteq \pi(\{\mathcal{F}_n : n \in \omega\backslash L\})$ and $\pi(\{\mathcal{F}_n : n \in \omega\backslash L\}) = \{\mathcal{F}_n : n \in \omega\backslash L\}$ finishing the proof of the claim.

We continue with the proof by considering the following two cases:

(a) There exists $Y \in X/\sim$ such that $|L| \geq 2$, where L is the set given in the claim. Replace $\mathfrak{F}$ with $\mathfrak{F}'$ where $\mathfrak{F}'$ is given as in the claim. Without loss of generality we may assume that $\mathfrak{F} = \mathfrak{F}'$ and for every $k \in L, Y \in \mathcal{F}_k$. Assume, for our convenience, that $0, 1 \in L$ and fix by (3.1) $F_0 \in \mathcal{F}_0$, $F_1 \in \mathcal{F}_1$, $F_0, F_1 \subseteq Y$ such that $F_0 \cap F_1 = \varnothing$. Since $\mathcal{F}_0, \mathcal{F}_1$ are not free ultrafilters of A (recall that A has no free ultrafilters), it follows that there exists partitions $\{F_{00}, F_{01}\}$ of F_0 and $\{F_{10}, F_{11}\}$ of F_1 into infinite sets such that $F_{00} \notin \mathcal{F}_0$, $F_{01} \notin \mathcal{F}_0$, $F_{10} \notin \mathcal{F}_1$, $F_{11} \notin \mathcal{F}_1$ and $\{F_{00}\} \cup \mathcal{F}_0$, $\{F_{01}\} \cup \mathcal{F}_0$, $\{F_{10}\} \cup \mathcal{F}_1$, $\{F_{11}\} \cup \mathcal{F}_1$ have the fip. Let π be a permutation of A given by:

$$\pi(F_{01}) = F_{10}, \pi(F_{10}) = F_{01} \text{ and for all } a \in A\backslash(F_{01} \cup F_{10}), \pi(a) = a.$$

By (3.3), $\pi \in H$. Since, $\pi(F_0) = F_{00} \cup F_{10}$ and $\pi(F_0) \cap F_0 = F_{00} \notin \mathcal{F}_0$, $\pi(F_0) \cap F_1 = F_{10} \notin \mathcal{F}_1$ it follows that $\pi(\mathcal{F}_0) \neq \mathcal{F}_0$ and $\pi(\mathcal{F}_0) \neq \mathcal{F}_1$. Assume, aiming for a contradiction, that $\pi(\mathcal{F}_0) = \mathcal{F}_k$ for some $k > 1$. Since $\pi(Y) = Y \in \mathcal{F}_0$, it follows that $Y \in \mathcal{F}_k$ and consequently $k \in L$. Fix $W_0 \in \mathcal{F}_0, W_k \in \mathcal{F}_k$ such that $W_0 \subseteq F_0$, $W_k \subseteq Y$ and $W_0 \cap W_k = \varnothing$. Fix $F \in \mathcal{F}_0$ such that $\pi(F) = W_k$. Clearly, $U = F \cap W_0 \in \mathcal{F}_0$ satisfies:

$$\pi(U) = \pi(F) \cap \pi(W_0) = W_k \cap \pi(W_0) \subseteq W_k.$$

Since $\{F_{00}\} \cup \mathcal{F}_0$, $\{F_{01}\} \cup \mathcal{F}_0$ have the fip and $U \in \mathcal{F}_0$ it follows that $U_0 = F_{00} \cap U \neq \varnothing$ and $U_1 = F_{01} \cap U \neq \varnothing$. We have:

$$\pi(U) = \pi(U_0 \cup U_1) = \pi(U_0) \cup \pi(U_1) = U_0 \cup \pi(U_1) \subseteq W_k.$$

Since $W_0 \cap W_k = \varnothing$ and $U \subseteq W_0$ we see that $U_0 = (U_0 \cup \pi(U_1)) \cap U = \varnothing$. Contradiction! So (a) cannot be the case.

(b) For all $Y \in X/\sim$, $|L| < 2$. If this is the case then it is easy to see that there exists a $k \in \omega$ such that for all $n \geq k, X^c \in \mathcal{F}_n$. Without loss of generality we may assume that $X^c \in \mathcal{F}_0 \cap \mathcal{F}_1$. As in case (a) we fix $F_0 \in \mathcal{F}_0$, $F_1 \in \mathcal{F}_1$, $F_0, F_1 \subseteq X^c$ such that $F_0 \cap F_1 = \varnothing$. Let $\{F_{00}, F_{01}\}$,

$\{F_{10}, F_{11}\}$ and π be as in the proof of case (a). Similarly with the proof of case (a) we can show (b) cannot be the case.

Since one of the cases (a) and (b) must hold true, we have arrived at a contradiction. Hence, there is no $H \in \mathcal{H}$ with $H \subseteq sym(\mathfrak{F})$ meaning that $\mathfrak{F} \notin \mathcal{N}51$.

(iii) This follows at once from part (ii). **NAS**, hence by Proposition 2.1 $\mathbf{IFBI}_2$ also holds true in $\mathcal{N}51$ but by part (ii) **IFBI** fails. An application of the Jech-Sochor Embedding Theorem (Theorem 6.1 in [4]) at this point yields a **ZF** model satisfying $\mathbf{IFBI}_2$ and the negation of **IFBI**.

The following diagram summarizes the web of implications and non-implications between the principles, as well as the questions left open. □

$$\begin{array}{ccccccc}
\boxed{\mathbf{IDI}} & \begin{array}{c}\rightarrow\\ \not\leftarrow\end{array} & \boxed{\mathbf{IWDI}} & & & & \\
(\uparrow ?)\ \not\downarrow & (\nearrow ?)\ \not\swarrow & \downarrow (\uparrow ?) & & & & \\
\boxed{\begin{array}{c}\mathbf{IEI}(T_1, T_2)\\ \rceil\mathbf{EUF}\end{array}} & \begin{array}{c}\not\leftarrow\\ (\rightarrow ?)\end{array} & \boxed{\begin{array}{c}\mathbf{IEI}_\omega\\ \mathbf{IEI}_\omega^{cof}\\ \mathbf{IFBI}\end{array}} & \begin{array}{c}\rightarrow\\ \not\leftarrow\end{array} & \boxed{\mathbf{IFBI}_2} & \leftrightarrow & \boxed{\mathbf{NAS}}
\end{array}$$

Diagram 1

References

[1] P. Howard and J. E. Rubin, *Consequences of the axiom of choice,* Math. Surveys and Monographs, **59** A.M.S. Providence R.I., 1998.

[2] H. Herrlich, K. Keremedis, *Extending compact topologies to compact Hausdorff topologies in* **ZF**, Topology and its applications **158** (2011), 2279–2286.

[3] K. Keremedis, *Non-discrete metrics in* **ZF** *and some notions of finiteness*, to appear in MLQ.

[4] T. Jech, *The Axiom of Choice.* North-Holland Publishing Co., (1973)

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