Hausdorff closedness in the convergence settings

by

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ABSTRACT. We use convergence theory as the framework for studying H-closed spaces and H-sets in topological spaces. From this viewpoint, it becomes clear that the property of being H-closed and the property of being an H-set in a topological space are pretopological notions. Additionally, we define a version of H-closedness for pretopological spaces and investigate the properties of such a space.

1. INTRODUCTION AND PRELIMINARIES

The early development of general topology was guided in part by the desire to develop a framework in which to discuss different notions of convergence found in analysis. In 1948, G. Choquet [4] laid out the theory of *convergence spaces*, general enough to contain the classes of topological spaces and closure spaces while unifying the desired notions of convergence.

Once an agreed-upon definition of topological space was arrived at, the concept of compactness revealed itself to be deserving of much study and subsequently of generalization. One of the most fruitful variations of compactness is that of a *Hausdorff closed space*, defined in [1] by

P. Alexandroff and P. Urysohn in 1928. In this paper we use the common abbreviation H-closed when discussing Hausdorff closed spaces. One particular advantage of considering H-closed spaces is that, in contrast with compact spaces, every Hausdorff topological space can be densely embedded in an H-closed space. Much later, in 1968, N. V. Veličko [17] relativized H-closedness to subspaces by defining the *H-sets* of a topological space X. In this same paper, Veličko gives us the tools needed to consider H-closedness and H-sets as purely convergence-theoretic properties. In [5] R.F. Dickman and J.R. Porter use these tools to define the particular convergence we will use to discuss H-closed spaces and H-sets in the convergence setting.

Our first task here will be to place H-closed spaces and H-sets in the convergence theoretic framework. In Section 2, we give preliminary definitions and results pertaining to H-closed spaces and H-sets in the usual topological setting. This is followed in Section 3 by the basic definitions and results necessary to consider the convergence theoretic point of view. Particularly of interest will be the definition of pretopological spaces, which is the subcategory of convergence spaces in which we will mainly work. At this point we will frame H-closed spaces and H-sets as pretopological notions. In particular, Theorem 3.12 points to the potential advantages of this point of view.

In Section 4, we define a purely convergence-theoretic notion which parallels that of H-closedness for topological spaces. The basic properties of the so-called *PHC spaces* (short for pretopologically H-closed spaces) are investigated. Additionally, we develop a technique for constructing new PHC spaces using images of compact pretopological spaces.

Lastly, we will discuss extensions of convergence spaces. Much work has been done in this area, in particular by D.C. Kent and G.D. Richardson, who catalogued much of the early progress in the field in [12]. In [12] an axiom is used in the definition of a convergence space which we will not assume. This axiom will be given in 4.2 under the name property **(R)** and we will be explicit when making use of this axiom in the results of that section. We investigate PHC extensions of a pretopological space X. These extensions prove to be of interest in that for any pretopological space X, there is a PHC extension of X which is projectively larger than any compactification of X. This is not true of compactifications, as a convergence space X does not in general have a largest compactification.

We now take a moment to normalize some notations and give preliminary definitions to be used in the sequel. The Greek letters τ and π will always represent a topology and a pretopology, respectively.

If (X, τ) is a topological space, then the closure operator with respect to the topology will be denoted cl_τ and for $x \in X$, the family of neighborhoods of x will be given by $\mathcal{N}_\tau(x)$. A function $f : (X, \tau) \to (Y, \sigma)$ between topological spaces is θ-*continuous* if for each $x \in X$ and $V \in \mathcal{N}_\sigma(f(x))$, there exists $U \in \mathcal{N}_\tau(x)$ such that $f[cl_\tau U] \subseteq cl_\sigma V$. Every continuous function is θ-continuous. If (Y, σ) is regular, then the notions coincide.

Let X be a set, and $\mathcal{L} \subseteq 2^X$ a family of subsets of X such that $(\mathcal{L}, \subseteq)$ is a lattice. An $\mathcal{L}$-*filter* on X is an isotone family of elements of $\mathcal{L}$ closed under finite meets. If $\mathcal{L} = 2^X$, then $\mathcal{F}$ is simply called a *filter*. If (X, τ) is a topological space and $\mathcal{L} = \tau$, then $\mathcal{F}$ is called an *open filter*. If $\mathcal{B}$ is a family of subsets of X with the finite intersection property, then we use $\langle \mathcal{B} \rangle$ to denote the smallest filter on X generated by $\mathcal{B}$. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{L}$-filters on X, we say that $\mathcal{G}$ is *finer than* $\mathcal{F}$, written $\mathcal{F} \leq \mathcal{G}$, if $\mathcal{F} \subseteq \mathcal{G}$. An $\mathcal{L}$-filter $\mathcal{U}$ which is maximal with respect to $\leq$ is called an $\mathcal{L}$-*ultrafilter*. Again, if $\mathcal{L} = 2^X$, we say simply that $\mathcal{U}$ is an ultrafilter and if $\mathcal{L} = \tau$, we say that $\mathcal{U}$ is an open ultrafilter.

If (X, τ) is a topological space, $x \in X$ and $\mathcal{F}$ is a filter on X, it is said that $\mathcal{F}$ τ-*converges* to x if $\mathcal{N}_\tau(x) \subseteq \mathcal{F}$. The *adherence of* $\mathcal{F}$ is defined to be $\bigcap_{F \in \mathcal{F}} \mathrm{cl}_\tau F$.

2. H-closed Spaces and H-sets

A Hausdorff topological space is *H-closed* if it is closed in every Hausdorff topological space in which it is embedded. The following well-known characterizations of H-closed spaces are useful and will be used interchangeably as the definition of H-closed.

Theorem 2.1. *Let X be a Hausdorff topological space. The following are equivalent.*

(1) *X is H-closed,*
(2) *Whenever $\mathcal{C}$ is an open cover of X, there exist $C_1, ..., C_n \in \mathcal{C}$ such that $X = \bigcup_{i=1}^n \mathrm{cl}_\tau C_i$,*
(3) *Every open filter on X has nonempty adherence,*
(4) *Every open ultrafilter on X has a convergence point.*

Veličko [17] relativized the concept of H-closed to subspaces in the following way: If X is a Hausdorff topological space and $A \subseteq X$, we say that A is an *H-set* if whenever $\mathcal{C}$ is a cover of A by open subsets of X, there exist $C_1, ..., C_n \in \mathcal{C}$ such that $A \subseteq \bigcup_{i=1}^n \mathrm{cl}_\tau C_i$. We say that a filter $\mathcal{F}$ *meets* a set A if $F \cap A \neq \varnothing$ for each $F \in \mathcal{F}$. If $\mathcal{F}$ meets A we will sometimes write $\mathcal{F} \# A$. We note the following well-known characterizations of H-sets which mirror the above theorem.

Proposition 2.2. *Let X be a topological space and $A \subseteq X$. The following are equivalent.*

(1) *A is an H-set in X,*
(2) *If $\mathcal{F}$ is an open filter on X which meets A, then $\mathrm{adh}_X \mathcal{F} \cap A \neq \varnothing$,*
(3) *If $\mathcal{U}$ is an open ultrafilter on X which meets A, then $\mathrm{adh}_X \mathcal{U} \cap A \neq \varnothing$.*

It is important to note that the property of H-closeness is not closed-hereditary. Also, note that the definition of an H-set is dependent on the ambient space being considered. In particular, not every H-set is H-closed. The following example, due to Urysohn, points to this distinction. Recall that a space X is *semiregular* if the regular-open subsets of X form an open base.

Example 2.3. Let $X = \mathbb{N} \times \mathbb{Z} \cup \{-\infty, +\infty\}$. Define $U \subseteq X$ to be open if

$$+\infty \in U \text{ implies that there is } n_U \in \mathbb{N} \text{ such that}$$

$$\{(n, k) \in \mathbb{N} \times \mathbb{Z} : n > n_U, k > 0\} \subseteq U$$

$$-\infty \in U \text{ implies that there is } n_U \in \mathbb{N} \text{ such that}$$

$$\{(n, k) \in \mathbb{N} \times \mathbb{Z} : n > n_U, k < 0\} \subseteq U$$

$$(n, 0) \in U \text{ implies that there is some } k_U \in \mathbb{N} \text{ such that}$$

$$\{(n, k) \in \mathbb{N} \times \mathbb{Z} : |k| > k_U\} \subseteq U.$$

Then X is H-closed and semiregular. Let $A = \{(n, 0) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}\} \cup \{+\infty\}$. Notice that A is a closed discrete subset of X and that A is an H-set in X. However, with the subspace topology, $A \cong \mathbb{N}$, and thus is not H-closed.

3. Convergence Spaces

For a basic reference on convergence theory, see [9]. Given a relation ξ between filters on X and elements of X, we write $x \in \lim_\xi \mathcal{F}$ whenever $(\mathcal{F}, x) \in \xi$ and say that x is a *ξ-limit point* of $\mathcal{F}$. If $A \subseteq X$, let $\langle A \rangle$ be the principal filter generated by A. We abbreviate $\langle \{x\} \rangle$ by $\langle x \rangle$. A *convergence space* is a set X paired with a relation ξ between filters on X and points of X satisfying

(1) $x \in \lim_\xi \langle x \rangle$, and
(2) if $\mathcal{F} \subseteq \mathcal{G}$ and $x \in \lim_\xi \mathcal{F}$, then $x \in \lim_\xi \mathcal{G}$.

The relation ξ is called a convergence on X. Notice that, thanks to (1), the set X is equal to the range of the relation ξ. Therefore, the underlying set is determined by the convergence.

Clearly, a topological space (X, τ) paired with the usual topological notion of convergence in which $x \in \lim_\tau \mathcal{F}$ if and only if $\mathcal{N}_\tau(x) \subseteq \mathcal{F}$ is an example of a convergence space. Since topological convergence is determined by the topology τ, we will abuse notation and use the symbol τ for both the family of open subsets of X and the convergence determined by τ. The class of convergence structures on a set X can be given a lattice structure. We say that σ is *coarser than* ξ, written $\sigma \leq \xi$ if $\lim_\sigma \mathcal{F} \supseteq \lim_\xi \mathcal{F}$ for each filter $\mathcal{F}$ on X. In this case we also say that ξ is finer than σ.

Example 3.1. Throughout this paper, if X is a topological space, let θ be the convergence on X defined by $x \in \lim_\theta \mathcal{F}$ if and only if $\mathrm{cl}_\tau U \in \mathcal{F}$ for each $U \in \mathcal{N}_\tau(x)$. If there is any possibility for confusion, we will write θ_X. This type of convergence was studied extensively under the name "almost convergence" in [5]. We will frequently come back to this example of a convergence space.

Two filters $\mathcal{F}$ and $\mathcal{G}$ *meet* if $F \cap G \neq \varnothing$ for each $F \in \mathcal{F}$ and $G \in \mathcal{G}$, in which case we write $\mathcal{F} \# \mathcal{G}$. Given a filter $\mathcal{F}$ on a convergence space (X, ξ), the *adherence of* $\mathcal{F}$ is defined to be

$$\mathrm{adh}_\xi \mathcal{F} = \bigcup \{\lim_\xi \mathcal{G} : \mathcal{G} \# \mathcal{F}\}.$$

For $A \subseteq X$, we write $\mathrm{adh}_\xi A$ to abbreviate $\mathrm{adh}_\xi \langle A \rangle$. We will also define the *inherence* of a set A by

$$\mathrm{inh}_\xi A = X \setminus \mathrm{adh}_\xi (X \setminus A).$$

These two concepts will function as versions of topological closure and interior generalized to convergence spaces.

A convergence ξ is *Hausdorff* if every filter has at most one limit point.

Topological spaces are now seen as a particular instance of convergence spaces. In fact, if (X, τ) is a topological space, then $\mathrm{adh}_\tau A = \mathrm{cl}_\tau A$ for any $A \subseteq X$ and $\mathrm{adh}_\tau \mathcal{F} = \bigcap_{F \in \mathcal{F}} \mathrm{cl}_\tau F$. Two other important classes of convergences are *pseudotopologies* and *pretopologies*. If $\mathcal{F}$ is a filter on X, let $\beta \mathcal{F}$ denote the set of all ultrafilters on X containing $\mathcal{F}$. A convergence ξ is a pseudotopology if $\lim_\xi \mathcal{F} \supseteq \bigcap \{\lim_\xi \mathcal{U} : \mathcal{U} \in \beta \mathcal{F}\}$. In [10], Herrlich, Lowen-Colebunders and Schwatz discuss the categorical advantages of working in the category of pseudotopological spaces. We will discuss the usefulness of working with pretopological spaces to characterize H-closed space and H-sets in the next subsection.

A convergence space (X, ξ) is *compact* if every filter on X has nonempty adherence. The following notions of compactness for filters will allow us to get at compactness of subspaces.

Definition 3.2. Let (X, ξ) be a convergence space, $\mathcal{F}$ a filter on X and $A \subseteq X$. We say that $\mathcal{F}$ is *compact at* A if whenever $\mathcal{G}$ is a filter on X and $\mathcal{G} \# \mathcal{F}$, $\operatorname{adh}_\xi \mathcal{G} \cap A \neq \varnothing$. In particular, a filter $\mathcal{F}$ is *relatively compact* if it is compact at X.

If $\mathcal{B}$ is a family of subsets of X, then $\mathcal{F}$ is *compact at* $\mathcal{B}$ if whenever $\mathcal{G} \# \mathcal{F}$, $\operatorname{adh}_\xi \mathcal{G} \# \mathcal{B}$. A filter is *compact* if $\mathcal{F}$ is compact at itself.

Using this definition, $A \subseteq X$ is *compact* if whenever $\mathcal{G}$ is a filter on X which meets A, we have that $\operatorname{adh}_\xi \mathcal{G} \cap A \neq \varnothing$. Notice that for topological spaces this also characterizes the compact subspaces.

Let (X, ξ) and (Y, σ) be convergence spaces. A function $f : (X, \xi) \to (Y, \sigma)$ is *continuous* if $f[\lim_\xi \mathcal{F}] \subseteq \lim_\sigma f(\mathcal{F})$ for each filter $\mathcal{F}$ on X, where $f(\mathcal{F})$ is the filter generated by $\{f[F] : F \in \mathcal{F}\}$. Notice that if X and Y are topological spaces, then $f : X \to Y$ is θ-continuous if and only if $f : (X, \theta_X) \to (Y, \theta_Y)$ is continuous in the sense of convergence spaces.

Given $A \subseteq X$ and a convergence ξ on X, we can define the *subconvergence* on A as follows: If $\mathcal{F}$ is a filter on A, let $\langle \mathcal{F} \rangle$ be the filter on X generated by $\mathcal{F}$. Define $\lim_{\xi|_A} \mathcal{F} = \lim_\xi \langle \mathcal{F} \rangle \cap A$. This is also the initial convergence of A generated by the inclusion map $i : A \to (X, \xi)$; that is, the coarsest convergence making the inclusion map continuous. Thus, A is a compact subset of (X, ξ) is equivlent to $(A, \xi|_A)$ is a compact convergence space.

3.1. **Pretopologies, H-closed Spaces and H-sets.** For each $x \in X$, the *vicinity filter at* x is defined

$$\mathcal{V}_\xi(x) = \bigcap \{\mathcal{F} : x \in \lim_\xi \mathcal{F}\}.$$

If $A \subseteq X$, then

$$\mathcal{V}_\xi(A) = \{V \subseteq X : A \subseteq \operatorname{inh}_\xi V\}.$$

A convergence ξ on X is a *pretopology* if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for each $x \in X$. We take a moment to gather several well-known facts and definitions pertaining to pretopological spaces here:

Proposition 3.3. *If (X, π) is a pretopological space, then the adherence operator satisfies each of the following*

   (1) $\operatorname{adh}_\pi \varnothing = \varnothing$,
   (2) $A \subseteq \operatorname{adh}_\pi A$ *for each* $A \subseteq X$,
   (3) $\operatorname{adh}_\pi(A \cup B) = \operatorname{adh}_\pi A \cup \operatorname{adh}_\pi B$ *for any* $A, B \subseteq X$.

Additionally, $U \in \mathcal{V}_\pi(x)$ if and only if $x \in \operatorname{inh}_\pi U$ and $x \in \operatorname{adh}_\pi \mathcal{F}$ if and only if $\mathcal{V}_\pi(x) \# \mathcal{F}$.

In particular, this proposition shows that the closure spaces of [3] and pretopological spaces as defined above are equivalent.

Proposition 3.4. *If (X, π) is a pretopological space, then X is Hausdorff if and only if whenever x_1, $x_2 \in X$ and $x_1 \neq x_2$, there exists $U_i \in \mathcal{V}_\pi(x_i)$ $(i = 1, 2)$ such that $U_1 \cap U_2 = \varnothing$.*

Proposition 3.5. *Let $f : (X, \pi) \to (Y, \sigma)$. The following are equivalent*

(1) *f is continuous*
(2) *$f[\mathrm{adh}_\pi \mathcal{F}] \subseteq \mathrm{adh}_\sigma f(\mathcal{F})$ for each filter $\mathcal{F}$ on X*
(3) *$f[\mathrm{adh}_\pi A] \subseteq \mathrm{adh}_\sigma f[A]$ for each $A \subseteq X$*
(4) *$f^{\leftarrow}[\mathrm{inh}_\sigma B] \subseteq \mathrm{inh}_\pi f^{\leftarrow}[B]$ for each $B \subseteq Y$*
(5) *For each $x \in X$, if $V \in \mathcal{V}_\sigma(f(x))$, there exists $U \in \mathcal{V}_\pi(x)$ such that $f[U] \subseteq V$.*

Definition 3.6. A collection $\mathcal{C}$ of subsets of a pretopological space (X, π) is a π-*cover* (or simply *cover* if there is no possible confusion) if for each $x \in X$, $\mathcal{C} \cap \mathcal{V}_\pi(x) \neq \varnothing$. For $A \subseteq X$, we say that $\mathcal{C}$ is a *cover of A* if for each $x \in A$, $\mathcal{C} \cap \mathcal{V}_\pi(x) \neq \varnothing$.

Proposition 3.7. *Let (X, π) be a pretopological space, $\mathcal{F}$ a filter on X and $A \subseteq X$. Then $\mathcal{F}$ is compact at A if and only if whenever $\mathcal{C}$ is a π-cover of A, there exists $F \in \mathcal{F}$ and $C_1, ..., C_n \in \mathcal{C}$ such that $F \subseteq \bigcup_{i=1}^{n} C_i$.*

The notion of covers has been studied before (see, for example, [9]) and it is well known that this definition of a pretopological cover is a specific case of the more general notion for convergence spaces. For pretopological spaces – and more generally for convergence spaces – the characterization of compact in terms of covers in Proposition 3.7 is weaker than the notion of *cover-compactness* found in Definition 3.13. These characterizations coincide for topological spaces.

For an example of a pretopology which is not in general a topology, we return to Example 3.1. In this case, (X, θ) is a pretopological space and $\mathcal{V}_\theta(x)$ is the filter generated by $\{\mathrm{cl}_\tau U : U \in \mathcal{N}_\tau(X)\}$. For $A \subseteq X$, $\mathrm{adh}_\theta A$ is the well-known θ-closure. Explicitly,

$$x \in \mathrm{adh}_\theta A \Leftrightarrow \forall (U \in \mathcal{N}_\tau(X))\, \mathrm{cl}_\tau U \cap A \neq \varnothing.$$

To see why this convergence is not in general a topology, we make use of Proposition 22 from [9].

Proposition 3.8. *Let (X, ξ) be a convergence space. If ξ is a topology, then the adherence operator adh_ξ is idempotent on subsets of X.*

Now let X be the topological space defined in Example 2.3 equipped with the pretopology θ described in Example 3.1. Consider the following subset B of X :

$$B = \{(n, m) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}, m > 0\}.$$

Then
$$\mathrm{adh}_\theta\, B = B \cup \{(n,0) \in \mathbb{N} \times \mathbb{Z} : n \in \mathbb{N}\} \cup \{+\infty\}.$$
However,
$$\mathrm{adh}_\theta\, \mathrm{adh}_\theta\, B = \mathrm{adh}_\theta\, B \cup \{-\infty\}$$
and by Proposition 3.8, (X, θ) is not a topological space.

We can now characterize both H-closed spaces and H-sets in the terms of the pretopological convergence θ. The following theorem is well-known. The first part is due to Veličko [17] and the second can be found in [5].

Theorem 3.9. *Let X be a Hausdorff topological space and $A \subseteq X$. Then*
  (1) *X is H-closed if and only if $\mathrm{adh}_\theta\, \mathcal{F} \neq \varnothing$ for every filter $\mathcal{F}$ on X.*
  (2) *A is an H-set in X if and only if $\mathrm{adh}_\theta\, \mathcal{F} \cap A \neq \varnothing$ for each filter $\mathcal{F}$ which meets A.*

We can restate Theorem 3.9 using Definition 3.2. The following then characterizes both H-closed spaces and H-sets as pretopological notions.

Theorem 3.10. *Let X be a Hausdorff topological space and $A \subseteq X$.*
  (1) *X is H-closed if and only if (X, θ_X) is a compact pretopological space.*
  (2) *A is an H-set in X if and only if A is a compact subset of (X, θ_X).*

Just as immediate, but perhaps more interesting, is the case of H-sets in Urysohn spaces. Recall that a topological space X is Urysohn if distinct points have disjoint closed neighborhoods. For every Hausdorff space X, there exists an extremally disconnected, Tychonoff space EX, called the *absolute of X*, and a perfect, irreducible, θ-continuous map $k_X : EX \to X$. Explicitly, the space EX has as its points the open ultrafilters on X and for an open ultrafilter $\mathcal{U}$, $k_X(\mathcal{U})$ is the unique adherent point of $\mathcal{U}$ in X. Moreover, the absolute of X is unique in a sense. For a full treatment of absolutes, see [15]. The following theorem is due to Vermeer [18] and makes use of this construction.

Theorem 3.11. *Let X be H-closed and Urysohn and let $A \subseteq X$. Then A is an H-set if and only if $k_X^\leftarrow[A]$ is a compact subset of EX.*

In the same paper, Vermeer gives an example of an H-closed non-Urysohn space X which has an H-set which is not the image under k_X of any compact subspace of EX. A more general phrasing of the above theorem of Vermeer is that if A is an H-set in an H-closed Urysohn space, then there exists a compact Hausdorff topological space K and a θ-continuous function $f : K \to X$ such that $f[K] = A$. Vermeer then asked if this was true for an H-set in any Hausdorff topological space; i.e. if X is a Hausdorff topological space and A is an H-set in X, does there exist a compact,

Hausdorff topological space K and a θ-continuous function $f : K \to X$ such that $f[K] = A$? The answer, it turns out, is no. This was shown first by Bella and Yaschenko in [2]. Later, in [13], McNeill showed that it is in addition possible to construct a Urysohn space containing an H-set which is not the θ-continuous image of a compact, Hausdorff topological space. This makes the following observation interesting.

Theorem 3.12. *Let X be a Urysohn topological space. Then A is an H-set if and only if $(A, \theta|_A)$ is a compact, Hausdorff pretopological space, where $\theta|_A$ is the subconvergence on A inherited from (X, θ). In particular, if X is a Urysohn topological space and $A \subseteq X$ is an H-set, then there exists a compact, Hausdorff pretopological space (K, π) and a continuous function $f : (K, \pi) \to (X, \theta)$ such that $f[K] = A$.*

The question remains – if X is a Hausdorff topological space and A is an H-set in X, is there a compact, Hausdorff pretopological space (K, π) and a continuous function $f : (K, \pi) \to (X, \theta)$ such that $f[K] = A$? More broadly, is there a pretopological version of the absolute?

3.2. **Perfect Maps.** Much of the following can be seen as generalizing the results of [5] to pretopological spaces. Throughout this subsection, let (X, π) and (Y, σ) be pretopological spaces. The results below will be used in the construction of the θ-quotient convergence in Section 4.

Definition 3.13. A function $f : (X, \pi) \to (Y, \sigma)$ is *perfect* if $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$ whenever $y \in \lim_\sigma \mathcal{F}$.

In the case of topological spaces, this definition was shown by Whyburn [20] to be equivalent to the usual definition of a perfect function for topological spaces; that is, a function which is closed and has compact fibers.

Proposition 3.14. *A function $f : (X, \pi) \to (Y, \sigma)$ is perfect if and only if $f(\operatorname{adh}_\pi \mathcal{F}) \supseteq \operatorname{adh}_\sigma f(\mathcal{F})$ for each filter $\mathcal{F}$ on X.*

Proof. Suppose that f is perfect. Let $\mathcal{F}$ be a filter on X and let $y \in \operatorname{adh}_\sigma f(\mathcal{F})$. By way of contradiction, suppose that $f^{\leftarrow}(y) \cap \operatorname{adh}_\pi \mathcal{F} = \varnothing$. Since σ is a pretopology, $y \in \lim_\sigma \mathcal{V}_\sigma(y)$ and since f is perfect, it follows that $f^{\leftarrow}(\mathcal{V}_\sigma(y))$ is compact at $f^{\leftarrow}(y)$. Since $y \in \operatorname{adh}_\sigma f(\mathcal{F})$, $\mathcal{V}_\sigma(y) \# f(\mathcal{F})$. It follows that $f^{\leftarrow}(\mathcal{V}_\sigma(y)) \# \mathcal{F}$. Thus, it must be that $\operatorname{adh}_\pi \mathcal{F} \cap f^{\leftarrow}(y) \neq \varnothing$, a contradiction. Hence, $y \in f(\operatorname{adh}_\pi \mathcal{F})$.

Conversely, suppose $\mathcal{F}$ is a filter on Y and $y \in \lim_\sigma \mathcal{F}$. Let $\mathcal{G}$ be a filter on X such that $\mathcal{G} \# f^{\leftarrow}(\mathcal{F})$. Then $f(\mathcal{G}) \# \mathcal{F}$. Since $y \in \lim_\sigma \mathcal{F}$, it follows that $y \in \operatorname{adh}_\sigma f(\mathcal{G}) \subseteq f[\operatorname{adh}_\pi \mathcal{G}]$. So, we can find $x \in \operatorname{adh}_\pi \mathcal{G}$ such that $f(x) = y$. In other words, $\operatorname{adh}_\pi \mathcal{G} \cap f^{\leftarrow}(y) \neq \varnothing$, and $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. $\qquad \square$

To get a similar characterization to that of perfect functions between topological spaces for perfect functions between pretopological spaces we need the concept of *cover-compact* sets, a strengthening of compact sets. This characterization can be found in [6], but we feel it is worthwhile to lay out the details in this less technical setting.

Definition 3.15. Let (X, π) be a pretopological space and $A \subseteq X$. Then A is *cover-compact* if whenever $\mathcal{C}$ is a cover of A, there exist $C_1, ..., C_n \in \mathcal{C}$ such that $A \subseteq \text{inh}_\pi \left(\bigcup_{i=1}^n C_i \right)$.

Proposition 3.16. *Let (X, π) be a pretopological space and $A \subseteq X$. The following are equivalent,*

(1) *For any filter $\mathcal{F}$ on X, $\text{adh}_\pi \mathcal{F} \cap A = \varnothing$ implies that there exists some $F \in \mathcal{F}$ such that $\text{adh}_\pi F \cap A = \varnothing$,*

(2) *A is cover-compact,*

(3) *$\text{adh}_\pi \mathcal{F} \cap A = \varnothing$ implies there exists $V \subseteq X$ and $F \in \mathcal{F}$ such that $A \subseteq \text{inh}_\pi V$ and $V \cap F = \varnothing$ for any filter $\mathcal{F}$ on X.*

Proof. Suppose that A is cover-compact and let $\mathcal{C}$ be a cover of A. Suppose that no finite subcollection exists as needed. Then $\mathcal{F} = \{X \setminus (C_1 \cup ... \cup C_n) : C_i \in \mathcal{C}, i \in \mathbb{N}\}$ is a filterbase on X. Note that $\text{adh}_\pi \mathcal{F} \subseteq X \setminus \bigcup_{C \in \mathcal{C}} \text{inh}_\pi C$ and as such $\text{adh}_\pi \mathcal{F} \cap A = \varnothing$. Since A is cover-compact, we can find $F \in \mathcal{F}$ such that $\text{adh}_\pi F \cap A = \varnothing$. However, $F = X \setminus (C_1 \cup ... \cup C_n)$ for some $C_1, ..., C_n \in \mathcal{C}$, so we have that $A \subseteq \text{inh}_\pi(C_1 \cup ... \cup C_n)$, a contradiction.

Suppose that $\mathcal{F}$ is a filter on X and $\text{adh}_\pi \mathcal{F} \cap A = \varnothing$. Then, for each $x \in A$, fix $V_x \in \mathcal{V}_\pi(x)$ and $F_x \in \mathcal{F}$ such that $V_x \cap F_x = \varnothing$. Then $\{V_x : x \in A\}$ is a cover of A. By assumption, we can choose $x_1, ..., x_n \in A$ such that $A \subseteq \text{inh}_\pi \left(\bigcup_{i=1}^n V_{x_i} \right)$. Therefore, $V = \bigcup_{i=1}^n V_{x_i} \in \mathcal{V}_\pi(A)$ and $V \cap (F_{x_1} \cap ... \cap F_{x_n}) = \varnothing$. Since $F_{x_1} \cap ... \cap F_{x_n} \in \mathcal{F}$, we have shown that (c) holds.

Lastly, let $\mathcal{F}$ be a filter on X such that $\text{adh}_\pi \mathcal{F} \cap A = \varnothing$. By assumption, we can find $V \in \mathcal{V}_\pi(A)$ and $F \in \mathcal{F}$ such that $V \cap F = \varnothing$. For each $x \in A$, $V \in \mathcal{V}_\pi(x)$, so $x \notin \text{adh}_\pi F$. It follows immediately that $A \cap \text{adh}_\pi F \neq \varnothing$. $\square$

It is useful to note that if $A \subseteq X$ is cover-compact, then $\text{adh}_\pi A = A$.

Theorem 3.17. *Let $f : (X, \pi) \to (Y, \sigma)$ be a map between pretopological spaces satisfying (a) $f[\text{adh}_\pi A] \supseteq \text{adh}_\sigma f[A]$ for any $A \subseteq X$ and (b) $f^\leftarrow(y)$ is cover-compact for each $y \in Y$. Then f is perfect.*

Proof. Let $\mathcal{F}$ be a filter on Y which σ-converges to some $y \in Y$. Let $\mathcal{G}$ be a filter on X which meets $f^\leftarrow(\mathcal{F})$. Then $f(\mathcal{G})$ meets $\mathcal{F}$. Since $y \in \lim_\sigma \mathcal{F}$, $\mathcal{F}$ is compact at y. Therefore, $y \in \text{adh}_\sigma f(\mathcal{G}) = \bigcap_{G \in \mathcal{G}} \text{adh}_\sigma f[G]$.

By assumption (a), for each $G \in \mathcal{G}$, $f[\mathrm{adh}_\pi G] \supseteq \mathrm{adh}_\sigma f[G]$. Therefore, $\mathrm{adh}_\pi G \cap f^{\leftarrow}(y) \neq \varnothing$ for each $G \in \mathcal{G}$. By assumption (b), $f^{\leftarrow}(y)$ is cover-compact, so $\mathrm{adh}_\pi \mathcal{G} \cap f^{\leftarrow}(y) \neq \varnothing$. In other words, $f^{\leftarrow}(F)$ is compact at $f^{\leftarrow}(y)$ and f is perfect. $\qquad \square$

Theorem 3.18. *Let $f : (X, \pi) \to (Y, \sigma)$ be perfect and continuous. Then f satisfies (a) and (b) of 3.17.*

Proof. By Proposition 3.5(3) and Proposition 3.14, $f[\mathrm{adh}_\pi A] = \mathrm{adh}_\sigma f[A]$ for each $A \subseteq X$. Thus, a property stronger than (a) holds. To see that (b) holds, fix $y \in Y$ and let $\mathcal{F}$ be a filter on X such that $\mathrm{adh}_\pi \mathcal{F} \cap f^{\leftarrow}(y) = \varnothing$. By Proposition 3.14, $y \notin f[\mathrm{adh}_\pi \mathcal{F}] \supseteq \mathrm{adh}_\sigma f(\mathcal{F})$. Thus, we can find $V \in \mathcal{V}_\sigma(y)$ and $F \in \mathcal{F}$ such that $V \cap f(F) = \varnothing$. It follows that $f^{\leftarrow}[V] \cap F = \varnothing$. Since f is a continuous function, for each $x \in f^{\leftarrow}(y)$, fix $U_x \in \mathcal{V}_\pi(x)$ such that $f[U_x] \subseteq V$. Then $\bigcup_{x \in f^{\leftarrow}(y)} U_x \subseteq f^{\leftarrow}[V]$ and thus $\bigcup_{x \in f^{\leftarrow}(y)} U_x \cap F = \varnothing$. So, $\mathrm{adh}_\pi F \cap f^{\leftarrow}(y) = \varnothing$, as needed. $\qquad \square$

Corollary 3.19. A continuous function $f : (X, \pi) \to (Y, \sigma)$ is perfect if and only if it satisfies (a) and (b) of 3.17.

4. PHC Spaces

In this section we will define a variation of H-closed spaces for pretopological spaces. After establishing some basic facts about the so-called *PHC spaces*, we will describe a method for constructing PHC pretopologies and PHC extensions.

The following definition appears in [7].

Definition 4.1. Let (X, π) be a pretopological space. The *partial regularization $r\pi$* of π is the pretopology determined by the vicinity filters $\mathcal{V}_{r\pi}(x) = \{\mathrm{adh}_\pi U : U \in \mathcal{V}_\pi(x)\}$.

Notice that if (X, τ) is a topological space, then $r\tau$ is the usual θ-convergence on X. Thus, a Hausdorff topological space (X, τ) is H-closed if and only if $(X, r\tau)$ is compact. This inspires the following definition, aiming to generalize the notion of H-closed spaces to pretopological spaces.

Definition 4.2. Let (X, π) be a Hausdorff pretopological space. The pretopology π is *PHC (pretopologically H-closed)* if $(X, r\pi)$ is compact. Without the assumption of Hausdorff, we will use the term *quasi PHC*.

For $n \in \mathbb{N}$ and $A \subseteq X$, let $\mathrm{inh}_\pi^n A$ be the nth iteration of the inherence operator on A. Given a filter $\mathcal{F}$ on a pretopological space (X, π) let

$$i_\pi \mathcal{F} = \{F : \mathrm{inh}_\pi F \in \mathcal{F}\}.$$

Inductively, define

$$i_\pi^n \mathcal{F} = \{H : \mathrm{inh}_\pi^n H \in \mathcal{F}\},$$

and finally

$$i_\pi^\omega \mathcal{F} = \bigcap_{n \in \mathbb{N}} i_\pi^n \mathcal{F}.$$

We use the convention $i_\pi^0 \mathcal{F} = \mathcal{F}$ and $i_\pi^1 \mathcal{F} = i_\pi \mathcal{F}$. Notice then that $i_\pi^n \mathcal{F} = i_\pi \left(i_\pi^{n-1} \mathcal{F} \right)$ for each $n \in \mathbb{N}$.

Lemma 4.3. *Let (X, π) be a pretopological space and let $\mathcal{F}$ be a filter on X such that $i_\pi \mathcal{F} = \mathcal{F}$. Then $\mathrm{adh}_\pi \mathcal{F} = \mathrm{adh}_{r\pi} \mathcal{F}$.*

Proof. To begin, since $r\pi \leq \pi$, $\mathrm{adh}_\pi \mathcal{F} \subseteq \mathrm{adh}_{r\pi} \mathcal{F}$. Now, $x \notin \mathrm{adh}_\pi \mathcal{F}$ if and only if we can find $F \in \mathcal{F}$ and $U \in \mathcal{V}_\pi(x)$ such that $U \cap F = \varnothing$. Since $U \cap F = \varnothing$, if $y \in \mathrm{inh}_\pi F$, then $y \notin \mathrm{adh}_\pi U$. In other words, $\mathrm{adh}_\pi U \cap \mathrm{inh}_\pi F = \varnothing$. Since $\mathcal{F} = i_\pi \mathcal{F}$, $\mathrm{inh}_\pi F \in \mathcal{F}$ and by definition $x \notin \mathrm{adh}_{r\pi} \mathcal{F}$, as needed.   □

Lemma 4.4. *Let (X, π) be a pretopological space and let $\mathcal{F}$ be a filter on X. Then $\mathrm{adh}_{r\pi} i_\pi^n \mathcal{F} = \mathrm{adh}_\pi i_\pi^{n+1} \mathcal{F}$ for each $n \in \mathbb{N}$.*

Proof. We begin by showing the lemma holds for $n = 0$. Recall that $i_\pi^0 \mathcal{F} = \mathcal{F}$. Suppose that $x \notin \mathrm{adh}_{r\pi} \mathcal{F}$. Then there exists $U \in \mathcal{V}_\pi(x)$ and there exists $F \in \mathcal{F}$ such that $\mathrm{adh}_\pi U \cap F = \varnothing$. So, $F \subseteq X \setminus \mathrm{adh}_\pi U = \mathrm{inh}_\pi(X \setminus U)$. By definition, it follows that $X \setminus U \in i_\pi \mathcal{F}$. Since $U \cap X \setminus U = \varnothing$, we have that $x \notin \mathrm{adh}_\pi i_\pi \mathcal{F}$. Conversely, if $x \notin \mathrm{adh}_\pi i_\pi \mathcal{F}$, then there exists $U \in \mathcal{V}_\pi(x)$ and $F \in i_\pi \mathcal{F}$ such that $U \cap F = \varnothing$. As we have seen before, it follows that $\mathrm{adh}_\pi U \cap \mathrm{inh}_\pi F = \varnothing$. Since $F \in i_\pi F$, we know that $\mathrm{inh}_\pi F \in \mathcal{F}$. It follows that $x \notin \mathrm{adh}_{r\pi} \mathcal{F}$, as needed.

The remainder of the lemma follows easily by setting $\mathcal{F} = i_\pi^n \mathcal{F}$, in which case $i_\pi \mathcal{F} = i_\pi^{n+1} \mathcal{F}$.   □

Definition 4.5. A filter $\mathcal{F}$ on a pretopological space is *inherent* if $\mathrm{inh}_\pi F \neq \varnothing$ for each $F \in \mathcal{F}$. If $\mathcal{U}$ is maximal with respect to the property of being inherent, we say that $\mathcal{U}$ is an *inherent ultrafilter*.

Theorem 4.6. *For a Hausdorff pretopological space (X, π), the following are equivalent.*

    (1) *X is PHC*
    (2) *whenever $\mathcal{C}$ is a π-cover of X, there exists $C_1, ..., C_n \in \mathcal{C}$ such that $X = \bigcup_{i=1}^n \mathrm{adh}_\pi C_i$*
    (3) *each inherent filter $\mathcal{F}$ on X has nonempty adherence*
    (4) *$\mathrm{adh}_\pi i_\pi \mathcal{F} \neq \varnothing$ for each filter $\mathcal{F}$ on X.*

Proof. Let $\mathcal{C}$ be a π-cover of X. Without loss of generality, assume that $\mathcal{C} = \{U_x : x \in X\}$ where each $U_x \in \mathcal{V}_\pi(x)$. Suppose no such finite subcollection exists. Then $\mathcal{A} = \{X \setminus \mathrm{adh}_\pi U_x : x \in X\}$ has the finite intersection property. Let $\mathcal{F}$ be the filter generated by $\mathcal{A}$. For each

$x \in X$, $x \in \mathrm{inh}_{r\pi} \mathrm{adh}_{\pi} U$ if and only if there exists $V \in \mathcal{V}_{\pi}(x)$ such that $\mathrm{adh}_{\pi} V \subseteq \mathrm{adh}_{\pi} U$. Therefore, for each $x \in X$, $x \in \mathrm{inh}_{r\pi} \mathrm{adh}_{\pi} U_x$. Thus, $\mathrm{adh}_{r\pi} \mathcal{F} = X \setminus \bigcup_{x \in X} \mathrm{inh}_{r\pi} \mathrm{adh}_{\pi} U_x = \varnothing$, a contradiction.

Next, let $\mathcal{F}$ be a filter on X such that $\mathrm{inh}_{\pi} F \neq \varnothing$ for each $F \in \mathcal{F}$. Suppose that $\mathrm{adh}_{\pi} \mathcal{F} = \varnothing$. Then $\mathcal{C} = \{X \setminus F : F \in \mathcal{F}\}$ is a π-cover of X. By assumption, there exist $F_1, ..., F_n \in \mathcal{F}$ such that $\mathrm{adh}_{\pi}(X \setminus F_1 \cup ... \cup X \setminus F_n) = X \setminus \mathrm{inh}_{\pi}(F_1 \cap ... \cap F_n) = X$. However, $F_1 \cap ... \cap F_n \in \mathcal{F}$ and thus by assumption $F_1 \cap ... \cap F_n$ has nonempty inherence, a contradiction.

Let $\mathcal{F}$ be a filter on X. Notice that $i_{\pi} \mathcal{F}$ is a filter on X such that $\mathrm{inh}_{\pi} F \neq \varnothing$ for each $F \in i_{\pi} \mathcal{F}$. Then by assumption, $\mathrm{adh}_{\pi} i_{\pi} \mathcal{F} \neq \varnothing$.

Let $\mathcal{F}$ be a filter on X. Then $\mathrm{adh}_{r\pi} \mathcal{F} = \mathrm{adh}_{\pi} i_{\pi} \mathcal{F} \neq \varnothing$ by Lemma 4.4. Thus, we have shown that $(X, r\pi)$ is compact and the theorem is proven. $\square$

4.1. θ-quotient Convergence.

Let (X, π) be a compact Hausdorff pretopological space, Y a set and $f : (X, \pi) \to Y$ a surjection such that $f^{\leftarrow}(y)$ is cover-compact for each $y \in Y$. For $A \subseteq X$, let $f^{\#}[A] = \{y \in Y : f^{\leftarrow}(y) \subseteq A\}$. Define the θ-quotient convergence $f^{\#}\pi$ on Y as follows: a filter $\mathcal{F}$ on Y $f^{\#}\pi$-converges to y if and only if $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$.

Lemma 4.7. *Let $\mathcal{F}$ be a filter on Y. Then $y \in \lim_{f^{\#}\pi} \mathcal{F}$ if and only if $f^{\leftarrow}(\mathcal{F}) \supseteq \mathcal{V}_{\pi}(f^{\leftarrow}(y))$.*

Proof. Suppose that $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. Then whenever $\mathcal{C}$ is a cover of $f^{\leftarrow}(y)$, there exists $F \in \mathcal{F}$ and $C_1, ..., C_n \in \mathcal{C}$ such that $f^{\leftarrow}[F] \subseteq \bigcup_{i=1}^{n} C_i$. Let $V \in \mathcal{V}_{\pi}(f^{\leftarrow}(y))$. By definition, $f^{\leftarrow}(y) \subseteq \mathrm{inh}_{\pi} V$. In other words, $\{V\}$ is a one-element cover of $f^{\leftarrow}(y)$. Thus, $V \in f^{\leftarrow}(\mathcal{F})$, as needed.

Conversely, let $\mathcal{C}$ be a cover of $f^{\leftarrow}(y)$. Since $f^{\leftarrow}(y)$ is cover-compact, we can find $C_1, ..., C_n \in \mathcal{C}$ such that $f^{\leftarrow}(y) \subseteq \mathrm{inh}_{\pi} (\bigcup_{i=1}^{n} C_i)$. By definition, $C = \bigcup_{i=1}^{n} C_i \in \mathcal{V}_{\pi}(f^{\leftarrow}(y))$. Thus, $C \in f^{\leftarrow}(\mathcal{F})$ and there exists $F \in \mathcal{F}$ such that $f^{\leftarrow}[F] \subseteq \bigcup_{i=1}^{n} C_i$ and $f^{\leftarrow}(\mathcal{F})$ is compact at $f^{\leftarrow}(y)$. $\square$

Lemma 4.8. *Let (X, π) be a Hausdorff pretopology. If $A, B \subseteq X$ are disjoint cover-compact subsets of X, then there exist disjoint vicinities $U \in \mathcal{V}_{\pi}(A)$, $V \in \mathcal{V}_{\pi}(B)$.*

Proof. First we show this holds for $B = \{x\}$. For each $z \in A$, choose disjoint $U_z \in \mathcal{V}_{\pi}(z)$ and $V_z \in \mathcal{V}_{\pi}(x)$. Since A is cover-compact, we can choose $z_1, ..., z_n \in A$ such that $A \subseteq \mathrm{inh}_{\pi} (\bigcup_{i=1}^{n} U_{z_i})$. Thus, $U = \bigcup_{i=1}^{n} U_{z_i} \in \mathcal{V}_{\pi}(A)$. Also, $V = \bigcap_{i=1}^{n} V_{z_i} \in \mathcal{V}_{\pi}(x)$ and $U \cap V = \varnothing$. It is a straightforward exercise to now show this holds for disjoint cover-compact sets, A and B. $\square$

Proposition 4.9. $(Y, f^{\#}\pi)$ *is a Hausdorff pretopology. Furthermore, for each* $y \in Y$,

$$\mathcal{V}_{f^{\#}\pi}(y) = \langle \{f^{\#}[W] : W \in \mathcal{V}_{\pi}(f^{\leftarrow}(y))\} \rangle.$$

Proof. We first show that $f^{\#}\pi$ is indeed a pretopology. Notice that for $y \in Y$,

$$\bigcap\{\mathcal{F} : y \in \lim_{f^{\#}\pi} \mathcal{F}\} = \bigcap\{\mathcal{F} : V \in \mathcal{V}_{\pi}(f^{\leftarrow}(y)) \text{ implies } f^{\#}[V] \in \mathcal{F}\}.$$

It follows that $\mathcal{V}_{f^{\#}\pi}(y)$ is the filter generated by $\{f^{\#}[U] : U \in \mathcal{V}_{\pi}(f^{\leftarrow}(y))\}$. For any $A \subseteq X$, $f^{\leftarrow}[f^{\#}[A]] \subseteq A$. It follows easily that $f^{\leftarrow}(\mathcal{V}_{f^{\#}\pi}(y)) \supseteq \mathcal{V}_{\pi}(f^{\leftarrow}(y))$. By Lemma 4.7, then, $y \in \lim_{f^{\#}\pi} \mathcal{V}_{f^{\#}\pi}(y)$ and $f^{\#}\pi$ is a pretopology with the stated vicinity filters.

Now, if $y_1 \neq y_2$, by Lemma 4.8, for $i = 1, 2$, we can find $U_i \in \mathcal{V}_{\pi}(f^{\leftarrow}(y_i))$ such that $U_1 \cap U_2 = \varnothing$. It is immediate that $f^{\#}[U_1] \cap f^{\#}[U_2] = \varnothing$ and $f^{\#}\pi$ is Hausdorff. $\square$

Definition 4.10. Let (X, π) and (Y, σ) be pretopological spaces. A function $f : (X, \pi) \to (Y, \sigma)$ is *strongly irreducible* if there exists $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U \cap V$ for any subsets U and V of X with nonempty inherence such that $U \cap V \neq \varnothing$.

The function f is *weakly θ-continuous (wθ-continuous* for short) if $f : (X, \pi) \to (Y, r\sigma)$ is continuous.

Theorem 4.11. *If* (X, π) *is a compact, Hausdorff pretopological space,* $f : (X, \pi) \to Y$ *a strongly irreducible surjection such that* $f^{\leftarrow}(y)$ *is cover-compact for each* $y \in Y$ *and* $f^{\#}\pi$ *is the θ-quotient pretopology on* Y, *then* $f : (X, \pi) \to (Y, f^{\#}\pi)$ *is wθ-continuous and* $(Y, f^{\#}\pi)$ *is a PHC Hausdorff pretopological space.*

Proof. For $x \in X$, let $V \in \mathcal{V}_{f^{\#}\pi}(f(x))$. Without loss of generality, we can assume that $V = f^{\#}[W]$ for some $W \in \mathcal{V}_{\pi}(f^{\leftarrow}(f(x)))$. Note that in this case $x \in \text{inh}_{\pi} W$. Supose that $w \in W$ and $f(w) \in f^{\#}[U]$ for some $U \in \mathcal{V}_{\pi}(f^{\leftarrow}(f(w)))$. Notice that $w \in W \cap U$, so $W \cap U \neq \varnothing$. Since f is strongly irreducible, we can find $y \in f^{\#}[U] \cap f^{\#}[W]$. Therefore, $f(w) \in \text{adh}_{\pi} f^{\#}[W]$. In particular, $f[W] \subseteq \text{adh}_{\pi} f^{\#}[W]$ and f is wθ-continuous.

Since the continuous image of a compact space is again compact, $(Y, rf^{\#}\pi)$ is compact and by definition $(Y, f^{\#}\pi)$ is PHC. $\square$

4.2. PHC Extensions of X. Let (X, π) be a pretopological space. By an *extension* of π, we mean a convergence ξ on a set Y such that (X, π) is a subspace of (Y, ξ) and $\text{adh}_{\xi} X = Y$. There is an ordering on the family extensions of X. If ξ and ζ are extensions of π, we say that ξ *is projectively larger than* ζ, written $\xi \geq_{\pi} \zeta$ if there exists a continuous

map $f : (Y, \xi) \to (Z, \zeta)$ which fixes the points of X. In the comment following the definition of a convergence, we noted that the underlying set of a convergence space is determined by the convergence itself. For coherence of notation, when discussing extensions of convergence spaces we will often refer to the convergence without reference to the underlying set. This is not a problem thanks to the aforementioned comment.

We borrow from topology the concepts of *strict* and *simple* extensions. If ξ is an extension of π, we define ξ^+ a new extension of π on the same underlying set as ξ. For $p \in Y$,

$$\mathcal{V}_{\xi^+}(p) = \langle \{\{p\} \cup U : \exists W \in \mathcal{V}_\xi(p), W \cap X = U\} \rangle.$$

If $\xi = \xi^+$, then we say ξ is a *simple extension* of π.

In a similar way, we define $\xi^\#$, an extension of π on the same set as Y. If $A \subseteq X$, let

$$oA = \{p \in Y : \exists W \in \mathcal{V}_\xi(p), W \cap X = A\}.$$

If $p \in Y$, then $V_{\xi^\#}(p)$ is the filter generated by $\{oA : \exists V \in \mathcal{V}_\xi(p), V \cap X = A\}$. If $\xi = \xi^\#$, then we sy that ξ is a *strict extension* of π.

Lemma 4.12. *If ξ is an extension of π, then $\xi^\# \leq \xi \leq \xi^+$.*

Proof. In both cases it is straight-forward to check that the identity map is continuous and fixes X. $\qquad\square$

Proposition 4.13. *Suppose that (X, π) is a Hausdorff pretopological space and ξ is a pretopology and a compactification of π. Then ξ^+ is PHC.*

Proof. Recall that by compactification, we mean a compact extension. Fix $p \in Y$ and let $\{p\} \cup U \in \mathcal{V}_{\xi^+}(p)$. Then $\mathrm{adh}_{\xi^+}(\{p\} \cup U) = oU \cup \mathrm{adh}_\pi U$. So, in the partial regularization of ξ^+, the vicinity filters are generated by sets of the form $oU \cup \mathrm{adh}_\pi U$ for $U \subseteq X$. In particular, this shows that $\mathcal{V}_{r\xi^+}(p) \subseteq \mathcal{V}_{\xi^\#}(p)$ for each $p \in Y$. Since $\xi^\#$ is a coarser pretopology than ξ, it follows that the partial regularization of ξ^+ is coarser than ξ. Since (Y, ξ) is compact, so is $(Y, r\xi^+)$ and by definition, ξ^+ is PHC. $\qquad\square$

For any Hausdorff convergence space (X, σ), Richardson [16] constructs a compact, Hausdorff convergence space (X^*, σ^*) in which X is densely embedded. It should be noted that Richardson's definition of a convergence includes the following third axiom in addition to the two in our definition:

(R) $\qquad\qquad$ If $x \in \lim_\sigma \mathcal{F}$, then $x \in \lim_\sigma(\langle x \rangle \cap \mathcal{F})$.

We will make use of assumption (R) in Theorem 4.16. Note that if σ is a pretopology, then σ already satisfies (R). If σ is a pretopology, then so is σ^*. It is said that (X, ξ) is *regular* if $x \in \lim_\xi \mathcal{F}$ implies that $x \in \lim_\xi \{\mathrm{adh}_\xi F : F \in \mathcal{F}\}$. Richardson [16] proves the following:

Theorem 4.14. *If (X, σ) is a Hausdorff convergence space, (Y, ξ) is a compact, Hausdorff, regular convergence space and $f : (X, \sigma) \to (Y, \xi)$ is continuous, then there exists a unique continuous map $F : (X^*, \sigma^*) \to (Y, \xi)$ extending f.*

We seek to circumvent the assumption of regularity on (Y, ξ). For a Hausdorff pretopological space (X, π), let $(\kappa_\pi X, \kappa\pi) = (X^*, (\pi^*)^+)$. By the above proposition, $\kappa\pi$ is PHC. Additionally, $\kappa\pi$ has the following property.

Theorem 4.15. *Let (X, π) and (Y, ξ) be Hausdorff pretopological spaces spaces. If $f : (X, \sigma) \to (Y, \xi)$ is continuous, then there exists a continuous function $F : (\kappa_\pi X, \kappa\pi) \to (\kappa_\xi Y, \kappa\xi)$ which extends f.*

Proof. For each free ultrafilter $\mathcal{U}$ on X, $f(\mathcal{U})$ is an ultrafilter on Y. Define $F(\mathcal{U})$ as follows:

- If $y \in \lim_\xi f(\mathcal{U})$ for some $y \in Y$, let $F(\mathcal{U}) = y$.
- If $f(\mathcal{U})$ is free in (Y, ξ), let $F(\mathcal{U}) = f(\mathcal{U})$.

We show that F is continuous. Since f is continuous, if $x \in X$ and $F(x) \in \mathcal{V}_{\kappa\pi}(f(x)) = \langle \mathcal{V}_\pi(f(x)) \rangle$, then we can find $U \in \mathcal{V}_\pi(x)$ such that $f[U] \subseteq V$. Suppose $\mathcal{U} \in \kappa X \setminus X$. If $F(\mathcal{U}) \in Y$, let $V \in \mathcal{V}_\pi(F(\mathcal{U}))$. Since $y \in \lim_\xi f(\mathcal{U})$, $V \in f(\mathcal{U})$. Therefore, for some $U \in \mathcal{U}$, $f(U) \subseteq V$. It follows that $F[\{\mathcal{U}\} \cup U] \subseteq V$. Lastly, suppose that $F(\mathcal{U}) \in \kappa Y \setminus Y$ and fix $V \in F(\mathcal{U}) = f(\mathcal{U})$. Then for some $U \in \mathcal{U}$, $f[U] \subseteq V$. So, $F[\{U\} \cup U] \subseteq \{F(\mathcal{U})\} \cup V$ and F is continuous. $\qquad\square$

The pretopological space $\kappa_\pi X$ is a variation on the Katětov extension of a topological space. However, the corresponding version of Theorem 4.15 does not hold for topological spaces. What we mean to say is that it is possible to find topological spaces X and Y and a continuous function $f : X \to Y$ which does not extend to the (topological) Katětov extensions of X and Y. See 5A in [15] for an example. Thus, Theorem 4.15 is surprising in much the same way as Theorem 3.12 and shows the value of broadening our perspective to include pretopological spaces when considering problems usually thought of as topological.

In [12], it is shown that a convergence ξ has a projective maximum compactification if and only if ξ has only finitely many free ultrafilters. In contrast with this, we have the following facts:

Theorem 4.16. *If (X, π) is a Hausdorff pretopological space, (Y, ξ) is a compact Hausdorff convergence space satisfying **(R)** and $f : (X, \pi) \to (Y, \xi)$ is continuous, then there exists a continuous map $F : (\kappa_\pi X, \kappa \pi) \to (Y, \xi)$ extending f.*

Proof. We define $F : (\kappa_\pi X, \kappa \pi) \to (Y, \xi)$ as we did in the proof of Theorem 4.15. However since (Y, ξ) is compact, for each free ultrafilter $\mathcal{U}$ on X, there exists $y_{\mathcal{U}} \in Y$ such that $y_{\mathcal{U}} \in \lim_\xi f(\mathcal{U})$. Let $F(\mathcal{U}) = y_{\mathcal{U}}$. Since π is a pretopology, to show that F is continuous it is enough to show that for each $p \in \kappa_\pi X$, $F(p) \in \lim_\xi F(\mathcal{V}_{\kappa \pi}(p))$.

If $x \in X$, then $F(\mathcal{V}_{\kappa \pi}(x)) \supseteq F(\mathcal{V}_\pi(x)) = f(\mathcal{V}_\pi(x))$. Since f is continuous by assumption, we have that $F(x) \in \lim_\xi F(\mathcal{V}_{\kappa \pi}(x))$.

If $\mathcal{U} \in \kappa_\pi X \setminus X$, then $F(\mathcal{V}_{\kappa \pi}(\mathcal{U})) = \langle \{F(\mathcal{U}) \cup F[H] : H \in \mathcal{U}\} \rangle = \langle F(\mathcal{U}) \rangle \cap f(\mathcal{U})$. By construction, $F(\mathcal{U}) \in \lim_\xi f(\mathcal{U})$ and thus by (R), $F(\mathcal{U}) \in \lim_\xi F(\mathcal{V}_{\kappa \pi}(\mathcal{U}))$, as needed. $\qquad\square$

Corollary 4.17. *If (X, π) is a pretopological space, then $\kappa \pi \geq_\pi \xi$ for any Hausdorff compactification ξ of π.*

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