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TOPOLOGY
PROCEEDINGS

Volume 49, 2017
Pages 179-198

http://topology.nipissingu.ca/tp/

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Electronically published on October 21, 2016

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

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Volume 49 (2017)
Pages 179-198

E-Published on October 21, 2016

THE DETERMINATION OF CERTAIN HIGHER DERIVED FUNCTORS OF MOMENT ANGLE COMPLEXES

DAVID ALLEN AND JOSÉ LA LUZ

Abstract. Given an m-gon, P, for $m \geq 4$ and the moment angle complex Z_P, we compute the first derived functor of the indecomposable functor, $L_1Q(-)$ of the algebra $H^*(Z_P)$. As a consequence of this calculation and the work of Buchstaber, Panov and McGavaran, we compute the first derived functor of the indecomposable functor for $H^*(\sharp_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\sharp(k-2)\binom{m-2}{k-1}})$ and determine an upper bound for the rank of $L_1Q(H^*(Z_P))$ by relying on the combinatorics.

Acknowledgements

The first author is grateful to the organizers of the 29th Summer Conference on Topology and its Applications held at the College of Staten Island, CUNY. The venue provided an open atmosphere where researchers in various fields of Topology could exchange ideas. In particular, the first author would like to thank Professor Prabudh Misra for his assistance and support and Professor Abhijit Champanerkar for running/organizing the Geometric Topology session where results related to this paper were presented.

2010 *Mathematics Subject Classification.* Primary 14M25; Secondary 57N65.

Key words and phrases. Moment Angle Complexes, Complement of Coordinate Subspace Arrangement, Toric Space, Higher Derived Functors .

1. Introduction

We take as our main reference [10]–although, there is an updated Toric Topology book located in the Arxiv [11]. Moment angle manifolds have appeared in numerous places, but from an Algebraic Topological perspective we recall the definitions highlighted in the seminal works [18] and [10].

We set some notation in the context for which we are working noting that Moment angle manifolds (or Moment angle complexes more generally speaking) can be defined quite generally. In what follows k will be a field and we let K be a simplicial complex on $[m]$ that is, with vertex set $\{1, ..., m\}$. When reference is made to a simplicial complex it's meant to be one of this type. In regards to the dual complex we mean the simplicial complex dual to a polyhedron whereby each vertex is dual to a codimension one face of the polyhedron. Higher dimensional simplicies are dual to certain faces of P and the interested reader can refer to [18] for additional details. When reference to P is made then we write K_P as the dual, if required, and if the context is clear, we simply write K.

Let P be an n-dimensional simple convex polytope with facets $\{F_1, ..., F_m\}$. Assign a coordinate torus to each facet by the assignment $F_i \to S^1_i$, the circle in the i^{th} factor of $S^1_1 \times \cdots \times S^1_m$. To every face there is a coordinate subtorus consisting of the product of circle factors coming from the corresponding faces. More specifically, if F is an $(n-l)$-face, then it is a codimension l-face; hence, the intersection of l-facets. It is the product of circle factors coming from this intersection that we are interested in. The following definition of the Moment angle complex, Z_P, will suffice for our purposes [10, 11, 18]. Specifically, recall from page 85 [10] the following: $Z_P = (T^m \times P)/\sim$ where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_G$. Observe that T_G is the coordinate torus that corresponds to the face G that contains p in its interior. The equivalence relation collapses certain circle factors: those corresponding to the face G. This definition is analogous to the one used to define a *Quasitoric manifold* [18]. However, the torus in that case is linked to a certain subgroup of $\mathbb{Z}^m$ that is a direct summand. This subgroup, in the case of Quasitoric manifolds, depends on the torus action in a very explicit way as realized through the existence of a regular sequence $\lambda_1, ..., \lambda_n$ that encodes a portion of each stabilizer subgroup for the T^n-action on P, as above. Both of these manifolds (or spaces in a more general setting) are central to the study of Toric Topology and they are related in many important ways. More details can be found in the references mentioned above.

For a general simplicial complex K, there is a formulation of Z_K as a colimit [11].

$$Z_K = \bigcup_{I \in K} (\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1). \tag{1.1}$$

The orbit of the action $T^m \times Z_K \to Z_K$ is a cubical complex [11] pg 137. The curious reader can refer to page 138 to view specific decompositions of Z_K, specifically example 5.

More recently, (1.1) has been generalized in a way that unifies many Toric Topological constructions for which the Moment angle complex is a particular example. We recall some terminology from [11]. Let K be a given, fixed, simplicial complex and suppose there is a collection of spaces (X_i, A_i) such that for each $i = 1, ..., m, \quad A_i \subset X_i$. The *polyhedral product* is:

$$(X, A)^K = \bigcup_{I \in K} (\prod_{i \in I} X_i \times \prod_{i \notin I} A_i). \tag{1.2}$$

The construction for the Moment angle complex given by (1.1) and (1.2) are related by $Z_K = (D^2, S^1)^K$. For research in this direction, the reader should consult the seminal paper [7].

The cohomology ring of Z_P has been and continues to be the focus of intense research [17, 21]. First, we recall a critical definition. Let R' be a commutative ring with unit; the Stanley-Reisner ring (also referred to as the Face ring) $R'(K)$ is a polynomial algebra on indeterminants of degree two indexed by the facets of P modulo the square free ideal generated by those monomials that come from trivial intersections of facets. It was proven in [10] that $H^*(ET^m \times_{T^m} Z_K) \cong R'(K)$.

We note that for P as above, the dual simplicial complex is an $(n-1)$-dimensional simplicial sphere. In terms of decompositions, this property has useful cohomological implications in regards to the cohomology ring of the Moment angle complex; namely, the realization of the fundamental class as a cup product of lower dimensional cohomology classes with restrictions coming from the combinatorics of the orbit. The interested reader can refer to Theorem 7.18 [10] pg. 110.

The simpler structure of the Stanley-Reisner ring $R'(K)$ has been leveraged in [4, 5, 6] to make calculations of the higher derived functors of certain non-additive functors (of an algebra related to $R'(K)$) using simplicial/cosimplicial methods. One crucial element is the fact that the ideal is generated by square free monomials which are directly linked to the combinatorics of the orbit. The authors had difficulty in trying to make similar computations for $H^*(Z_P)$ for a variety of reasons, least of which was the lack of a mechanism to keep track of cycles in complicated resolutions. Difficulties were amplified by the reliance on the notion of a *projective extension sequence* [26], which is, roughly speaking, a short exact sequence in the the category of graded algebras over a commutative ring. Projective extension sequences induce long exact sequences of certain higher derived functors. These sequences can produce isomorphisms, under certain conditions and in certain cases; however, for the cases under analysis in this setting–$H^*(Z_P)$ these projective extension sequences are not particularly helpful. The hypotheses are not satisfied and it is not all obvious how to leverage the Toric Topology to rectify this. To overcome these obstacles, the authors develop a combinatorial bookkeeping mechanism that respects the simplicial/cosimplicial structure that manifests when trying to make such computations [6]. One can make specific low dimensional computations so long as the relations and relations among relations in the algebra can be enumerated and explicitly determined.

One of the main thrusts of this paper is to do exactly that for the case when P is an m-gon and the resulting moment angle complex is $\sharp_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\sharp(k-2)\binom{m-2}{k-1}})$. The information obtained from computing these higher derived functors can be fed into a certain Composite Functor Spectral Sequence [8] whose E_∞–term converges to the E_2–term of the Unstable Adams Novikov Spectral Sequence, which converges to the Homotopy groups of the space localized at a prime p. The authors have not seen any work in the literature that attempts to set-up these spectral sequences, let alone compute the required datum for complicated connect sums of sphere products as those highlighted above. It is the additional structure given by the combinatorics and Toric Topology that makes such computations feasible.

2. The Higher Derived Functors of the Indecomposable Functor

The material in this section can be found in [6]. Let R be a commutative ring with unit. We assume that all R-algebras A are graded, free R-modules endowed with a product map m and a unit $\eta : R \to A$. We also require that $\eta|_R \to A_0$ to be an isomorphism and we let $\overline{A} = Coker(R \xrightarrow{\eta} A)$.

Definition 2.1. The module of indecomposables of A is defined and denoted by

$$Q(A) = \overline{A}/\overline{A}^2.$$

This defines a non-additive functor from the category of R-algebras to the category of R-modules.

Let F be the free, commutative algebra functor with unit over R from the category of free R-modules to the category of R-algebras. The functor F comes equipped with a natural transformation $s : 1 \to F$. If M is a free R-module and B is an R-algebra with an R-module map $f : M \to B$, then there is a unique R-algebra map, $\bar{f} : F(M) \to B$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[dr, "f"'] & F(M) \arrow[d, dashed, "\bar{f}"] \\ & B \end{tikzcd} .$$

Let $M = A$ and $f = id$. Then we let $s_{-1} = s$ and $d_0 = \overline{id}$. With this we obtain an augmented simplicial object over the category of R-algebras, $\mathbf{F}^{\bullet}(A)$

$$\begin{tikzcd} \cdots F^3(A) \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & F^2(A) \arrow[l, bend right, "s_0"'] \arrow[l, bend left, "s_1"] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & F(A) \arrow[l, bend right, "s_0"'] \arrow[r, "d_0"] & A \end{tikzcd} ,$$

where $d_i = F^n((d_0)_{F^{n-i}(B)}) : F^n(B) \to F^{n-1}(B)$ for $0 \le i \le n$ and for $0 \le i \le n-1$ we have $s_i = F^n((s_{-1})_{F^{n-i}(B)}) : F^n(B) \to F^{n+1}(B)$. Applying the functor Q we obtain an un-augmented chain complex, $ch_u(Q\mathbf{F}^{\bullet}(B))$ with the boundary map given by:

$$\delta_n = \sum_{i=0}^{n} (-1)^i Q(d_i) \ . \tag{2.1}$$

When the context is clear we will simply write δ for the boundary map.

Definition 2.2.

$$L_i Q(A; R) = H_i(ch_u(Q\mathbf{F}^{\bullet}(A))) \ .$$

Referring to [6] we have

Theorem 2.3. *For any algebras A, B and C*

i) $L_0 Q(A; R) = QA$
ii) If A is a free R-algebra, then $L_i Q(A; R) = 0$ for $i > 0$
iii) $L_i Q(A \otimes_R B; R) \cong L_i Q(A; R) \oplus L_i Q(B; R)$

Remark 2.4. In [23], the authors define, for an augmented algebra B, the module of indecomposables as $Q(A) = R \otimes_R \overline{A}$ and, for an augmented coalgebra C, the module of primitives as $P(C) = R \square_C \overline{C}$. Because our algebras and coalgebras are augmented, it is easy to see that the definitions given by the authors for these modules coincide with the definition in [23]. Using proposition 3.2 (2) of [23], we see that $Q(B)^* = P(B^*)$.

From [6] we list for the convenience of the reader, notational conventions that will be used throughout. In this section we introduce a method to describe and track elements in the complex $Q\mathbf{F}^\bullet(B)$. First, we set some notation.

Notational Convention: In terms of the higher left derived functors, we write $L_nQ(B)$ for $L_nQ(B; R)$.

The elements of the free algebra generated by A, $F(A)$, can be described as $[a_1]\cdots[a_n]$ where $a_i \in A$. From this it follows that $QF(A)$, is the R-span of the set $\{[a_1 \cdots a_n] | a_i \in A\}$. The map $s_{-1} : A \to F(A)$ takes a to $[a]$ and $d_0([a_1]\cdots[a_n]) = a_1 \cdots a_n$. If D is another algebra and $g : A \to D$ is an algebra map then the induced map is: $F(g)([m]) = [g(m)]$. Given a collection of brackets:

$$\left[\cdots\left[[\boldsymbol{a}_{\iota_1}]\cdots[\boldsymbol{a}_{\iota_p}]\right]\cdots\right]$$

A *bracket of depth i* is the ith bracket from the outermost bracket. When referring to the depth of a bracket it is understood to mean a pair of brackets where the left and right sides are simultaneously at depth i. For example, a bracket of depth zero is the outermost bracket: that is, the outermost pair of brackets. A bracket of depth one is the bracket that is one place over from the bracket of depth zero. Put another way, it is the second outermost bracket. This mechanism allows for an explicit formulation of the face maps. The map $d_i : QF^n(A) \to QF^{n-1}(A)$ removes the bracket in depth i. The map $s_i : QF^n(A) \to QF^{n+1}(A)$ will add a bracket in depth i.

It is useful to observe that elements in $QF^n(A)$ will have n brackets. For illustrative purposes we write out a few elements then we compute a few of the face maps in low degrees.

Example 2.5. Consider the following element in $QF^3(A)$.

$$\left[\left[\underbrace{[}_{\textit{\textbf{depth 2}}}\ \boldsymbol{a}_1][\boldsymbol{a}_2]\right]\left[[\boldsymbol{a}_3]\right]\right]$$

$$d_0\left(\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right) = \left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]$$

Observe that the $im\ d_0 \in QF^2(A)$ and as such it is equal to zero.

$$d_1\left(\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right) = \left[[\boldsymbol{a_1}][\boldsymbol{a_2}][\boldsymbol{a_3}]\right]$$

$$d_2\left(\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right) = \left[[\boldsymbol{a_1}\boldsymbol{a_2}][\boldsymbol{a_3}]\right]$$

$$s_0\left(\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right) = \left[\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right]$$

$$s_1\left(\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right) = \left[\left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]\right].$$

Notational Convention: We define $[x]^{(k)}$ inductively. Let $[x]^{(1)} = [x]$ and $[x]^{(k+1)} = \left[[\boldsymbol{x}]^{(\boldsymbol{k})}\right]$.

Definition 2.6. Let $x = [a_1 \cdots a_n] \in QF(A)$. The core of x, $c(x)$, is $\{a_1, \cdots, a_n\}$ and $|x| = |c(x)| = \sum_i |a_i|$. For $x_n \in QF^n(A)$, let $x_n = [x_{\iota_1} \cdots x_{\iota_j}]$ where $x_{\iota_k} \in F^{n-1}(A)$. The core of x_n, $c(x_n) := \cup_{i=1}^{j} c(x_{\iota_i})$ and the degree is the sum of the degrees of the elements contained in the union.

Example 2.7. Let $x_n \in QF^n(A)$ and suppose it is of the form:

$$x_n = \left[\cdots\left[[\boldsymbol{x_{\iota_1}}]\cdots[\boldsymbol{x_{\iota_p}}]\right]\cdots\right]$$

$c(x_n) = \{x_{\iota_1}, ..., x_{\iota_p}\}$ and its degree is $|x_{\iota_1}| + \cdots + |x_{\iota_p}|$. Simply put, the core is nothing more than the set of elements contained within the innermost bracket.

We give a more explicit example below:

Example 2.8. Given the element in $QF^3(A)$

$$x_3 = \left[\left[[\boldsymbol{a_1}][\boldsymbol{a_2}]\right]\left[[\boldsymbol{a_3}]\right]\right]$$

its core is: $\{a_1, a_2, a_3\}$ and $|x| = |a_1| + |a_2| + |a_3|$.

Remark 2.9. As the filtration increases the degree of the core remains constant. For computational purposes, it is useful to recall that the d_i are degree preserving maps.

3. Spectral Sequence Considerations

For the convenience of the reader we list, in this section, material from [2]. The unstable spectral sequences cited below will provide some additional motivation as to how the calculations in this paper fit into the broader scheme of Homotopy Theory. We begin by describing the construction of the E_2-term of the Bousfield-Kan spectral sequence. See [15] for details.

Let $E = \{\underline{\mathrm{E}}_k\}$ be a multiplicative ring spectrum such that $E_*(\underline{\mathrm{E}}_k)$ is a free E_*-module (we are mainly interested in BP) and let $\mathcal{M}_{E_*}$ be the category of free (positively graded) E_*-modules. Then there is a functor $G = G_E : \mathcal{M}_{E_*} \to \mathcal{M}_{E_*}$. In addition there are natural transformations $\epsilon : G \to I$ and $\delta : G \to G^2$ that makes (G, δ, ϵ) a cotriple over $\mathcal{M}_{E_*}$ (see §4 from [9]). Given the cotriple (G, δ, ϵ) we define the a G-coalgebra as an object M in $\mathcal{M}_{E_*}$ with a map $M \to G(M)$ that commute with the G-structure in the obvious way. The category $\mathcal{G}$ is referred to as the *category of unstable G-coalgebras* [9]. Under certain conditions on E and a space X, [15] gives an isomorphism between $E_2^{s,t}(X)$ and $Ext^s_{\mathcal{G}}(E_*(S^t), E_*(X))$ for the Bousfield-Kan spectral sequence.

Generally, the category is not suitable for performing calculations. For instance, it is generally very challenging to compute the coaction (G-Coalgebra structure map: $M \to G(M)$ where M is the E–homology of X). To simplify calculations, we introduce a new cotriple.

Definition 3.1. For $M \in \mathcal{G}$ we define

$$U(M) = (P \circ G)(M)$$

where P is the primitive element functor.

Of course, the *indecomposable functor*, Q, is related to P and exactly how that is the case is elucidated in [5] and in the previous section. Let $\mathcal{A}$ be the category E_*-modules, M, with a map $M \to U(M)$. This is called the category of *unstable* $E_*(E)$-comodules.

Theorem 3.2. *Let $BP_*(X)$ be a free BP_*-module then there is a spectral sequence*

$$Ext^r_{\mathcal{A}}(BP_*(S^t), R^s P(BP_*(X))) \Rightarrow Ext^{r+s}_{\mathcal{G}}(BP_*(S^t), BP_*(X)) \ .$$

In the case that $BP_*(X)$ is nice i.e., $R^i(BP_*(X)) = 0$ for $i > 1$, then the spectral sequence on the left–the Composite Functor Spectral Sequence, collapses to two rows inducing a long exact sequence:

$$\cdots \to Ext^s_{\mathcal{A}}(BP_*(S^t), PBP_*(X)) \to Ext^s_{\mathcal{G}}(BP_*(S^t), BP_*(X)) \to$$
$$\to Ext^{s-1}_{\mathcal{A}}(BP_*(S^t), R^1P(M)) \to Ext^{s+1}_{\mathcal{A}}(BP_*(S^t), PBP_*(X)) \to \cdots$$

and in the case that $BP_*(X)$ is cofree i.e., $R^i(BP_*(X)) = 0$ for $i > 0$, then the long exact sequence reduces to an isomprphism

$$Ext^s_{\mathcal{A}}(BP_*(S^t), PBP_*(X)) \cong Ext^s_{\mathcal{G}}(BP_*(S^t), BP_*(X)).$$

Examples of nice coalgebras would include $BP_*[v_1, ..., v_m]/\langle v_{\iota_1} \cdots v_{\iota_s}\rangle$ [3, 12] where the degree of each v_i is two. The Toda sphere (e.g., $\widehat{S}^{2n}$, adhering to the notation used in [9]) is a space such that its BP homology is a nice coalgebra [9].

We wish to elaborate upon Theorem 3.2. The E_2-term on the left is the one that we seek to compute as a first step toward making homotopy calculations. In this paper, we have computed, under mild assumptions, the zeroth and first Right Higher Derived Functor for $H^*(Z_K)$ when K is a polygon for the functor Q. That is, $L_0QH^*(Z_K)$ and $L_1QH^*(Z_K)$. In fact, these calculations are over rings more general than BP_*, but for the purposes of making possible spectral sequence calculations we can reduce to this case when necessary. Ordinarily, the next step in computing the E_2-term (on the left) is to compute the coaction formulae, that is, to compute the maps $M \to U(M)$ in the category $\mathcal{A}$. In our setting, this takes the following form, as we just follow the arguments highlighted in [2], $BP_*(Z_K) \to U(R^iP(BP_*(Z_K)))$ using the fact that $L_iQBP^*(Z_K)$ and $R^iPBP_*(Z_K)$ are intimately related [5]. However, the Toric Topology provides additional structure that provides some insight. Recall, the Borel space: $B_TP = ET^m \times_{T^m} Z_K$ and the fibration $Z_K \to ET^m \times_{T^m} Z_K \to BT^m$ [10]. From this and basic properties of $\mathbb{C}P^\infty$ we obtain, for $* > 2$, the isomorphism: $\pi_*(B_TP) \cong \pi_*(Z_K)$. This isomorphism implies that it is sufficient to study the homotopy groups of B_TP to obtain information on the homotopy of Z_P and this was done in [2].

It was shown that classes in the category $\mathcal{A}$ are of the form $*_{j\in J}\overline{\beta_{1,j}}$ where the $\overline{\beta}$ are the duals of products of v (generators of the E–Face ring, where E is a complex orientable theory) and the subscripts are just a bookkeeping mechanism to keep track of certain classes of interest. It is worth mentioning that the star products come from those products in the E–Face ring that come from the missing faces. To elucidate, we recall that trivial intersections of facets in the underlying polytope give rise to relations in the Face ring. One key argument in that paper was the identification of the map (the reader will recognize this as the coaction map): $R^1PBP_*(B_TP) \to U(R^1PBP_*(B_TP))$ with a map in a certain resolution that comes from the functor G described above. This was done by constructing a certain commutative diagram [2] and deducing that the previous map can be interpreted as d: $G(F) \to U(G(F))$ where F is the dual of the BP cohomology of the Face ring. This "F" is not to be confused with the notation used earlier. For the sake of keeping notation constant we use this notation as first highlighted in [2] so that the reader can refer to that paper. The image of this map (the coaction), is: $d(*_{j\in J}\overline{\beta_{1,j}}) = 1 \otimes (*_{j\in J}\overline{\beta_{1,j}})$. This says that the coaction is trivial and from this key calculation it is shown in [2] that the homtopy of the Borel space (and by extension the Moment angle complex, in sufficiently large degrees) is given by the homotopy groups of a product of spheres indexed by the cardinality of the generating set of the ideal I in the E–Face ring. The dimensions of these spheres depend on the degrees of the generators of I [2] (cf. Theorem 6.35). In referring to that paper, one limitation to the calculations was the term $\Re_{min}$–a relation among relations of minimal degree coming from the relations in the ideal I. This term was the obstruction to the determination of the right higher derived functors of the primitive element functor without the range restriction as described in [2].

This interpretation makes sense since the generators of $R^1PBP_*(B_TP)$ (cycles) are certain classes in the coalgebra $G(F)$ (cf. [2] pg. 461). To expand on this a bit, consider the example of the square $P = \Delta^1 \times \Delta^1$. Here, the Face ring has two relations coming from the combinatorics, $v_1v_3 = 0$ and $v_2v_4 = 0$. In the coalgebra $G(BP_*(B_TP))$ these relations have the names $\overline{\beta_{1,1}} * \overline{\beta_{1,3}}$ and $\overline{\beta_{1,2}} * \overline{\beta_{1,4}}$. This notation is explained very precisely in [2]. Furthermore, these star products are generators of the First Right Higher Derived Functor of the Primitive Element Functor of the coalgebra $BP_*(B_TP)$ (i.e., $R^1PBP_*(B_TP)$). This is consistent with the results in this paper which show that for the Moment Angle complex, the First Left Higher Derived Functor of the Indecomposable Functor comes from the relations in the cohomology of Z_K.

One obstruction to more complete derived functor calculations in [2] was the reliance on using injective extension sequences and $\Re_{min}$. This sequence induces a long exact sequence of these higher derived functors and in certain cases, specific computations can be made [12]. However, in this paper no such restriction is present since we make computations by brute force calculations in the corresponding resolution. The same methods can be applied to the E–cohomology of B_TP to extend the higher derived functor computation and related calculations of [2]. In this case, the range restriction would be lifted. In fact, such calculations are made in [6] and they are used to classify torus actions.

Computing differentials and resolving potential extension problems in a spectral sequence can be delicate and difficult. For the spectral sequences listed in Theorem 3.2 it is even worse since the category $\mathcal{G}$ is mysterious and $\mathcal{A}$ is not trivial to work with either. Given these substantial obstacles we can use Theorem 5.1 to deduce a few facts about the classes (that appear in that Theorem). Since the homotopy groups of the Borel space $B_T P$ give the homotopy of Z_K we can focus our attention to the arguments above for $B_T P$. That said, the relations that come from the differences can not survive to the homotopy since the homotopy classes are generated by products that show up in the Face ring. This implies that these differences are either killed by a differential in the Composite Functor Spectral Sequence (on the left side in Theorem 3.2) or they survive this spectral sequence and are in the image of a differential in the Adams Spectral Sequence (on the right side in Theorem 3.2).

The products of classes (we are moving back and forth between the functors P and Q by relying on the results of [5]) in the cohomology of Z_K that give zero do not survive to the homotopy either simply by degree arguments. They too must be in the image of a differential in either spectral sequence or they are killed in an extension. The authors do not know which is the case and this would be one of the next steps in the programme. The point here is that we can use the homotopy of the Borel space to reverse engineer, at least, in terms of degrees, the possible location of classes and differentials in the E_2–term of the E–homology of Z_K. It is also a non-trivial matter to compute the E–homology of Z_K since the Atiyah-Hirzebruch Spectral Sequence is more complicated due to the existence of odd-dimensional classes, if one were to try a direct calculation. Assuming this line of attack were taken, the next step in the program would be to compute the unstable coaction formulae for the coalgebra $H_*(Z_K; E)$ by augmenting the arguments in [2] to those classes that are not just the duals to the products v^J. To do so, would require understanding how the Koszul resolution as described in [10] fits into the homotopy theoretic scheme as outlined in [8] and applied in [2].

4. The Cohomology of Z_K

We begin by briefly reviewing some constructions that will be useful for our purposes. For the convenience of the reader, we now recall the construction of the Koszul resolution often used in Toric Topology [10] pg. 41 and we follow their exposition. Fix a simplicial complex K on $[m]$ and let k be a field thought of as a $k[v_1, \ldots, v_m]$-module via the map that sends $v_i \mapsto 0$. Let R be the bi-graded complex

$$\Lambda[u_1, \ldots, u_m] \otimes k[v_1, \ldots, v_m] \ ,$$

where Λ refers to the exterior algebra on the u_i. The bi-degrees are as follows: $bideg(u_i) = (-1, 2), bideg(v_i) = (0, 2)$ with $d(u_i) = v_i$ and $d(v_i) = 0$, requiring that d be a derivation. The following two theorems–often-quoted–relate the Koszul resolution to the Moment angle complex and are at the heart of many calculations concerning the cohomology of Z_K. The proofs can be found in [10] pg 103 Theorems 7.6 and 7.7; we list both of them below.

Theorem 4.1. *There is an isomorphism as bi-graded algebras:*

$$H^{*,*}(Z_K) \cong H[\Lambda[u_1, \ldots, u_m] \otimes k(K), d]$$

and

Theorem 4.2. *There is an isomorphism as algebras:*

$$H^*(Z_K) \cong Tor_{k[v_1,\ldots,v_m]}(k(K), k) \ .$$

In [11], there is an algebra, $R^*(K)$ (see page 150) that is quite useful when trying to explicitly write down classes in the cochain complex. We now recall some key facts and features.

$$R^*(K) = (\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}(K))/\langle v_i^2 = u_i v_i = 0,\ 1 \leq i \leq m\rangle \ .$$

We observe that the differential is given by: $d(u_i) = v_i$ and $d(v_i) = 0$. Buchstaber and Panov explain that there is a basis for $R^*(K)$ as an abelian group and it is given by $v_I u_J$ where $I \cap J = \emptyset$, $I \in K$ and $J \subset [m]$. Summarizing Lemma 4.5.1 [11].

Lemma 4.3. *There is an additive isomorphism $H[R^*(K)] \cong H^*(Z_K)$.*

For the convenience of the reader we recall Proposition 7.23 (pg 112) of [10]; the proof can be found there. We let P be the m–gon for $m \geq 4$ and write Z instead of Z_P. Finally, we remark that given P, there is a dual simplicial complex $K_P = K$. So, in the previous section when Z_K was written instead of Z_P (to avoid confusion with the primitive element functor P, the K is meant to be the one dual to P).

Remark 4.4. If $I = \{i_1, i_2, ..., i_k\}$ and $\tau = \{j_1, j_2, ..., j_s\}$, then $v_I = v_{i_1} v_{i_2} \cdots v_{i_k}$ and $u_\tau = u_{j_1} u_{j_2} \cdots u_{j_s}$ [10]. In the cohomology of Z, such classes are denoted with brackets as described in [10]–Proposition 4.5 below. However, for reasons of convenience this is slightly changed (see below). Moreover, in lieu of Proposition 4.5, $|I| = 1, 2$ and in such cases we simply write $v_{i_1} v_{i_2}$ instead of v_I. Furthermore, v_I and u_τ in the Koszul resolution do not survive to the cohomology of Z.

Proposition 4.5. *The only non-zero bi-graded cohomology groups of Z are:*

- $H^0(Z) = H^{0,0}(Z)$
- $H^{p+2}(Z) = H^{-p,2(p+1)}(Z)$ *generated by* $[v_i u_\tau]$ *subject to:* $|\tau| = p$ *and* $i \notin \tau$ *and* $i \pm 1 \notin \tau$ *for* $p \in \{3, \ldots, m-3\}$
- $H^{m+2}(Z) = H^{-m+2,2m}(Z)$ *generated by* $[v_1 v_2 u_3 \cdots u_m]$.
- *The relations are given by* $[v_{i_1} u_{\tau_1}][v_{i_2} u_{\tau_2}] = 0$ *if* $\{i_1, i_2\} \cup \tau_1 \cup \tau_2 \neq [m]$.

5. Main Results

We assume that we are working over a field k of nonzero characteristic. Let P be an m-gon for $m \geq 4$; we fix P and let K denote the dual of P with vertex set $[m]$. Let $\tau \subset [m]$ and $i \in K$. By $|\tau|$ we mean the cardinality of the subset. When we write $i < \tau$ we mean there is a $j \in \tau$ such that $i < j$. Given P as above let $Z_P = Z$ (or when suitable, Z_K when referrals are made to the references). Recall that Z is an $m+2$ dimensional manifold whose top dimensional class is generated by the fundamental class.

Notational Convention: Observe that the classes in Proposition 4.5 are bracketed. For the sake of clarity we will write such classes using the notation $\overline{v_i u_\tau}$. Below, the brackets are used to keep track of the filtration level in the simplicial resolution.

If $\overline{v_i u_\tau}$ is a non-zero element in $H^*(Z)$, then let $\Delta(\tau) = min\{|j_1 - j_2|| j_1, j_2 \in \tau\}$.

When reference is made to the chain complex coming from the simplicial resolution of $H^*(Z)$ (refer to §2 for additional details) we will simply write $QF^\bullet$. For reasons of clarity, we will, when required, refer to specific modules in that chain complex by referencing them as QF^i for a particular i and this i is not to be confused with the i in the subscripts in the terms for classes in the cohomology of Z; the context should make this clear. Furthermore, reference to δ means (2.1).

For the sake of brevity in the statement and proof of the main theorem we consider the following conditions on the indexing sets for the classes $\overline{v_{i_q} u_{\tau_q}}$ where $q = 1, 2$. We have the following cases:

(A) (a) $\{i_1, i_2\} \cup \tau_1 \cup \tau_2 \neq [m]$

(b) if $|\tau_1||\tau_2| \equiv_2 1$ then $\tau_1 \neq \tau_2$

(c) if $\Delta(\tau_q) > 1$ then $i_q < \tau_q$ for $q = 1, 2$

(B) (a) $\{i_1, i_2\} \cup \tau_1 \cup \tau_2 = [m]$

(b) if $|\tau_{i_q}||\tau_{i_{q+1}}| \equiv_2 1$ then $i_j \neq i_{j+1}$ or $\tau_q \neq \tau_{q+1}$ for $q = 1, 2, 3$

(c) if $\Delta(\tau_q) > 1$ then $i_q < \tau_q$ for $q = 1, \ldots, 2$.

When we say that indexing sets satisfy cases (A) or (B) it is meant to mean the above. See Remark 5.4 for additional commentary concerning (c).

Theorem 5.1. *Let P be the m-gon and Z the corresponding moment angle complex, then*

$$L_1Q(H^*(Z)) \cong \begin{cases} k\left\{ \left[[\overline{\boldsymbol{v_{i_1} u_{\tau_1}}}][\overline{\boldsymbol{v_{i_2} u_{\tau_2}}}] \right] \,\middle|\, (A) \textit{ holds} \right\} \\ \\ k\left\{ \left[[\overline{\boldsymbol{v_{i_1} u_{\tau_1}}}][\overline{\boldsymbol{v_{i_2} u_{\tau_2}}}] \right] - \left[[\overline{\boldsymbol{v_{i_3} u_{\tau_3}}}][\overline{\boldsymbol{v_{i_4} u_{\tau_4}}}] \right] \,\middle|\, (B) \textit{ holds} \right\} \end{cases}$$

Remark 5.2. We note that the classes in Theorem 5.1 are a set of generators, not a basis for $L_1Q(H^*(Z))$.

Remark 5.3. Given Remark 4.4 and the comments above, we observe that v_I and u_τ in the Koszul resolution do not survive to the cohomology of Z; hence, we do not use the bar notation (i.e., $\overline{v_I}$ and $\overline{u_\tau}$). Recall, we reserve the bar notation for those classes in the cohomology of Z. In the language of [6], classes such as

$$\left[[\boldsymbol{v_I}][\boldsymbol{u_\tau}] \right]$$

do not show up in QF^2. Hence, they can not populate bracket(s).

Proof. We observe that for conditions (A) and (B), (b) ensures that we do not get trivial products in QF^2. We note that item (a) in regards to (A) produces the relations since such indexing sets produce trivial products by [10]. In trying to determine what elements are cycles in the complex $QF^\bullet$ we consider products of the following form where the indexing sets satisfy (A):

$$\left[[\overline{v_{i_1}u_{\tau_1}}][\overline{v_{i_2}u_{\tau_2}}]\right] \tag{5.1}$$

More generally, if we assume that there is a product of the form:

$$\left[[\overline{v_{i_1}u_{\tau_1}}]\cdots[\overline{v_{i_n}u_{\tau_n}}]\right]$$

where $n > 2$, then this lies in the image of:

$$-\left[\left[[\overline{v_{i_1}u_{\tau_1}}]\cdots[\overline{v_{i_{n-1}}u_{\tau_{n-1}}}]\right][\overline{v_{i_n}u_{\tau_n}}]^{(2)}\right]$$

Suppose that we have the following type of product:

$$\left[[\overline{v_{i_1}v_{i_2}u_{\tau_1}}][\overline{v_{i_2}u_{\tau_2}}]\right]$$

where $\{i_1, i_2\} \cup \tau_1 = [m]$, then we can write, the class: $\overline{v_{i_1}v_{i_2}u_{\tau_1}}$ as a product $\overline{v_{i_1}u_{\tau_1'}} \cdot \overline{v_{i_2}u_{\tau_2'}}$ so long as $\tau_1^{'} \cup \tau_2^{'} = \tau_1$ and $i_k \notin \tau_k'$. Then the element is homologous to

$$\left[[\overline{v_{i_1}u_{\tau_1'}}][\overline{v_{i_2}u_{\tau_2'}}][\overline{v_{i_2}u_{\tau_2}}]\right] \tag{5.2}$$

via the element

$$\left[\left[[\overline{v_{i_1}u_{\tau_1'}}][\overline{v_{i_2}u_{\tau_2'}}]\right][\overline{v_{i_2}u_{\tau_2}}]^{(2)}\right].$$

We observe that (5.2) is a cycle. We now assert that (5.1) is not in the image. The only possible element in QF^3 that can hit this element in this case is:

$$\left[[\overline{v_{i_1}u_{\tau_1}}]^{(2)}[\overline{v_{i_2}u_{\tau_2}}]^{(2)}\right]. \tag{5.3}$$

We observe that δ on the element (5.3) is zero and we note that populating the innermost brackets in any other way will result in the class mapping to zero too, assuming such a class exists.

We now address those classes of the type satisfying (B):

$$\left[[\overline{\boldsymbol{v_{i_1}u_{\tau_1}}}][\overline{\boldsymbol{v_{i_2}u_{\tau_2}}}]\right] - \left[[\overline{\boldsymbol{v_{i_3}u_{\tau_3}}}][\overline{\boldsymbol{v_{i_4}u_{\tau_4}}}]\right]. \tag{5.4}$$

We remark that these elements come from the fundamental class as can be seen by taking the products of the elements populating the innermost brackets. Observe that δ on (5.4) is zero for the reasons just given, so it suffices to show that this class is not in the image. Recall, the boundary maps in $QF^\bullet$ are degree preserving module maps; therefore, referring to the previous arguments, the only possible class that could hit (5.4) is:

$$\left[[\overline{v_{i_1}u_{\tau_1}}]^{(2)}[\overline{v_{i_2}u_{\tau_2}}]^{(2)}\right] - \left[[\overline{v_{i_3}u_{\tau_3}}]^{(2)}[\overline{v_{i_4}u_{\tau_4}}]^{(2)}\right]. \tag{5.5}$$

A simple verification shows that this is not the case. □

The additional structure given by the Toric Topology makes this calculation plausible; without it, there would be many more cases to check in the chain complex. We list a few examples to illustrate.

Remark 5.4. Condition (c) arises because if we have $\tau = \{j_1, \ldots, j_r\}$ and $j_0, j_1 \pm 1 \in \tau$, such that $\Delta(\tau) > 1$, then $d(u_{\{j_0\}\cup\tau})$ will give a linearly dependent relation between $\overline{v_{j_0}u_{j_1}\cdots u_{j_r}}, \overline{v_{j_1}u_{j_0}\ldots u_{j_r}}, \ldots, \overline{v_{j_r}u_{j_0}\ldots u_{j_{r-1}}}$. This observation will be used in Proposition 6.2.

Example 5.5. For $P = \Delta^1 \times \Delta^1$, the Moment angle complex is $S^3 \times S^3$ [10, 11]. By [9] and [6], $L_1Q(H^*(Z)) = 0$ (since the dual of $H^*(S^3)$ is cofree as a coalgebra). We verify this result in this setting. The class $\overline{v_1u_3}$ is a generator of $H^3(Z)$ and its square is zero. Consider the class in $QF^2(H^*(Z))$:

$$\left[[\overline{\boldsymbol{v_1u_3}}][\overline{\boldsymbol{v_1u_3}}]\right].$$

This class does not survive to $L_1QH^*(Z)$ since d_1 (recall, $\delta = d_0 - d_1$) on the class gives

$$[\overline{\boldsymbol{v_1^2u_3^2}}].$$

But, the term v_1^2 is the image of v_1u_1 in the Koszul. Hence, this class does not survive to $H^*(Z)$. A similar argument holds for $\overline{v_2u_4}$– the class coming from the other factor of S^3; hence we conclude that $L_1QH^*(Z) = 0$.

We remark that powers of brackets of relations do not survive to $L_1Q(H^*(Z))$ for the reasons mentioned above. We describe a slightly more complicated example concerning the pentagon.

Example 5.6. There is a relation, $\overline{(v_1u_3u_4)}\ \overline{(v_3u_5u_1)} = 0$ where each term in the product is a generator of $H^4(Z)$. Following the same procedure as in the previous example, the class

$$\left[[\overline{\boldsymbol{v_1 u_3 u_4}}][\overline{\boldsymbol{v_3 u_5 u_1}}]\right]$$

survives to the first higher derived functor. If the following class existed in $QF^\bullet$ then it would hit the above. As described in the remark above, it does not (this is why we surround it with quotation marks. We refrain from the "bar" notation since the class does not survive to the cohomology of Z. Such notation is reserved for those classes that are products of v's and u's in the cohomology of Z.)

$$\text{“}\left[\left[[\boldsymbol{v_1}][\boldsymbol{u_3 u_4}]\right]\left[[\boldsymbol{v_3}][\boldsymbol{u_5 u_1}]\right]\right] - \left[\left[[\boldsymbol{v_1}][\boldsymbol{v_3}]\right]\left[[\boldsymbol{u_3 u_4}][\boldsymbol{u_5 u_1}]\right]\right]\text{”}$$

From Theorem 4.6.12 [11] pg 161, the Moment angle complex is known to be homeomorphic to $\sharp_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\sharp(k-2)\binom{m-2}{k-1}}$. We have the first derived functor for connect sums for certain sphere products.

Corollary 5.7. $L_1Q(H^*(\sharp_{k=3}^{m-1}(S^k \times S^{m+2-k})^{\sharp(k-2)\binom{m-2}{k-1}})) \cong$

$$\begin{cases} k\left\{ \left[[\overline{\boldsymbol{v_{i_1} u_{\tau_1}}}][\overline{\boldsymbol{v_{i_2} u_{\tau_2}}}]\right] \middle| \text{(A) holds} \right\} \\ k\left\{ \left[[\overline{\boldsymbol{v_{i_1} u_{\tau_1}}}][\overline{\boldsymbol{v_{i_2} u_{\tau_2}}}]\right] - \left[[\overline{\boldsymbol{v_{i_3} u_{\tau_3}}}][\overline{\boldsymbol{v_{i_4} u_{\tau_4}}}]\right] \middle| \text{(B) holds} \right\}. \end{cases}$$

6. An Upper Bound to the Rank of the First Derived Functor

In this section we compute a gross upper bound on $L_1Q(H^*(Z))$ for Z as in §4. We rely heavily upon the description of the cohomology classes given in Proposition 4.5. The counting formulae are determined with replacement at each step in the summation.

Proposition 6.1. *An upper bound for the elements of type* $[[\overline{v_{i_1} u_{\tau_1}}][\overline{v_{i_2} u_{\tau_2}}]]$ *in the first derived functor is given by*

$$m^2(2^{m-3}-1)^2 - m \sum_{i=0}^{\lfloor \frac{m-4}{2} \rfloor} \binom{m-3}{2i+1}.$$

Proof. We begin by counting the number of possible products in $H^*(Z)$ that do not form a partition of $[m]$. For a fixed v_i there are $\binom{m-3}{j}$ possible elements $u_\tau \in \Lambda(u_1, \ldots, u_m)$ that can be multiplied by v_i for $1 \leq j \leq m-3$. Summing this we have

$$\binom{m-3}{1}+\cdots+\binom{m-3}{m-3} = \binom{m-3}{0}+\binom{m-3}{1}+\cdots+\binom{m-3}{m-3}-1\,.$$
$$= 2^{m-3}-1$$

Since there are m possible ways of choosing v_i we obtain an upper bound for the bracketed elements in $H^*(Z)$ and it is $m(2^{m-3}-1)$. Since the elements of the form $[[\overline{v_{i_1}u_{\tau_1}}][\overline{v_{i_2}u_{\tau_2}}]]$ are the product of these elements (except when one of the elements inside one of the inner brackets is the biggest possible product) we have $m^2(2^{m-3}-1)^2$ minus the square of odd elements–elements where the indexing set has odd cardinality. Using a similar argument, the square of odd elements can be enumerated and we obtain:

$$m\left[\binom{m-3}{1}+\cdots+\binom{m-3}{2\lfloor\frac{m-4}{2}\rfloor+1}\right] = m\sum_{i=0}^{\lfloor\frac{m-4}{2}\rfloor}\binom{m-3}{2i+1}$$

since there are $\lfloor\frac{m-4}{2}\rfloor$ odd numbers between 0 and $m-3$ □

Proposition 6.2. *An upper bound for the elements of type* $[[\overline{v_{i_1}u_{\tau_1}}][\overline{v_{i_2}u_{\tau_2}}]]-[[\overline{v_{i_3}u_{\tau_3}}][\overline{v_{i_4}u_{\tau_4}}]]$ *in the first derived functor is given by*

$$\binom{m}{2}\sum_{i=1}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-4}{i}-1\,.$$

Proof. We first determine the number of possible elements of the form $[[\overline{v_{i_1}u_{\tau_1}}][\overline{v_{i_2}u_{\tau_2}}]]$ where $\{i_1,i_2\}\cup\tau_1\cup\tau_2=[m]$. Since we have one v within each inner bracket then there are $\binom{m}{2}$ possible pairs of v's. We now have to count the number of possible u's. Since the u's in the first bracket will determine the ones in the second, we focus on one of the brackets. There are $m-4$ possible u's to fill in the bracket (there has to be at least one u in the second bracket). Therefore, there are $\binom{m-4}{i}$ possibilities with i choices for u where $0<i\le\lfloor\frac{m-2}{2}\rfloor$. This gives

$$\binom{m}{2}\sum_{i=1}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-4}{i}$$

Let $x_1,\ldots,x_2,\ldots,x_t$ be the elements of the required form, then by Remark 5.4, $x_1-x_2, x_1-x_3,\ldots,x_1-x_t$ is a linearly independent set that generates all possible differences; there are $\binom{m}{2}\sum_{i=1}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-4}{i}-1$ such elements. □

We do not assert that the upper bound(s) derived in Propositions 6.1 and 6.2 are best possible. In fact, research continues to find the best possible upper bound.

Remark 6.3. Finally, we note that the calculations in this paper are related to the work of [1]. These explicit structural theorems and upper bound statements have a cotangent complex interpretation.

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