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Jeffrey T. Denniston, Austin Melton, and Stephen E. Rodabaugh

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# ASYMMETRY IN MANY-VALUED TOPOLOGY: SPECTRA OF QUANTALES AND SEMIQUANTALES

#### JEFFREY T. DENNISTON, AUSTIN MELTON, AND STEPHEN E. RODABAUGH\*

ABSTRACT. The symmetry vis-a-vis asymmetry issue naturally occurs in two settings of traditional topology: the symmetry axiom of a metric space, and the symmetry conditions satisfied by specialization orders associated with certain topological spaces. In each case, the issue of symmetry relates to separation: each metric space is  $T_2$  and symmetry is explicitly invoked for the disjointness of separating neighborhoods; and for  $T_0$  topological spaces, specialization order symmetry is equivalent to  $T_1$ . This paper studies asymmetry for many-valued or L-topological spaces—with various conditions imposed on L—via two "standard" specialization orders (and their duals) associated with such spaces; and special emphasis is placed on spaces which are  $L\mbox{-spectra}$  of semiquantales, sometimes with additional restrictions on L or the represented semiquantales; these additional restrictions may involve left- and/or right-residuations as well as with special involutive, isotone, anti-automorphisms. For consistent L: the L-spectrum of a semiquantale with at least two related distinct primes is asymmetric; the left and right topologies of  ${\mathbb R}$  and the left and right L-topologies of the "fuzzy" real line  $\mathbb{R}(L)$  are all asymmetric; and the L-spectrum of each of these (L-)topologies is asymmetric. On the other hand, for each (complete) DeMorgan algebra  $L, \mathbb{R}(L)$  with the canonical topology  $\tau(L)$ is symmetric w.r.t. the first specialization order; and the "alternative" fuzzy real line  $\mathbb{R}^{*}(L)$ —the L-spectrum of the standard topology on  $\mathbb{R}$ —is symmetric in the same sense if L is a complete Boolean algebra. Under appropriate conditions, the well-known  $G_{\chi}$ ,  $M_L$ ,  $\omega_L$ ,  $\iota_L$  functors produce and/or reflect both asymmetric and symmetric L-topological spaces.

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<sup>\*</sup>corresponding author.

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#### 1. MOTIVATION AND PRELIMINARIES

One way in which the issue of asymmetry arises in traditional topology is in connection with the study of quasimetric spaces. To make this connection clear, recall that a *metric space* (X, d) is a set X equipped with a mapping  $d: X \times X \to \mathbb{R}$  satisfying these properties:

- (M1)  $\forall x, y \in X, d(x, y) \ge 0$  (positivity);
- (M2)  $\forall x, y \in X$ ,  $[d(x, y) > 0 \text{ or } d(y, x) > 0] \Leftrightarrow x \neq y$  (weak positive definiteness);
- (M3)  $\forall x, y \in X, d(x, y) = d(y, x)$  (symmetry);
- (M4)  $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$  (triangle-inequality).

There are two axioms M2a, M2b related to M2 which are useful to this discussion:

(M2a)  $\forall x, y \in X, d(x, y) = 0$  if x = y (zero-distance on diagonal);

(M2b)  $\forall x, y \in X, d(x, y) > 0 \Leftrightarrow x \neq y$  (strong positive definiteness).

It is easy to see that M2b is the rewriting of M2 with "or" replaced by "and", and hence M2b implies M2; the converse does not generally hold, but M2 and M2b are equivalent in the presence of M3, a preliminary indication that these forms of positive definiteness are linked to symmetry—cf. Propositions 1.1(1,2) and 1.2(3) below. Also, in the presence of M1, each of M2 and M2b implies M2a; neither of these implications reverse, even with the added presence of M3. Finally, assuming that d is bounded by 1, M2a is equivalent to the similarity relation s(x, y) = 1 - d(x, y) being 1 on the diagonal, namely that s interpreted as an [0, 1]-relation is reflexive [9].

The pair (X, d) is a *pseudometric space* if M1, M2a, M3, M4 are satisfied, a *quasimetric space* if M1, M2b, M4 are satisfied, and a *hemimetric space* if M1, M2a, M4 are satisfied. If (X, d) is a hemimetric space, then there is an associated topology  $\mathfrak{T}_d$  constructed from d analogously to how it is done in the metric case via the notion of  $\varepsilon$ -balls—

$$B_{\varepsilon}(x) := \left\{ y \in X : d(x, y) < \varepsilon \right\}.$$

The assumptions of M2a and M4 are precisely what is needed to assure that  $\{B_{\varepsilon}(x) : x \in X, \varepsilon > 0\}$  is a basis for  $\mathfrak{T}_d$ .

The following proposition is useful in understanding the discussion of asymmetry in traditional topology from the standpoint of distance functions:

**Proposition 1.1.** Let (X, d) be a hemimetric space. The following hold:

- (1)  $(X, \mathfrak{T}_d)$  is  $T_0 \Leftrightarrow d$  satisfies M2.
- (2)  $(X, \mathfrak{T}_d)$  is  $T_1 \Leftrightarrow d$  satisfies M2b.
- (3)  $(X, \mathfrak{T}_d)$  is  $T_2$  if d satisfies M2 and M3 (and so is a metric).

The characterizations of 1.1(1, 2) fail in the necessity direction without the assumption of M2a. Both M2 and M2b make use of the symmetric relation  $x \neq y$ , but M2b is implicitly more "symmetric" than M2, since the former uses a conjunction on the left-hand side of the predicate; this is also indicated by the strengthening from  $T_0$  to  $T_1$  in 1.1(1, 2) and is better understood below by considering specialization orders. It should be noted that M3 is non-superfluous in Proposition 1.1(3) in the sense that it plays a critical role in showing, by contradiction, that the separating  $\varepsilon$ -balls are in fact disjoint. On the other hand, it should also be noted that Proposition 1.1(3) does not reverse, i.e., M3 is not necessary for Hausdorff to hold: let  $X = \{x, y\}$ , and put  $d: X \times X \to [0, 1]$  by

$$d(x,x) = d(y,y) = 0, \quad d(x,y) = 1/3, \quad d(y,x) = 2/3;$$

and note

$$B_{1/3}(x) = \{x\}, \quad B_{1/3}(y) = \{y\}, \quad \mathfrak{T}_d = \{\emptyset, \{x\}, \{y\}, X\}.$$

Hence  $(X, \mathfrak{T}_d)$  is  $T_2$ , but M3 does not hold. Further, it may be observed that an equivalent quasimetric may be given for  $(X, \mathfrak{T}_d)$  which is in fact symmetric (and a metric).

To summarize, in the presence of M1, M2a, M4, it is the case that M2 characterizes  $T_0$ , M2b characterizes  $T_1$ , and M3 (with M2) is sufficient for Hausdorff and strategically needed for Hausdorff (though not logically necessary). Altogether,  $T_1$  and Hausdorff are to a certain degree tied to notions of symmetry from the standpoint of metric spaces. Working in non- $T_1$  or non-Hausdorff metric-like settings necessitates working with asymmetric hemimetrics or asymmetric quasimetrics.

There is another (perhaps more) general way to frame the issue of asymmetry, namely via *specialization orders*. To review, let  $(X, \mathfrak{T})$  be a topological space, and put

$$x \leq_{\mathfrak{T}} y \quad \Leftrightarrow \quad y \in \{x\}.$$

The dual of this order is commonly called specialization order—e.g., see [28]. With respect to issues of antisymmetry, symmetry, and asymmetry, the two forms of this order are logically equivalent. The authors prefer the above formulation for reasons stemming from topological systems and sets enriched by po-monoids viewed as categories [9] and for the fact that such notions induce many-valued specialization orders for many-valued topological spaces consistent with the above formulation. It can be noted that the above formulation of specialization orders is implicit in [1] when identifying non- $T_1$  coseparators for **Top**<sub>0</sub>—see Examples 7.18(5) on p. 105 of [1].

The following proposition is useful in understanding issues of asymmetry in traditional topology from the standpoint of specialization orders: **Proposition 1.2.** Let  $(X, \mathfrak{T})$  be a topological space. The following hold:

- (1)  $\leq_{\mathfrak{T}}$  is a preorder on X.
- (2)  $\leq_{\mathfrak{T}}$  is antisymmetric, and hence a partial order on X, if and only if  $(X, \mathfrak{T})$  is  $T_0$ .
- (3)  $\leq_{\mathfrak{T}}$  is antisymmetric and symmetric, and hence trivially the equality on X, if and only if  $(X,\mathfrak{T})$  is  $T_1$ .

From the standpoint of specialization orders, symmetry for  $T_0$  spaces is logically equivalent to  $T_1$  separation: this seems consistent with the fact that the  $T_1$  axiom is an axiom of symmetric separation—one can both separate x from y and y from x. Working with non- $T_1$  spaces, e.g., with certain sober spaces, means working with asymmetric specialization orders. The mention of sober spaces is intentional, since these are the spectra of locales; and the spectrum functor produces many asymmetric spaces—the spectrum of the co-finite topology on  $\mathbb{R}$  is sober and not  $T_1$ and hence asymmetric with respect to its specialization order.

It is the purpose of this paper to extend [9] by exploring the issue of asymmetry and associated issues of separation for many-valued topological spaces using notions of many-valued specialization orders and manyvalued spectra. Our results include: a characterization of when manyvalued spectra are not  $(L-)T_1$ , where  $T_1$  separation for many-valued topology is defined by imposing antisymmetry and symmetry on many-valued specialization orders; general conditions for when many-valued spectra fail to be symmetric, i.e.,  $T_1$ ; and an inventory of examples of manyvalued spaces and spectra which are not  $T_1$ . Such results show that many spaces produced by many-valued spectrum functors are not  $T_1$ , including many-valued spectra produced by common topologies on  $\mathbb R$  taken as locales—including the co-finite topology referred to above, extending those results obtained by the traditional spectrum functor. Our results also show that under appropriate conditions, the well-known  $G_{\chi}$ ,  $M_L$ ,  $\omega_L$ ,  $\iota_L$  functors produce and/or reflect both asymmetric and symmetric ric L-topological spaces. Altogether, these results show that asymmetry seems as much at home in "fuzzy" topology as in traditional topology.

The remainder of this paper is organized as follows: needed ordertheoretic foundations and topological notions are given in Section 2; Section 3 reviews the construction of the many-valued spectrum of a semiquantale; two specialization orders (and their duals) are stated in Section 4 for many-valued topological spaces and their use in characterizing the well-established  $L-T_0$  axiom is given; in Section 5, the  $L-T_1$  axiom of [32] for many-valued topology—dubbed  $L-T_1$  (1) in this paper—is rediscovered

using symmetry of the first of these specialization orders, and an apparently new L- $T_1$  axiom for many-valued topology—dubbed L- $T_1$  (2) in this paper—is introduced using symmetry of the second of these specialization orders; Section 6 gives the main results relating the L- $T_1$  axioms of Section 5 and their negations to the many-valued spectra of Section 3 and the  $G_{\chi}$ ,  $M_L$ ,  $\omega_L$ ,  $\iota_L$  functors; and Section 7 gives detailed summaries and an extensive list of open questions for asymmetry research in fuzzy topology.

An important preliminary comment: throughout this paper, use is made of AFT( $\bigvee$ ), namely that version of the Special Adjoint Functor Theorem (AFT) which constructs arbitrary  $\bigwedge$  preserving right-adjoints for arbitrary  $\bigvee$  preserving maps, as well as the dual AFT( $\bigwedge$ ), namely that version of AFT which constructs arbitrary  $\bigvee$  preserving left-adjoints for arbitrary  $\bigwedge$  preserving maps; see [28]. The notation for right and left adjoints follows [59]: the right-adjoint of f is denoted  $f^{\vdash}$ , so that  $f \dashv f^{\vdash}$ ; and the left-adjoint of g is denoted  $g^{\dashv}$ , so that  $g^{\dashv} \dashv g$ .

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#### 2. Order-theoretic and many-valued topological notions

This section gives needed order-theoretic tools, powerset monads for many-valued topology as used in this paper, and basic notions of (fixedbasis) many-valued topology and topological spaces.

2.1. Order-theoretic tools, IIA operators, and example classes. Used throughout this paper are several key notions catalogued from [5, 62, 21, 18, 24, 15, 63, 59] and which follow the format of [9]. These notions are augmented in later sections as needed.

**Definition 2.1.1.** A magma or groupoid  $(X, \otimes)$  is a set X equipped with a binary operation  $\otimes : X \times X \to X$  with no other assumptions made concerning  $\otimes$ ; if  $\otimes$  is associative, then  $(X, \otimes)$  is a semigroup; if  $\otimes$  has a two-sided identity or unit e, then  $(X, \otimes)$  or  $(X, \otimes, e)$  is unital; and  $(X, \otimes)$ is a monoid if it is a unital semigroup. A semiquantale  $(L, \leq, \otimes)$  is a complete lattice  $(L, \leq)$  for which  $(L, \otimes)$  is a groupoid, in which case  $\otimes$  is sometimes called a *tensor product*, in keeping with [55, 59]; this usage is broader than that of [2], but coincides with [2] in some of the examples of 2.1.3 below (e.g., (4) and (5)). The term semiquantale is related to the term quantale (see 2.1.2(5) below) analogously to how the terms semiframe and semilocale are related, respectively, to the terms frame and locale (see paragraph below 2.1.2). The "flat" structure of semiquantales proves quite useful in characterizing various kinds of theories which can be used to build foundations for many-valued topology in [59, 60] and subsequent literature.

**Definition 2.1.2.** A *po-groupoid*, also called *partially ordered-groupoid* or *ordered groupoid*, is a triple  $(L, \leq, \otimes)$  which satisfies the properties that  $(L, \leq)$  is a poset,  $(L, \otimes)$  is a groupoid, and  $\otimes$  is isotone in both variables. Additional conditions which may be imposed on a po-groupoid  $(L, \leq, \otimes)$  include those of Definition 2.1.1 along with the following:

- (L, ≤, ∞) is unital if (L, ∞) is unital; it is integral if (L, ≤) has a top element ⊤ which is the identity for ∞; and it is commutative if ∞ is commutative.
- (2) An involutive, isotone, anti-automorphism on  $(L, \leq, \otimes)$  is an operator  $*: L \to L$  which satisfies the following axioms (cf. [44, 45]):
  - (a) \* is *involutive*, i.e.,  $\forall a \in L, a^{**} := (a^*)^* = a;$
  - (b) \* is *isotone*, i.e.,  $\forall a, b \in L, a \leq b \Rightarrow a^* \leq b^*$ ; and
  - (c) \* interchanges with  $\otimes$ , i.e.,  $\forall a, b \in L$ ,

$$(a \otimes b)^* = b^* \otimes a^*.$$

For brevity, \* is called an *IIA operator* (pronounced "I-I-A operator") and  $(L, \leq, \otimes, *)$  is called an *IIA po-groupoid*. An element  $a \in L$  is *self-adjoint* or *hermitian* if  $a^* = a$ . The label "antiautomorphism" is a consequence of interchange axiom (c) and the fact that \* turns out to be an order-isomorphism.

(3)  $(L, \leq, \otimes)$  is left-residuated [right-residuated] if there exists a binary operation  $\searrow [\swarrow] : L \times L \to L$ , called the right implication [left implication], such that

 $\forall a,b,c\in L, \ a\searrow b\geq c \ \Leftrightarrow \ a\otimes c\leq b \quad \left[b\swarrow a\geq c \ \Leftrightarrow \ c\otimes a\leq b\right].$ 

 $(L,\leq,\otimes)$  is residuated if it is both left-residuated and right-residuated.

- (4) (L,≤,∞) is complete if (L,≤) is a complete lattice. It should be noted [9] that a complete po-groupoid is the same as an ordered semiquantale.
- (5)  $(L, \leq, \otimes)$  is a quantale if it is a complete po-semigroup such that  $\otimes$  distributes from both sides across arbitrary  $\bigvee$ , i.e.,  $\forall a \in L, \forall B \subset L$ ,

$$a \otimes \left(\bigvee_{b \in B} b\right) = \bigvee_{b \in B} (a \otimes b)$$

and

$$\left(\bigvee_{b \in B} b\right) \otimes a = \bigvee_{b \in B} (b \otimes a)$$

This means that a quantale may be viewed as a semiquantale in which the tensor product is associative and enjoys both of the above infinite distributive laws of  $\bigvee$  over  $\otimes$ .

- (6) A unital IIA quantale  $(L, \leq \otimes, e, *)$  or  $(L, \otimes)$  or L is a poset  $(L, \leq)$  equipped with a tensor product  $\otimes : L \times L \to L$  and IIA operator  $*: L \to L$  such that  $(L, \leq)$  is a complete lattice,  $\otimes$  is associative and distributes from both sides across arbitrary  $\bigvee$ , and  $\otimes$  has an identity or unit e. Such terminology comes from [9], though it should be noted that in [44, 45] such structures are called *involutive quantales*. In the many-valued context, involutions are typically antitone as well as lack the distinctive interchange property. The notion of an IIA operator is needed for structures more general than quantales; hence, this work follows the terminology of [9].
- (7) If in (5),  $\otimes$  is chosen to be the binary meet  $\wedge$  on L, then L is a *frame* or *locale* [28], in which case the tensor product is commutative and \* may be chosen to be  $id_L$ .
- (8) It should also be noted that, by the  $AFT(\bigvee)$ , a unital quantale may be equivalently described as a complete residuated pomonoid.
- (9) If in the notion of a unital semiquantale ⊗ is chosen to be the binary ∧, then the notion of a semiframe or semilocale results [55], language analogous to that of semiquantales vis-a-vis quantales.
- (10) Semiquantale morphisms are formally the same as quantale morphisms (preserving  $\bigvee$  and  $\otimes$ ), even as semiframe [semilocale] morphisms are formally the same as frame [locale] morphisms (preserving  $\bigvee$  and  $\wedge$  [dually preserving  $\bigvee$  and  $\wedge$ ]).
- (11) If L is any of the above structures, it is consistent if  $|L| \ge 2$ .

The condition of  $\otimes$  being commutative is a "kind" of symmetry, and the notion of an IIA operator is essential for a proper foundation of asymmetry in many-valued topology from the standpoint of *L*-valued specialization orders considered later in this paper [9]. Given a po-groupoid  $(L, \leq, \otimes)$ , it should be noted that if  $\otimes$  is commutative, then an allowable choice of IIA operator for  $\otimes$  is  $* = id_L$ , though there may be other choices as well. If  $* = id_L$  is an IIA operator for  $\otimes$ , then  $\otimes$  must be commutative. Thus the allowable choices of IIA operators for  $\otimes$  roughly gauge the deviation of  $\otimes$  from commutivity; and non-commutative  $\otimes$  and associated

IIA operators \* are fundamentally tied to expressing and tracking possible non-symmetries of *L*-valued specialization orders for *L*-topological spaces.

It is important to note that an extensive inventory of examples of the structures defined above, including IIA operators, is given in [9], from which a few are now recalled:

# Examples 2.1.3 (IIA po-groupoids/quantales)

(1) The set  $\mathcal{M}_{n \times n}$  all  $n \times n$  matrices with extended real values from  $[0, +\infty]$  with the order and arithmetic operations extended as follows:

$$\begin{aligned} \forall a \in [0, +\infty], \ a \leq +\infty, \\ \forall a \in [0, +\infty], \ a + (+\infty) = (+\infty) + a = +\infty, \\ \forall a > 0, \ a \cdot (+\infty) = (+\infty) \cdot a = +\infty, \\ 0 \cdot (+\infty) = (+\infty) \cdot 0 = 0. \end{aligned}$$

Then  $\mathcal{M}_{n \times n}$  is equipped with the entry-wise order,  $\otimes$  as (extended) matrix multiplication, and \* as the IIA operation which takes the transpose  $A^T$  of a matrix A. It follows that  $\mathcal{M}_{n \times n}$  is a non-commutative, unital, non-integral, IIA quantale if  $n \geq 2$ .

- (2) The following extensions of the examples in (1) above come from suggestions of U. Höhle and are based on ideas and notation from [45].
  - (a) First, let M<sub>n×n</sub> now denote the complex-valued square matrices, and let C be the scalar field. Note that these form a C\*-algebra in which the transposition and conjugation of matrices coincide with the formation of adjoint operators of a C\*-algebra.
  - (b) Next, let A be any C\*-algebra with unit, and form the complete lattice Max (A) comprising all closed linear subspaces of A, with the order, meets, and joins done as follows: the order is subset inclusion; (arbitrary) meets are intersections; and (arbitrary) joins are taken dually to meets, or, alternatively, as the intersection of all closed linear subspaces containing the union of a given family of closed linear subspaces. Given closed linear subspaces H and K of A, put

$$H \otimes K = \overline{span(HK)} = \overline{span(\{hk : h \in H, k \in K\})},$$

i.e., the (topological) closure of the linear hull of the set formed by applying the multiplication (from  $\mathcal{A}$ ) of the elements of H with the elements of K. Then  $Max(\mathcal{A})$  provided

with this tensor product is a unital quantale where the unit is the 1-dimensional subspace generated by the unit of  $\mathcal{A}$ .

(c) The IIA operator on  $Max(\mathcal{A})$  (in the sense of Definition 2.1.2(2) above) is the adjointness operator \* of the C\*-algebra  $\mathcal{A}$  applied element-wise to the closed linear subspaces: i.e.,

$$H^* = \{h^* : h \in H\}.$$

Then  $Max(\mathcal{A})$  so equipped is a IIA and unital quantale and is called the spectrum of the C\*-algebra  $\mathcal{A}$  in [45].

- (d) It should be noted that the linear subspace  $H^*$  is the *adjoint* subspace on H in the theory of C<sup>\*</sup>-algebras; see [29].
- (3) In preparation for the example class in (5) below, we now inventory examples of (complete) DeMorgan algebras, where a (complete) DeMorgan algebra  $(L,\leq,')$  is a complete lattice  $(L,\leq)$  equipped with an order-reversing involution ':  $L \to L$ . Sometimes ' is called a DeMorgan complementation or DeMorgan polarity.
  - (a) Each complete Boolean algebra is a DeMorgan algebra. Given a set X, ( $\wp(X), \subset, X - ()$ ), where X - () indicates set complementation, is a complete Boolean algebra, and hence a De-Morgan algebra, which is completely distributive. The regular open subsets from the usual topology on the real line form a complete Boolean algebra, and hence a DeMorgan algebra, which is weakly completely distributive but not completely distributive (since it is non-atomic and hence non-spatial).
  - (b) Given the usual real line interval [0, 1],  $([0, 1], \leq, ')$  is a complete DeMorgan algebra, where  $' : [0, 1] \rightarrow [0, 1]$ , defined by

$$a' = 1 - a,$$

is the well-known Łukasiewicz negation. This structure is completely distributive (since it is a complete chain).

(c) The five point "diamond"  $L = \{\perp, a, b, c, \top\}$ , equipped with the ordering given by

$$\perp < a, b, c < \top$$

and no two of  $\,a,b,c\,$  being related, and equipped with  $\,':\,L\to L\,$  defined by

$$\perp' = \top, \ \top' = \perp, \ a' = a, \ b' = c, \ c' = b,$$

is a (n) (orthocomplemented) DeMorgan algebra  $(L,\leq,\,')$  which is not distributive. (d) A variation on the construction of (c) yields a different De-Morgan algebra as follows. Define a new ordering by

 $\bot < b < c < \top, \ \bot < a < \top$ 

with a unrelated to b or c, but keep the same ':  $L \to L$  as in (c). Then this  $(L, \leq, ')$  is another (orthocomplemented) DeMorgan algebra which is not distributive.

(e) Let  $\mathcal{H}$  be a Hilbert space with the inner product  $\langle , \rangle$ , and let L be the family of all closed linear subspaces ordered by inclusion. Define ':  $L \to L$  by

$$A' = \{ x \in \mathcal{H} : a \in A \Rightarrow \langle x, a \rangle = 0 \}$$

Then (L, ') is also a(n) (orthocomplemented) DeMorgan algebra which is not distributive.

(4) The following construction is needed for the example class in (5) below. Let (L, ≤) be a complete lattice with bottom element ⊥ and top element ⊤. Following [59], based on comments of U. Höhle and motivated in part from B. Hutton's work [27] on many-valued uniformities comprising arbitrary join preserving operators, we put

$$S\left(L\right) = \left\{f: L \to L \mid f \text{ preserves arbitrary } \bigvee\right\},\$$

and equip S(L) with the pointwise order lifted from L and also denoted  $\leq$ . It follows from the associativity of joins in L that S(L) is closed under arbitrary  $\bigvee$  given by the pointwise lifting of the joins of L. Now S(L) has a bottom element  $\perp_{S(L)} =$  $\bigvee_{\gamma \in \mathscr{B}} \{f_{\gamma}\}$  which can be defined by  $\perp_{S(L)} (a) = \bot$  for each  $a \in L$ . Further, by duality, S(L) is closed under arbitrary  $\bigwedge$  given by

$$\bigwedge_{\gamma \in \Gamma} f_{\gamma} = \bigvee \left\{ g \in S(L) : g = \text{l.b.} \left\{ f_{\gamma} : \gamma \in \Gamma \right\} \right\},\$$

where  $\{g \in S(L) : g = \text{l.b. } \{f_{\gamma} : \gamma \in \Gamma\}\}$  is non-empty since it contains  $\perp_{S(L)}$ . It follows S(L) has a top element  $\top_{S(L)} = \bigwedge_{\gamma \in \emptyset} \{f_{\gamma}\}$  which can be defined by

$$\top_{S(L)} (a) = \begin{cases} \bot, & a = \bot \\ \top, & a \neq \bot \end{cases}$$

The tensor product on S(L) is chosen to be function composition  $\circ: S(L) \times S(L) \to S(L)$ . Then  $\circ$  is associative and, since the members of S(L) preserve arbitrary  $\bigvee$ , it easily follows that  $\circ$ distributes across arbitrary  $\bigvee$  from both sides. Also,  $id_L$  is the identity of  $\circ$  and  $id_L \neq \top_{S(L)}$  if  $|L| \geq 3$ . Further:

(a) If |L| = 2, then  $(S(L), \leq, \circ)$  is order-isomorphic to  $\mathbf{2} = \{\bot, \top\}$ .

- (b) If  $|L| \ge 3$ , then  $(S(L), \le, \circ)$  is a unital, non-integral, non-commutative quantale.
- (5) Let (L,≤,') be a DeMorgan algebra with bottom element ⊥ and top element ⊤, and construct (S(L), ≤, ∘) as in (4) above. We note by AFT(∨) that each f ∈ S(L) has a right adjoint, denoted f<sup>⊢</sup>: L → L, which is given by

$$f^{\vdash}(b) = \bigvee_{f(a) \le b} a$$

and which preserves arbitrary  $\bigwedge$ . Following [44], we define \* :  $S(L) \to S(L)$  at  $g \in S(L)$  by putting

$$g^*\left(a\right) = \left(g^{\vdash}\left(a'\right)\right)'.$$

It can be shown that \* is an IIA operator on S(L). In combination with (4) above, it is now concluded that  $(S(L), \leq, \circ, *)$  is a unital, non-integral, IIA quantale, which for  $|L| \geq 3$  is non-commutative. A rich inventory of such examples is assured by (3) above.

This subsection closes with results which note that the combination of an IIA operator and some form of residuation has synergistic effects quite useful for this paper.

**Lemma 2.1.4.** Let  $(L, \leq, \otimes, *)$  be a po-groupoid equipped with an IIA operator \*. Then L is left-residuated if and only if L is right-residuated; and any left-[right-]residuation induced using \* is independent of \* in the sense that any two IIA operators will induce the same left-[right-]residuation from any right-[left-]residuation.

*Proof.* ( $\Leftarrow$ ). Suppose L is right-residuated with  $\checkmark : L \times L \to L$ , and put  $\searrow : L \times L \to L$  by

$$a \searrow b = \left(b^* \swarrow a^*\right)^*.$$

It is to be shown that  $\forall a, b, c \in L$ ,

$$\forall a, b, c \in L, \ a \searrow b \ge c \ \Leftrightarrow \ a \otimes c \le b. \tag{L1}$$

We point out now that if (L1) were true, then  $\searrow$  would be independent of \*. To prove (L1), let  $a, b, c \in L$ . Now it is the case by right-residuation that

$$b^* \swarrow a^* \ge c^* \Leftrightarrow c^* \otimes a^* \le b^*.$$
 (L2)

Since \* is an order-isomorphism and interchanges with  $\otimes$ , we have each of the following:

$$a \searrow b \ge c \Leftrightarrow (b^* \swarrow a^*)^* \ge c \Leftrightarrow b^* \swarrow a^* \ge c^*;$$
 (L3)

$$c^* \otimes a^* \le b^* \Leftrightarrow (a \otimes c)^* \le b^* \Leftrightarrow a \otimes c \le b.$$
 (L4)

It should be clear that conjoining (L2, L3, L4) justifies the predicate of (L1), proving ( $\Leftarrow$ ) of the Lemma. The ( $\Rightarrow$ ) direction of the Lemma has an analogous proof. Now the independence of an induced left-[right-]residuation follows immediately from (L1) [analogue of (L1)] and the antisymmetry of the order  $\leq$  in L.

**Corollary 2.1.5.** Let  $(L, \leq, \otimes, e, *)$  be a complete po-monoid equipped with an IIA operator \*. Then L is left-residuated if and only if L is right-residuated if and only if L is a unital IIA quantale.

2.2. Powerset monadic and topological notions. This subsection inventories needed powerset monadic and topological ideas [55, 59]. Given semiquantale L and set X,  $L^X$  is the semiquantale which comprises all mappings from X to L—called (L-)subsets of X—being equipped with the pointwise lifting of  $\leq$  and  $\otimes$  and as top element the constant map  $\underline{\top}$  (the map from X to L which assigns the top element  $\overline{\top}$ ).  $(X, \tau)$  is a(n) (L-)topological space if  $\tau \subset L^X$  is closed under arbitrary  $\bigvee$  and  $\otimes$  and also contains  $\underline{\top}$ . The inclusion map  $\tau \hookrightarrow L^X$  preserves  $\bigvee$ ,  $\otimes$ , and  $\underline{\top}$ , making  $\tau$  a subsemiquantale of  $L^X$  in the sense of the category  $\mathbf{SQuant}_{\top}$  introduced at the beginning of Section 3 below. The structure  $\tau$  is a(n) (L-)topology and its members are (L-)open subsets of X.

Given a function  $f: X \to Y$ , then the image, preimage, lower image powerset operators  $f_L^{\to}: L^X \to L^Y$ ,  $f_L^{\leftarrow}: L^X \leftarrow L^Y$ ,  $f_{L\to}: L^X \to L^Y$ are, respectively, given by

$$f_{L}^{\rightarrow}\left(a\right)\left(y\right) = \bigvee_{f\left(x\right) = y} a\left(x\right), \quad f_{L}^{\leftarrow}\left(b\right) = b \circ f, \quad f_{L \rightarrow}\left(a\right) = \bigwedge_{a \leq f_{L}^{\leftarrow}\left(b\right)} b.$$

It is always the case that

$$f_L^{\to} \dashv f_L^{\leftarrow} \dashv f_{L \to}.$$

Further, for L consistent, given, respectively, the traditional image, preimage, lower image operators  $f^{\rightarrow}$ ,  $f^{\leftarrow}$ ,  $f_{\rightarrow}$ , and given  $A \subset X$ ,  $B \subset Y$ , it is the case that

$$f_L^{\rightarrow}(\chi_A) = \chi_{f^{\rightarrow}(A)}, \quad f_L^{\leftarrow}(\chi_B) = \chi_{f^{\leftarrow}(B)}, \quad f_{L\rightarrow}(\chi_A) = \chi_{f_{\rightarrow}(A)}.$$

The category *L*-**Top** has ground category **Set** and comprises all *L*-topological spaces along with all (L-)*continuous* maps, where  $f : (X, \tau) \to (Y, \sigma)$  is *L*-continuous if  $f : X \to Y$  is a function having the property that

$$\forall v \in \sigma, \ f_L^{\leftarrow}(v) \in \tau; \quad \text{i.e., } (f_L^{\leftarrow})^{\rightarrow}(\sigma) \subset \tau.$$

Each L-Top is a topological construct—see [1] and cf. the proofs of topologicity of fixed-basis topology in [55, 59].

A map f between the carrier sets of two spaces  $(X, \tau)$  and  $(Y, \sigma)$  is (L-)open if  $f_L^{\rightarrow}$  preserves L-open subsets; a map is a(n) (L-)embedding if it is injective, L-continuous, and relatively L-open —in the sense that  $f^{\mid f^{\rightarrow}(X)}$  is L-open as a map from  $(X, \tau)$  to  $(f^{\rightarrow}(X), \sigma_{f^{\rightarrow}(X)} =: \{v_{\mid f^{\rightarrow}(X)}: v \in \sigma\})$ , and f is a(n) (L-)homeomorphism if it is a surjective L-embedding.

2.3. **IIA invariant topologies.** Throughout this subsection,  $(L, \leq, \otimes, *)$  is a complete po-groupoid equipped with an IIA operator \* (Definition 2.1.2(2) above). The issue of IIA invariant topologies is seen later in this paper to be fundamentally related to possibly asymmetric *L*-valued specialization orders associated with *L*-topological spaces.

**Definition 2.3.1.** Let  $(X, \tau)$  be an *L*-topological space. Then  $(X, \tau)$  and  $\tau$  are \*-*invariant* if  $\forall u \in \tau$ ,  $u^* \in \tau$ , where

$$u^*: X \to L$$
 by  $u^*(x) = (u(x))^*$ .

Note that if  $\otimes$  is commutative and \* is chosen to be  $id_L$ , then  $(X, \tau)$  and  $\tau$  are trivially \*-invariant. Hence non-trivial \*-invariance is an issue primarily associated with non-commutative tensor products.

**Lemma** 2.3.2. For each L-topological space  $(X, \tau)$ ,  $\tau^*$  is an L-topology on X and  $(X, \tau^*)$  is an L-topological space, where

$$\tau^* = \{ u^* : u \in \tau \} \,.$$

*Proof.* It is the case that \* is an order-isomorphism, which implies that \* as lifted to  $L^X$  is an order-isomorphism, which in turn implies that the closure of  $\tau$  under  $\bigvee$  and  $\underline{\top}$  guarantees the closure of  $\tau^*$  under  $\bigvee$  and  $\underline{\top}$ . Further, \* interchanges with  $\otimes$ , and this is the case when both are lifted to  $L^X$ . It now follows that if  $u^*, v^* \in \tau^*$ , then

$$u^* \otimes v^* = (v \otimes u)^* \in \tau$$

since  $v \otimes u \in \tau$ . Hence,  $\tau^*$  is closed under  $\otimes$ . Hence  $\tau^*$  is an *L*-topology on *X*.

**Theorem 2.3.3.** Each L-topological space  $(X, \tau)$  generates a smallest L-topology T on X which contains  $\tau$  and is \*-invariant.

*Proof.* Given an L-topological space  $(X, \tau)$ , put

$$T = \tau \vee \tau^*.$$

This means, letting  $\mathbb{F}_{L}(X)$  denote the complete fibre of *L*-topologies on *X*, that

$$T = \bigcap \left\{ \sigma \in \mathbb{F}_L \left( X \right) : \sigma \supset \tau \cup \tau^* \right\}.$$

It is to be shown that  $T = T^*$ . By Lemma 2.3.2 above, it is the case that for any  $\sigma \subset L^X$ ,

$$\sigma \in \mathbb{F}_{L}(X) \Leftrightarrow \sigma^{*} \in \mathbb{F}_{L}(X).$$

Further, it is the case for  $\sigma \in \mathbb{F}_L(X)$  that

$$\sigma \supset \tau \cup \tau^* \ \Leftrightarrow \ \sigma^* \supset \tau \cup \tau^*.$$

Next, we note that  $*: L \to L$  is an order-isomorphism and lifts to an order-isomorphism  $*: L^X \to L^X$ , and this further lifts, as in the statement of the Lemma 2.3.2 above, to an order-isomorphism  $*: \wp(L^X) \to \wp(L^X)$ —this follows from this third \* being an involution. From all of the above, we have:

$$T^{*} = \left( \bigcap \left\{ \sigma \in \mathbb{F}_{L} \left( X \right) : \sigma \supset \tau \cup \tau^{*} \right\} \right)^{*}$$
$$= \bigcap \left\{ \sigma^{*} \in \mathbb{F}_{L} \left( X \right) : \sigma \supset \tau \cup \tau^{*} \right\}$$
$$= \bigcap \left\{ \sigma^{*} \in \mathbb{F}_{L} \left( X \right) : \sigma^{*} \supset \tau \cup \tau^{*} \right\}$$
$$= \bigcap \left\{ \sigma \in \mathbb{F}_{L} \left( X \right) : \sigma \supset \tau \cup \tau^{*} \right\}$$
$$= T.$$

**Remark 2.3.4**. It is the case that if  $\otimes$  is commutative and \* is chosen to be  $id_L$ , then T in the proof of Theorem 2.3.3 is simply  $\tau$  again; but if  $\otimes$  is not commutative, then \* cannot be  $id_L$ , in which case T is often a non-trivial extension of  $\tau$ . To justify this latter claim, we note that a rich inventory of examples of non-commutative complete po-groupoids with IIA operators is provided in Examples 2.1.3 above and in [9]. Now let  $(L, \leq, \otimes, *)$  be a complete po-groupoid with IIA operator  $*: L \to L$  such that  $\otimes$  is non-commutative, in which case \* is not  $id_L$ . Then  $\exists a \in L$ ,

$$a^* \neq a$$
.

Since \* is an order-isomorphism, it follows that

$$\bot^* = \bot, \quad \top^* = \top,$$

and, hence,

$$\perp \neq a \neq \top$$
.

Further, since \* is involutive,

 $\perp \neq a^* \neq \top.$ 

Let X be any nonempty set, and put

 $\tau = \{ \underline{\perp}, \underline{a}, \underline{\top} \},\$ 

where underscore indicates the constant map from X to L which always renders the indicated member of L. Then it follows that  $\tau$  is an L-topology on X which is not \*-invariant since  $\underline{a}^* \notin \tau$ . Letting T be the smallest \*invariant L-topology containing  $\tau$  constructed in Theorem 2.3.3 above, it follows that  $\underline{a}^* \in T$  and that T is a non-trivial extension of  $\tau$ .

#### 3. MANY-VALUED SPECTRA OF SEMILOCALES AND SEMIQUANTALES

This section summarizes the construction of a standard notion of L-spectrum, where L is a semiquantale, and, in doing so, borrows heavily from, and is a variation of, [51, 52, 53, 54, 56, 3, 9, 28, 47, 48, 49, 50] and their references. An alternate and different L-spectrum is also developed in [47, 48, 49, 50].

The sequel makes reference to the category  $\mathbf{SQuant}_{\top}$  of all semiquantales and those semiquantale morphisms which not only preserve arbitrary joins and tensor products, but also top elements.

Fix semiquantale L and let A be any semiquantale—no form of distributivity plays any role in this summary. Put

$$Lpt(A) = \mathbf{SQuant}_{\top}(A, L) = \left\{ p : A \to L \mid p \text{ preserves } \bigvee, \otimes, \top \right\}$$

A many-valued counterpart to the classical Stone "first comparison map" is  $\Phi_L: A \to L^{Lpt(A)}$  defined by

$$\Phi_L(a): Lpt(A) \to L \quad by \quad \Phi_L(a)(p) = p(a),$$

and we say A is (L-)spatial if  $\Phi_L$  is injective. An improvement which  $\Phi_L$  offers over M. H. Stone's first comparison map is that  $\Phi_L$  is explicitly in the form of an evaluation map, making the following critical lemma true essentially by inspection:

Lemma 3.1. The following hold:

- (1)  $\Phi_L : A \to L^{Lpt(A)}$  is a unital semiquantale morphism, i.e.,  $\Phi_L$  preserves  $\bigvee, \otimes, e$ .
- (2)  $(\Phi_L)^{\rightarrow}(A)$  as a subset of  $L^{Lpt(A)}$  is closed under  $\bigvee, \otimes, \underline{e}$  as lifted to  $L^{Lpt(A)}$  and hence is an L-topology on Lpt(A).
- (3)  $LPt(A) =: (Lpt(A), (\Phi_L)^{\rightarrow}(A))$  is an L-topological space.
- (4) A is (L-)spatial if and only if  $(\Phi_L)^{\mid (\Phi_L)^{\rightarrow}(A)}$  is a semiquantale isomorphism of A with the L-topology  $(\Phi_L)^{\rightarrow}(A)$ .
- (5)  $\forall$  L-topological space  $(X, \tau), \tau$  is an L-spatial semiquantale.

The space LPt(A) is the *L*-spectrum of *A*, setting up an object-level mapping LPt from **SQuant**<sup>op</sup><sub> $\top$ </sub> to *L*-**Top**. Now let

$$\varphi: A \to B$$

be a  $\mathbf{SQuant}_{\top}^{op}$  morphism between semiquantales, and put

$$LPt(\varphi): LPt(A) \to LPt(B)$$
 by  $LPt(\varphi)(p) = p \circ \varphi^{op}$ ,

where  $\varphi^{op} : A \leftarrow B$  is the concrete **SQuant**<sub> $\top$ </sub> morphism from *B* back to *A*. These notions yield the following lemma:

#### Lemma 3.2. The following hold:

- (1)  $LPt(\varphi) : LPt(A) \to LPt(B)$  is L-continuous.
- (2) LPt : L-**Top**  $\leftarrow$  **SQuant** $_{\top}^{op}$  is a covariant functor with the above object-level and morphism-level assignments.

Given an *L*-topological space  $(X, \tau)$ , the question arises: since  $\tau$  is a unital semiquantale, how does  $(X, \tau)$  compare with the *L*-spectrum  $(Lpt(\tau), (\Phi_L)^{\rightarrow}(\tau))$  of the topology  $\tau$ ? This leads to a many-valued counterpart to the classical Stone "second comparison map"  $\Psi_L$  which we first define on the underlying carrier sets. Put  $\Psi_L : X \to Lpt(\tau)$  by

$$\Psi_L(x): \tau \to L$$
 by  $\Psi_L(x)(u) = u(x)$ .

As in the case for  $\Phi_L$ , an improvement which  $\Psi_L$  offers over Stone's first comparison map is that  $\Psi_L$  is explicitly in the form of an evaluation map, making the following lemma true essentially by inspection:

**Lemma 3.3.**  $\forall x \in X, \Psi_L(x) \in Lpt(\tau), making \Psi_L : X \to Lpt(\tau) a$ well-defined map.

To grasp the impact of the above ideas, the following definition is very useful.

**Definition 3.4.** Let  $(X, \tau)$  be an *L*-topological space.

- (1)  $(X,\tau)$  is  $(L-)T_0$  if  $\forall x, y \in X, x \neq y \Rightarrow \exists u \in \tau, u(x) \neq u(y)$ .
- (2)  $(X,\tau)$  is  $(L-)S_0$  if  $\Psi_L$  is surjective; and  $(X,\tau)$  is (L-) sober if  $\Psi_L$  is bijective.

The case for "canonicity" of the L- $T_0$  axiom in many-valued topology is outlined in 7.0 of Section 7 below. In the traditional setting,  $S_0$  is sometimes referred to as "quasi-sober".

**Theorem 3.5.** Let  $(X, \tau)$  be an L-topological space. Then the following hold:

- (1)  $\Psi_L : (X, \tau) \to LPt(\tau)$  is L-continuous.
- (2)  $\Psi_L: (X, \tau) \to LPt(\tau)$  is relatively L-open, i.e.,

$$(\Psi_L)^{\mid (\Psi_L)^{\rightarrow}(X)} : (X,\tau) \rightarrow \left( (\Psi_L)^{\rightarrow}(X), \left[ (\Phi_L)^{\rightarrow}(\tau) \right]_{\mid (\Psi_L)^{\rightarrow}(X)} \right)$$

is L-open.

- (3)  $(X,\tau)$  is L-T<sub>0</sub> if and only if  $\Psi_L$  is injective if and only if  $\Psi_L$  is an L-embedding.
- (4)  $(X, \tau)$  is L-S<sub>0</sub> if and only if  $\Psi_L$  is an L-continuous, L-open surjection.
- (5)  $(X, \tau)$  is L-sober if and only if  $\Psi_L$  is an L-homeomorphism.
- (6)  $\forall A \in |\mathbf{SQuant}_{\top}^{op}|, LPt(A) \text{ is } L\text{-sober.}$

To ferret out the couniversal and universal properties of  $\Phi_L$  and  $\Psi_L$ , we need another functor, namely the left-adjoint of LPt. Put  $L\Omega$  : L- $\mathbf{Top} \to \mathbf{SQuant}_{\top}^{op}$  by

$$L\Omega(X,\tau) = \tau, \quad L\Omega[f:(X,\tau) \to (Y,\sigma)] = (f_L^{\leftarrow}:\tau \to \sigma)^{op}:\tau \to \sigma.$$

It can be shown that  $L\Omega$  is a covariant functor, leading to the following result:

### **Theorem 3.6**. The following hold:

- LΩ ⊣ LPt, with (left) units Ψ<sub>L</sub> and (right) counits Φ<sup>op</sup><sub>L</sub>.
   If L-SobTop is defined to be the full subcategory of L-Top comprising all L-sober topological spaces, if L-**SpatSQuant**<sup>op</sup><sub> $\top$ </sub> is defined to be the full subcategory of  $\mathbf{SQuant}^{op}_{\top}$  comprising all Lspatial unital semiquantales, and if  $L\Omega$ , LPt, respectively, the restrictions of  $L\Omega$ , LPt to these subcategories, then  $L\Omega \dashv LPt$  becomes a categorical equivalence, i.e.,  $L\Omega \sim LPt$ , and the categories L-SobTop and L-SpatSQuant<sup>op</sup><sub> $\top$ </sub> are categorically equivalent.

The above ideas provide a schema of spectrum theories and spectrum adjunctions indexed by the objects of **SQuant** $^{op}_{\tau}$ ; namely, we have the class

$$\{L\Omega \dashv LPt : L \in |\mathbf{SQuant}^{op}_{\top}|\}.$$

Does this class include the traditional spectrum theory? Let L = 2and note **Top**  $\approx L$ -**Top** via the isomorphisms  $G_{\chi}$  given, at various levels, by

$$\forall A \subset X, \ \chi_A(x) = \begin{cases} \ \top, \ x \in A \\ \bot, \ x \notin A \end{cases},$$
$$G_{\chi}(A) = \chi_A, \ G_{\chi}(\wp(X)) = \mathbf{2}^X, \ G_{\chi}(\mathfrak{T}) = \{\chi_U : U \in \mathfrak{T}\},$$
$$G_{\chi}(X, \mathfrak{T}) = (X, G_{\chi}(\mathfrak{T})), \ G_{\chi}(f) = f.$$

Now the functor  $\Omega$  : **Top**  $\rightarrow$  **Loc** of [28] can be written at the object level, using appropriate levels of  $G_{\chi}$ , as

$$\Omega = G_{\chi}^{-1} \circ \mathbf{2}\Omega \circ G_{\chi},$$

with similar formulations at the morphism level of  $\Omega$ ; and in analogous fashion, observing that 2pt = pt and  $G_{\chi} (\Phi^{\rightarrow} (\ )) = (\Phi_2)^{\rightarrow} (\ )$ , the functor  $Pt : \mathbf{Top} \leftarrow \mathbf{Loc}$  of [28] can be written in terms of 2Pt, namely,

$$\mathbf{2}Pt = G_{\chi} \circ Pt.$$

Further, the units  $\Psi$  and counits  $\Phi^{op}$  of  $\Omega \dashv Pt$  can be written, using appropriate levels of  $G_{\chi}$ , in terms of the units  $\Psi_2$  and counits  $\Phi_2^{op}$  of  $2\Omega \dashv 2Pt$ .

Thus, to answer the above question, the classical spectrum theory is included, up to categorical isomorphism, as the **2**-spectrum theory in the schema of many-valued spectrum theories outlined above; moreover, for each consistent L, the classical spectrum theory embeds into the L-valued spectrum theory.

#### 4. Many-valued specialization orders and the $L-T_0$ axiom

This and later sections deal with two specialization orders associated with many-valued topological spaces, one of which is a traditional (2-valued) preorder (Definition 4.2), and the second of which is a many-valued preorder (Definition 4.3). The latter notion in the restricted case  $\otimes = \wedge$  appears in [64, 14, 35] and is labeled "specialization" in [35]; see also [14, 24, 35, 46, 64, 67]. These conventions are followed for the rest of this paper: "preordered set" is abbreviated by "preset" [10], and, correspondingly, the category of all presets and isotone maps is designated **PreSet** [9].

It is in [9] that many-valued orders are presented in the full generality of tensors not necessarily commutative, and this is done as follows: first, generating and justifying the formal axioms of many-valued presets and associated morphisms from enriched categories; second, developing the formal axiom of symmetry within the framework of sets enriched by IIA meet-semilattice ordered groupoids; and third, developing the formal axiom of antisymmetry by constructing universal objects over many-valued presets.

This section generally follows [9], as do Sections 5 and 6 below; in contrast with [9], the axioms of many-valued preorders are not formally stated below, but rather emerge implicitly in the statement of Theorem 4.4(2) from the properties of many-valued specialization orders.

To motivate specialization orders for many-valued topology, recall from Section 1 that if  $(X, \mathfrak{T})$  is a (traditional) topological space,  $\leq_{\mathfrak{T}}$  is defined

on X by

$$x \leq_{\mathfrak{T}} y \iff y \in \{x\},$$

л

and that the choice of this order vis-a-vis its dual is addressed in Section 1 as well. Also recall from Proposition 1.2 that  $\leq_{\mathfrak{T}}$  (and its dual) is a preorder and can be used to fully capture  $T_0$  and  $T_1$  separation for traditional spaces, the former characterized by antisymmetry and the latter characterized by conjoining antisymmetry and symmetry.

We should point out that if  $f: (X, \mathfrak{T}) \to (Y, \mathfrak{S})$  is a continuous mapping between topological spaces, then  $f: (X, \leq_{\mathfrak{T}}) \to (Y, \leq_{\mathfrak{S}})$  is isotone, yielding a functor  $P: \mathbf{Top} \to \mathbf{PreSet}$ . On the other hand, if  $(X, \leq)$  is a preset, then the family  $\mathfrak{T}_{<}$  of the lower subsets of  $\leq$  is a topology on X; and if  $f:(X,\leq)\to (Y,\leq)$  is isotone, then  $f:(X,\mathfrak{T}_{\leq})\to (Y,\mathfrak{S}_{\leq})$  is a continuous mapping, yielding a functorial embedding from **PreSet** to **Top** which is the left adjoint to P in an adjunction which is a monoreflection (since  $\leq_{\mathfrak{T}_{<}} = \leq$ ). See [28].

Turning to the question of generalizing specialization orders, we note a major problem with generalizing  $\leq_{\mathfrak{T}}$ , as defined above, to the manyvalued context is that the underlying base L of truth values may not have a complementation appropriate for L-closed subsets or associated notions of L-closure—in this paper there is no standing assumption of DeMorgan (quasi-)complementation. The following proposition suggests a possible option for generalizing specialization orders to many-valued settings.

**Proposition 4.1.** Let  $(X, \mathfrak{T})$  be a (traditional) topological space. Then  $\forall x, y \in X$ , the following are equivalent:

- (1)  $x \leq_{\mathfrak{T}} y;$

- $\begin{array}{l} (1) & -2 & \overline{z} \\ (2) & \underline{y} \in \overline{\{x\}}; \\ (3) & \overline{\{y\}} \subset \overline{\{x\}}; \\ (4) & \forall U \in \mathfrak{T}, \ y \in U \Rightarrow x \in U. \end{array}$

Clearly Proposition 4.1(4), expressed solely in terms of open sets, gives us hope to generalize specialization orders to many-valued topology for a broad class of base lattices of truth values. To that end, let  $(X, \tau)$  be an Ltopological space, where initially L is a complete po-groupoid—additional restrictions are needed later; recall  $\tau \subset L^{X}$  is closed under arbitrary  $\bigvee$ and finite  $\otimes$  and contains  $\bot$ . Many generalizations of  $\leq_{\mathcal{T}}$  are possible: those given below occur in two dual pairs-two "crisp" specialization orders which are "duals", and two many-valued or "fuzzy" specialization orders which are "duals". Each such ordering below can be shown to be a generalization of the traditional  $\leq_{\mathfrak{T}}$ .

Definition 4.2. (L-Specialization Order). Given L-topological space  $(X,\tau)$  with L a complete po-groupoid, the (L-)specialization order  $\leq_{\tau}$  on X is defined by

 $x \leq_{\tau} y \Leftrightarrow \forall u \in \tau, \ u(y) \leq u(x).$ 

The dual (L-)specialization order would reverse the inequalities occurring in the right-hand side of the above display.

The *L*-specialization order  $\leq_{\tau}$  constructed in Definition 4.2 is a crisp relation induced by an *L*-topology on *X*. This order generalizes the traditional  $\leq_{\mathfrak{T}}$ , invoking the appropriate level of the  $G_{\chi}$  functor (from paragraphs following Theorem 3.6 above), as follows:

$$\begin{split} x \leq_{\mathfrak{T}} y & \Leftrightarrow & \forall U \in \mathfrak{T}, \ y \in U \Rightarrow x \in U \\ \Leftrightarrow & \forall U \in \mathfrak{T}, \ \chi_U(y) = \top \Rightarrow \chi_U(x) = \top \\ \Leftrightarrow & \forall U \in \mathfrak{T}, \ \chi_U(y) \leq \chi_U(x) \\ \Leftrightarrow & \forall u \in G_{\chi}(\mathfrak{T}), \ u(y) \leq u(x) \\ \Leftrightarrow & x \leq_{G_{\chi}(\mathfrak{T})} y. \end{split}$$

Thus for  $L = 2, \leq_{\mathfrak{T}}$  and  $\leq_{\tau}$  are the same up to order-isomorphism.

Noting that the relation  $\leq_{\tau}$  is crisp, i.e.,  $\leq_{\tau} \subset X \times X$  is a traditional subset of  $X \times X$ , it is also of interest to construct an *L*-valued specialization order on *X* (actually,  $X \times X$ ). For this construction *L* is strengthened to be a right-residuated complete po-monoid. Recall the right residuation  $\checkmark : L \times L \to L$ , defined by

$$b \swarrow a \ge c \Leftrightarrow c \otimes a \le b$$
,

or, alternatively,

$$b \swarrow a = \bigvee_{c \otimes a < b} c.$$

It is a consequence of  $AFT(\bigvee)$  that  $\otimes$  distributes across  $\bigvee$  from the right. Now given the inequality  $u(y) \leq u(x)$  appearing in the right-hand predicate of Definition 4.2, it follows that

 $u(y) \le u(x) \quad \Leftrightarrow \quad u(x) \swarrow u(y) \ge e.$ 

Noting the association of  $\forall$  with  $\bigwedge$ , the following definition is suggested:

**Definition 4.3.** (*L*-Valued Specialization Order). Given *L*-topological space  $(X, \tau)$  with *L* a right-residuated complete po-monoid, the (*L*-valued) specialization order  $P_{\tau} : X \times X \to L$  is defined by

$$P_{\tau}(x,y) = \bigwedge_{u \in \tau} \left( u\left(x\right) \swarrow u\left(y\right) \right).$$

The dual L-specialization order  $Q_{\tau} : X \times X \to L$ , with L a left-residuated complete po-monoid, is given by

$$Q_{\tau}(x,y) = \bigwedge_{u \in \tau} \left( u\left(x\right) \searrow u\left(y\right) \right).$$

The following theorem from [9] parallels and extends Proposition 1.2 (1,2) above and makes use of the L- $T_0$  axiom given in Definition 3.4(1) above.

**Theorem 4.4.** Let  $(X, \tau)$  be an L-topological space, with L a complete po-groupoid. The following statements hold:

- (1) The relation  $\leq_{\tau}$  is a preorder on X. Further, it is antisymmetric, and hence a partial order, if and only if  $(X, \tau)$  is L-T<sub>0</sub>. These statements remain true if  $\leq_{\tau}$  is replaced by its dual order.
- (2) For L a right-residuated complete po-monoid, the mapping  $P_{\tau}$ :  $X \times X \to L$  is an L-preorder on X in the sense that
  - (P1)  $\forall x \in X, P_{\tau}(x, x) \geq e$  ((L-)reflexivity) and
  - (P2)  $\forall x, y, z \in X, P_{\tau}(x, y) \otimes P_{\tau}(y, z) \leq P_{\tau}(x, z)((L)$ transitivity), and it is (L-)antisymmetric in the following sense—
  - (P3)  $\forall x, y \in X, P_{\tau}(x, y) \geq e, P_{\tau}(y, x) \geq e \Rightarrow x = y,$ and hence an L-partial order—if and only if  $(X, \tau)$  is L-T<sub>0</sub> (Definition 3.4).
- (3) For L a left-residuated complete po-monoid, the mapping Q<sub>τ</sub> : X×X → L is an L-preorder on X. Further, it is L-antisymmetric in the sense of (P3) above, and hence an L-partial order, if and only if (X, τ) is L-T<sub>0</sub>.
- (4) For L a right-[left-]residuated complete po-monoid,  $\leq_{\tau}$  is antisymmetric if and only if  $P_{\tau}$  [ $Q_{\tau}$ ] is L-antisymmetric. Hence, for L a unital quantale,  $\leq_{\tau}$  is antisymmetric if and only if  $P_{\tau}$  is L-antisymmetric if and only if  $Q_{\tau}$  is L-antisymmetric.
- (5) For L a DeMorgan frame with order-reversing involution ':  $L \to L$ , the L-valued hemimetric  $P'_{\tau}$  :  $X \times X \to L$  induced by the L-specialization order  $P_{\tau}$  :  $X \times X \to L$  satisfies

$$\forall x, y \in X, P_{\tau}(x, y) = P_{\tau}(y, x) = \perp \text{ if and only if } x = y,$$

*i.e.*, is positive definite, if and only if  $(X, \tau)$  is L-T<sub>0</sub>.

We note that Theorem 4.4(4) confirms that *L*-antisymmetry is an appropriate generalization of traditional antisymmetry; and 4.4(4) also reduces the complexity of checking the *L*-antisymmetry of  $P_{\tau}$  to that of checking the antisymmetry of  $\leq_{\tau}$ . This is a bit surprising since (for *L* 

a right-residuated complete po-monoid) it is always the case that  $\leq_{\tau}$  is contained in  $P_{\tau}$  in the sense that  $\chi^e_{\leq_{\tau}} \leq P_{\tau}$ , where

$$\chi^e_{\leq_\tau} \equiv \chi_{\leq_\tau} \land \underline{e} : X \times X \to \{\bot, e\} \subset L,$$

essentially a  $G_{\chi}$  kind of argument. Restated, the notion of *L*-antisymmetry given above, in the context of specialization for many-valued topology, is logically equivalent to traditional antisymmetry—this follows by comparing the crisp specialization order of an *L*-topological space against the fuzzy specialization order for the same space. Further, both antisymmetries are equivalent to *L*- $T_0$  separation. Thus 4.4 indicates that notions of antisymmetry, specialization,  $T_0$  separation, and many-valued topology fit well together. A detailed development of the *L*-antisymmetry axiom and its categorical properties is given in [9].

The complete success of the L-specialization and L-valued specialization orders in capturing the L- $T_0$  axiom, analogously to how the traditional specialization order captures the usual  $T_0$  axiom, suggests a way forward for generalizing the L- $T_1$  axiom for many-valued topology using Proposition 1.2(3) above as a template; this is taken up in Section 5 below.

The remainder of this section concerns categorical housekeeping particularly needed for Section 6 below.

**Definition 4.5** [9]. Letting L be a unital complete po-groupoid, the category L-**PreSet** comprises all L-presets (X, P) satisfying (P1, P2) of Theorem 4.4(2) above, along with all (L-)isotone mappings  $f : (X, P) \to (Y, Q)$  satisfying

$$\forall x, y \in X, P(x, y) \le Q(f(x), f(y)).$$

Further, *L*-**PreSet** is the full subcategory *L*-**PreSet** of all *L*-posets additionally satisfying (P3) of Theorem 4.4(2).

**Lemma 4.6.** Let L be a complete po-groupoid,  $(X, \tau)$ ,  $(Y, \sigma)$  be L-topological spaces, and  $f : (X, \tau) \to (Y, \sigma)$  be an L-continuous map. The following hold:

- If L is additionally unital and (X, ≤<sub>τ</sub>), (Y, ≤<sub>σ</sub>) are as defined in Definition 4.2 above, then f : (X, ≤<sub>τ</sub>) → (Y, ≤<sub>σ</sub>) is isotone.
- (2) If L is additionally unital, right-residuated, and monoidal, and  $(X, P_{\tau}), (Y, P_{\sigma})$  are as defined in Definition 4.3 above, then  $f : (X, P_{\tau}) \to (Y, P_{\sigma})$  is L-isotone.
- (3) If L is additionally unital, left-residuated, and monoidal, and  $(X, Q_{\tau}), (Y, Q_{\sigma})$  are as defined in Definition 4.3 above, then  $f : (X, Q_{\tau}) \to (Y, Q_{\sigma})$  is L-isotone.

*Proof.* Let  $x, y \in X$  and note that  $\tau \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma)$  from the *L*-continuity of *f*. Now for (1), we have that

$$\begin{aligned} x \leq_{\tau} y & \Leftrightarrow \quad \forall u \in \tau, \ u(y) \leq u(x) \\ & \Rightarrow \quad \forall v \in \sigma, \ f_{L}^{\leftarrow}(v)(y) \leq f_{L}^{\leftarrow}(v)(x) \qquad (\Rightarrow) \\ & \Leftrightarrow \quad \forall v \in \sigma, \ v(f(y)) \leq v(f(x)) \\ & \Leftrightarrow \quad f(x) \leq_{\sigma} f(y), \end{aligned}$$

and for (2), we have that

$$P_{\tau}(x,y) = \bigwedge_{u \in \tau} (u(x) \swarrow u(y))$$
  

$$\leq \bigwedge_{v \in \sigma} (f_{L}^{\leftarrow}(v)(x) \swarrow f_{L}^{\leftarrow}(v)(y)) \quad (\leq)$$
  

$$= \bigwedge_{v \in \sigma} (v(f(x)) \swarrow v(f(y)))$$
  

$$= P_{\sigma}(f(x), f(y)),$$

The proof of (3) is dual to that of (2).

**Corollary 4.7.** For *L* a complete po-groupoid [unital right-residuated complete po-monoid, unital left-residuated complete po-monoid, respectively], each of the following is a functor:

(1)  $P_L: L\text{-}\mathbf{Top} \to \mathbf{PreSet} \ by \ (X, \tau) \mapsto (X, \leq_{\tau}), \ f \mapsto f.$ 

(2) 
$$P_{LL}: L\text{-}\mathbf{Top} \to L\text{-}\mathbf{PreSet} \ by \ (X,\tau) \mapsto (X,P_{\tau}), \ f \mapsto f.$$

(3)  $Q_{LL}: L\text{-}\mathbf{Top} \to L\text{-}\mathbf{PreSet} \ by \ (X,\tau) \mapsto (X,Q_{\tau}), \ f \mapsto f.$ 

Each of  $P_L$ ,  $P_{LL}$ ,  $Q_{LL}$  preserves isomorphisms, and thus  $P_L$  takes Lhomeomorphisms to order-isomorphisms and each of  $P_{LL}$ ,  $Q_{LL}$  takes Lhomeomorphisms to L-order-isomorphisms, the latter meaning the bijections f in (2) and (3) respectively satisfy

$$P_{\tau}(x,y) = P_{\sigma}(f(x), f(y)), \quad Q_{\tau}(x,y) = Q_{\sigma}(f(x), f(y)).$$

*Proof.* That functors preserve isomorphisms is well-known [1]; and it can be noted that if, in the proof of Lemma 4.6,  $f : (X, \tau) \to (Y, \sigma)$  is assumed an *L*-homeomorphism, then " $\Rightarrow$ " in line ( $\Rightarrow$ ) in the proof of (1) becomes " $\Leftrightarrow$ " and " $\leq$ " in line ( $\leq$ ) of the proof of (2) becomes "=".

**Corollary 4.8.** Under the appropriate respective assumptions on L for  $\leq_{\tau}$ ,  $P_{\tau}$ ,  $Q_{\tau}$ , any property of L-topological spaces which can be framed in terms of the orders  $\leq_{\tau}$ ,  $P_{\tau}$ ,  $Q_{\tau}$  is necessarily an L-topological invariant. In particular, the L- $T_0$  separation axiom is an L-topological invariant.

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**Remark 4.9**. Assuming the appropriate conditions on L for  $\leq_{\tau}$ ,  $P_{\tau}$ ,  $Q_{\tau}$ , recalling the functor  $G_{\chi}: \mathbf{Top} \to L\text{-}\mathbf{Top}$  (fifth paragraph from the end of Section 3), recalling the functor  $P : \mathbf{Top} \to \mathbf{PreSet}$  (second paragraph of this section), and letting  $G_{\chi}$  also denote the isomorphism from **PreSet** to L-**PreSet** (via  $\leq \mapsto \chi_{\leq}$ ), the following hold:

- (1)  $P = P_L \circ G_{\chi};$
- (2)  $P_{LL} \circ G_{\chi} = G_{\chi} \circ P_L$  and  $Q_{LL} \circ G_{\chi} = G_{\chi} \circ P_L$ ; (3)  $P_{LL} = G_{\chi} \circ P_L$  and  $Q_{LL} = G_{\chi} \circ P_L$  if  $L = \mathbf{2}$ .

5. Many-valued asymmetry and L- $T_1$  axioms

Let  $(X, \tau)$  be an L-topological space. For L a right-residuated complete po-monoid, Theorem 4.4(4) records that  $\leq_{\tau}$  is antisymmetric if and only if  $P_{\tau}$  is antisymmetric, and these antisymmetries are equivalent to  $(X, \tau)$ being  $L-T_0$ , fully extending Proposition 1.2(1,2) above. This suggests using Proposition 1.2(3) as a template for generalizing the traditional  $T_1$  separation axiom to many-valued topology, namely by imposing with antisymmetry the additional condition that  $\leq_{\tau}$  is symmetric and/or  $P_{\tau}$ is L-symmetric. From the point of view of specialization orders,  $(X, \tau)$  or its topology  $\tau$  is asymmetric if it is L-T<sub>0</sub> and one or both of  $\leq_{\tau}$  or  $P_{\tau}$  are not symmetric. The notion of symmetry for  $\leq_{\tau}$ , since it is a traditional preorder, is the standard notion; the notion of (L-)symmetry for  $P_{\tau}$ , since it is a many-valued order, is more subtle and is defined below in Definition 5.1 after some needed discussion.

For the case of commutative tensor products  $\otimes$ , the formulation of symmetry for an L-valued preorder  $P: X \times X \to L$  is quite straightforward, namely the symmetry condition can be defined by

$$\forall x, y \in X, P(x, y) = P(y, x),$$

which may be interpreted as saying that for all  $x \in X$ , x precedes y to the same degree that y precedes x.

In the case when  $\otimes$  is not commutative, the issue of an L-valued preorder being (L-)symmetric is more delicate and makes formal use of IIA operators as defined in [9] and Section 2 above. Following [24, 19] as formatted in [9], we now define a general notion of symmetry for manyvalued relations which includes ordered structures of membership values whose tensor products need not be commutative.

**Definition 5.1**. Let X be a set and  $(L, \leq, \otimes, e, *)$  be a unital IIA pogroupoid. An L-valued relation  $P: X \times X \to L$  is (L-)symmetric if

(P4) 
$$\forall x, y \in X, P(x, y) = P^*(y, x),$$

where  $P^*(y, x)$  means  $(P(y, x))^*$  and the tag "(P4)" continues the numbering begun in Theorem 4.4(2) above.

It is important to notice that symmetry for *L*-valued relations is tied not only to the order, but also to the tensor product and the IIA operator \*. Clearly (P4) extends the usual notion of symmetry for a traditional relation: given  $\leq \subset X \times X$  and L = 2, where *L* is viewed as equipped with  $\otimes = \wedge$  (binary meet) and \* =  $id_L$ , then  $\chi_{\leq} : X \times X \to 2$ is **2**-symmetric if and only if  $\leq$  is symmetric. Further, it is the case that (P4) is a non-trivial extension of traditional symmetry by Examples 2.1.3 above.

We now state formally the notion of asymmetry for many-valued topology:

**Definition 5.1.1.** For *L* a right-residuated complete po-monoid, an *L*-topological space  $(X, \tau)$  or its topology  $\tau$  is *asymmetric* if it is *L*-*T*<sub>0</sub> and  $\leq_{\tau}$  or  $P_{\tau}$  is not symmetric.

Standing Assumption for Section 5. Taking into account Corollary 2.1.5 above, it is assumed for the remainder of this section that, unless stated otherwise,  $(L, \leq, \otimes, e, *)$  is a unital quantale with IIA operator \*.

The purpose of this section is now reiterated: use Proposition 1.2(3) as a template for generalizing the traditional  $T_1$  separation axiom to manyvalued topology, namely by imposing on  $\leq_{\tau} [P_{\tau}]$  both antisymmetry [*L*antisymmetry] and the additional condition that  $\leq_{\tau}$  is symmetric [respectively,  $P_{\tau}$  is *L*-symmetric in the sense of Definition 5.1 above]. Before moving forward with this approach, it is important to check whether the (traditional) symmetry of  $\leq_{\tau}$  is equivalent to *L*-symmetry of  $P_{\tau}$  in the sense of Definition 5.1; this checking makes use of the dual specialization order  $Q_{\tau}$  as well as spaces  $(X, \tau^*)$  and (X, T) induced from *L*-topological space  $(X, \tau)$ . See Subsection 2.3 and Section 4 above.

**Lemma 5.2**. Let  $x, y \in X$ . Then it is the case that:

(1)  $P^*_{\tau}: X \times X \to L$  by

$$P_{\tau}^{*}(x,y) = \bigwedge_{u \in \tau} \left( u^{*}(y) \searrow u^{*}(x) \right).$$

(2) 
$$P_{\tau}^{*}(y,x) = Q_{\tau^{*}}(x,y).$$

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- (3)  $P_{\tau}$  is symmetric if and only if  $P_{\tau} = Q_{\tau^*}$ , and  $Q_{\tau}$  is symmetric if and only if  $Q_{\tau} = P_{\tau^*}$ . Hence,  $P_{\tau}$  is symmetric if and only if  $Q_{\tau}$  is symmetric.
- (4)  $P_T$  is symmetric if and only if  $P_T = Q_T$  if and only if  $Q_T$  is symmetric.

*Proof.* For (1), the properties of \*, including its being an order-isomorphism, along with the Lemma 2.1.4 of Subsection 2.1 above, imply

$$P_{\tau}^{*}(x,y) = \left( \bigwedge_{u \in \tau} (u(x) \swarrow u(y)) \right)^{*}$$
$$= \bigwedge_{u \in \tau} (u(x) \swarrow u(y))^{*}$$
$$= \bigwedge_{u \in \tau} (u^{*}(y) \searrow u^{*}(x)).$$

As for (2), note

$$P_{\tau}^{*}\left(y,x\right)=\bigwedge_{u\in\tau}\left(u^{*}\left(x\right)\searrow u^{*}\left(y\right)\right)=\bigwedge_{u\in\tau^{*}}\left(u\left(x\right)\searrow u\left(y\right)\right)=Q_{\tau^{*}}\left(x,y\right).$$

Using the definition of symmetry for an *L*-valued preorder, (3) follows immediately from (2); and (4) follows from (3) using the fact that  $T = T^*$ .

## Theorem 5.3. The following hold:

- (1) Let  $P_{\tau}$  be symmetric. Then the following hold:
  - (a)  $\leq_{\tau}$  is symmetric.
  - (b)  $\leq_{\tau}$  coincides with  $\leq_{\tau^*}$ .
  - (c)  $\leq_{\tau^*}$  is symmetric.
  - (d)  $\leq_T$  coincides with  $\leq_{\tau}$  and hence is symmetric.
- (2) The converse to (1)(a) fails, even if  $(X, \tau)$  is additionally assumed to be L- $T_0$ ; restated,  $P_{\tau}$  need not be symmetric when  $\leq_{\tau}$  is symmetric, even for L- $T_0$  spaces.

*Proof.* Ad(1)(a). Assuming  $P_{\tau}$  is symmetric means assuming  $\forall x, y \in X$  that  $P_{\tau}(x, y) = P_{\tau}^*(y, x)$ . To show  $\leq_{\tau}$  is symmetric, let  $x, y \in X$ . It must be proved that

$$x \leq_{\tau} y \iff y \leq_{\tau} x,$$

namely that

$$\left[\forall u \in \tau, \ u\left(y\right) \le u\left(x\right)\right] \Leftrightarrow \left[\forall v \in \tau, \ v\left(x\right) \le v\left(y\right)\right]. \quad (\bullet)$$

We first demonstrate necessity in  $(\bullet)$ . To that end, we assume  $[\forall u \in \tau, u(y) \leq u(x)]$  and let  $v \in \tau$ . Then it follows that

 $e \otimes v(y) = v(y) \le v(x),$ 

which implies by right-residuation that

$$v(x) \swarrow v(y) \ge e.$$

Since  $v \in \tau$  is arbitrary, then universal generalization implies that

$$\forall v \in \tau, \ v(x) \swarrow v(y) \ge e.$$

It follows that

$$P_{\tau}(x,y) = \bigwedge_{v \in \tau} \left( v\left(x\right) \swarrow v\left(y\right) \right) \ge e.$$

But the symmetry of  $P_{\tau}$  now says that

$$P_{\tau}^{*}\left(y,x\right) \ge e,$$

which in turn, using that  $^{\ast}$  is involutive and isotone and that e is hermitian, implies

$$P_{\tau}(y,x) = P_{\tau}^{**}(y,x) \ge e^{*} = e,$$

establishing

$$\bigwedge_{v \in \tau} \left( v\left(y\right) \swarrow v\left(x\right) \right) \geq e.$$

It follows that  $\forall v \in \tau, v(y) \swarrow v(x) \ge e$ . Now let  $w \in \tau$  and note  $w(y) \swarrow w(x) \ge e$ . Again, invoking right-residuation, it follows that

$$w\left(x\right)=e\otimes w\left(x\right)\leq w\left(y\right),$$

justifying that  $\forall v \in \tau$ ,  $v(x) \leq v(y)$ . This completes the proof of necessity of (•). The sufficiency of (•) follows by a proof symmetric to that for necessity, completing the proof of (1)(a).

Ad(1)(b). To show that  $\leq_{\tau} = \leq_{\tau^*}$ , we first show  $\leq_{\tau} \subset \leq_{\tau^*}$ . Suppose  $x \leq_{\tau} y$ . Then  $\forall u \in \tau$ ,  $u(y) \leq u(x)$ , which, as seen in the proof of (1)(a), means that  $\forall u \in \tau$ ,  $u(x) \swarrow u(y) \geq e$ . Now applying the proof of Lemma 2.1.4 with the fact that e is hermitian yields that

$$\forall u \in \tau, \ u^{*}\left(y\right) \searrow u^{*}\left(x\right) = \left(u\left(x\right) \swarrow u\left(y\right)\right)^{*} \ge e^{*} = e,$$

from which it follows that  $\forall u \in \tau, \ u^*(y) \leq u^*(x)$ . Restated,  $\forall u \in \tau^*, \ u(y) \leq u(x)$ , i.e.,  $x \leq_{\tau^*} y$ . A similar and symmetric proof establishes the reverse inclusion, so that  $\leq_{\tau} = \leq_{\tau^*}$ .

Ad(1)((c),(d)). Statement (1)(c) is an immediate consequence of (a) and (b). As for Statement (1)(d), it is the case that  $\tau \subset T$ , which implies that  $\leq_T \subset \leq_{\tau}$ . To see the reverse inclusion, note the completeness and residuations of L provide infinite distributivity of  $\otimes$  over  $\bigvee$  from both sides, which insures that each member of T can be written as the join of members of the standard basis of T, where associativity of  $\otimes$  assures that this standard basis may be chosen to be

$$\{u_1 \otimes \ldots \otimes u_n : u_i \in \tau \cup \tau^*, n \ge 1\}.$$

Now suppose  $x \leq_{\tau} y$  and let  $u \in \tau$ . Then  $u(y) \leq u(x)$ ; and \* being an order-isomorphism implies  $u^*(y) \leq u^*(x)$ . It follows that  $\forall u \in \tau \cup \tau^*$ ,  $u(y) \leq u(x)$ . Letting  $u \in \tau, v \in \tau^*$ , the isotonicity of  $\otimes$  in both arguments implies that  $(u \otimes v)(y) \leq (u \otimes v)(x)$ . Thus for each basic member u of T,  $u(y) \leq u(x)$ .

Ad(2). Statement (2) follows from Examples 5.4(1) below.

**Examples 5.4.** We first construct an example in (1) below which shows that for *L*-topological spaces, even with *L* a unital commutative quantale and spaces which are *L*- $T_0$ , it is not necessarily the case that  $\leq_{\tau}$  is symmetric only if  $P_{\tau}$  is *L*-symmetric, i.e., as stated in 5.3(2), the converse to Theorem 5.3(1)(a) is false. Then an example is constructed in (2) which shows that there are *L*- $T_0$  spaces for which  $\leq_{\tau}$ , and hence  $P_{\tau}$ , are not symmetric. As will be seen in Corollary 5.6 below, the examples of (1) and (2), in combination with Theorem 5.3 above, show much more besides.

(1) Let  $X = \{x, y\}$  and L = 4, the four element Boolean algebra  $\{\perp, a, b, \top\}$ , where a, b are unrelated, i.e.,  $a \nleq b$  and  $b \nleq a$ . We note that  $\otimes = \wedge$  and is commutative, and therefore  $\searrow = \swarrow$ . We denote both residuations by  $\rightarrow$  and choose  $* = id_L$  to be the IIA operator—so that  $P_{\tau}^*(y, x) = P_{\tau}(y, x)$ .

Put  $u, v, o: X \to L$  by

$$u(x) = \bot, u(y) = a; v(x) = b, v(y) = \top; o(x) = b, o(y) = a.$$

Then it can be checked that  $\tau$  is an *L*-topology on *X*, where

$$\tau = \{ \underline{\perp}, \, u, \, v, \, o, \, \underline{\top} \} \,.$$

The conditions  $x\leq_{\tau} y$  and  $y\leq_{\tau} x$  need to be checked, namely the respective conditions

 $\left[\forall w \in \tau, \ w\left(y\right) \leq w\left(x\right)\right], \quad \left[\forall w \in \tau, \ w\left(x\right) \leq w\left(y\right)\right].$ 

The first condition fails since, for example,  $u(y) = a \nleq \bot = u(x)$ ; and the second condition fails since  $o(x) = b \nleq a = o(y)$ . Hence it is (vacuously) the case that  $x \leq_{\tau} y$  if and only if  $y \leq_{\tau} x$ , so  $\leq_{\tau}$  is symmetric. It is also the case that  $[x \leq_{\tau} y \text{ and } y \leq_{\tau} x] \Rightarrow x = y$ is vacuously true, so  $\leq_{\tau}$  is antisymmetric. It is noted immediately from Theorem 4.4 (1) that  $(X, \tau)$  is L- $T_0$ —which can also be checked directly from the topology  $\tau$ , and it is therefore noted from 4.4(2) that  $P_{\tau}$  is L-antisymmetric.

Now to check the *L*-symmetry of  $P_{\tau}$ , it is helpful to record the action of the residuation on *L*, which in this case is simply Boolean implication (constructed from Boolean complementation). Letting *c* be a variable ranging over *L*, we note:

$$\begin{split} & \bot \to c \,=\, \top, \quad \top \to c \,=\, c, \\ a \to a \,=\, a \to \top \,=\, \top, \quad a \to \bot \,=\, a \to b \,=\, b, \\ b \to b \,=\, b \to \top \,=\, \top, \quad b \to \bot \,=\, b \to a \,=\, a. \end{split}$$

We have the following computations:

$$P_{\tau}(x,y) = \bigwedge_{w \in \tau} (w(y) \to w(x))$$
  
=  $\bigwedge \{ \bot \to \bot, \top \to \top, a \to \bot, \top \to b, a \to b \}$   
=  $\bigwedge \{ \top, \top, b, b, b \} = b;$ 

$$P_{\tau}(y,x) = \bigwedge_{w \in \tau} (w(x) \to w(y))$$
  
=  $\bigwedge \{ \bot \to \bot, \top \to \top, \bot \to a, \ b \to \top, \ b \to a \}$   
=  $\bigwedge \{ \top, \top, \top, \top, a \} = a.$ 

Since  $a \neq b$ , it follows that  $P_{\tau}$  is not L-symmetric.

To sum up,  $(X, \tau)$  is L- $T_0, \leq_{\tau}$  is both antisymmetric and symmetric (and hence trivial), and  $P_{\tau}$  is L-antisymmetric but not L-symmetric.  $\Box$ 

(2) This second example is set up exactly as in (1) above except that the *L*-open subset o is not used; i.e., we choose the *L*-topology

$$\tau = \{\underline{\perp}, \, u, \, v, \, \underline{\perp}\} \,.$$

As in (1) above, the conditions  $x \leq_{\tau} y$  and  $y \leq_{\tau} x$  need to be checked, namely the respective equivalent conditions

$$\left[\forall w \in \tau, \ w\left(y\right) \le w\left(x\right)\right], \quad \left[\forall w \in \tau, \ w\left(x\right) \le w\left(y\right)\right].$$

The first condition fails since, for example,  $u(y) = a \nleq \bot = u(x)$ ; but the second condition is satisfied. This means  $x \nleq_{\tau} y$  and  $y \leq_{\tau} x$ ; so  $\leq_{\tau}$  is not symmetric, and hence  $P_{\tau}$  is also not *L*-symmetric by Theorem 5.3(1)(a) above. However, it is the case that  $[x \leq_{\tau} y \text{ and } y \leq_{\tau} x] \Rightarrow x = y$  is still vacuously true, so  $\leq_{\tau}$  is antisymmetric; and hence  $(X, \tau)$  is *L*- $T_0$  and  $P_{\tau}$  is antisymmetric by Theorem 4.4(1,2) above.

To sum up,  $(X, \tau)$  is L- $T_0, \leq_{\tau}$  is antisymmetric but not symmetric, and  $P_{\tau}$  is *L*-antisymmetric but not *L*-symmetric.  $\Box$ 

It follows from Theorem 5.3 and Examples 5.4 that if we pursue the rubric of Proposition 1.2(3)— $T_1$  is equivalent to the specialization order being both antisymmetric and symmetric, then the four specialization orders available with each L-topological space— $\leq_{\tau}$  and  $P_{\tau}$  and their duals—lead to essentially two distinct  $T_1$  axioms, which are now given in the following definition. That there are essentially two distinct such axioms is taken up below the definition.

**Definition 5.5** (*L*- $T_1$  separation axioms). Let  $(X, \tau)$  be an *L*-topological space.

- (1) Suppose L is a semiquantale. Then  $(X, \tau)$  is L- $T_1$  in the first sense, or is L-T<sub>1</sub>(1), if  $\leq_{\tau}$  is both antisymmetric and symmetric in the traditional senses, in which case  $\leq_{\tau}$  collapses to the equality on X.
- (2) Suppose L is a unital quantale equipped with an IIA operator \*. Then  $(X, \tau)$  is L-T<sub>1</sub> in the second sense, or is L-T<sub>1</sub>(2), if  $P_{\tau}$  is both L-antisymmetric and L-symmetric, i.e., if  $P_{\tau}$  satisfies both (P3) of Theorem 4.4(2) and (P4) of Definition 5.1. To summarize,  $(X, \tau)$  is L- $T_1(2)$  if  $P_{\tau}: X \times X \to \tau$  satisfies the following:
  - (P1)  $\forall x \in X, P_{\tau}(x, x) \geq e;$
  - $(P2) \quad \forall x, y, z \in X, \ P_{\tau}(x, y) \otimes P_{\tau}(y, z) \leq P_{\tau}(x, z);$
  - $\begin{array}{l} (\mathrm{P3}) \ \forall x, y \in X, \ P_{\tau}\left(x, y\right) \geq e, \ P_{\tau}\left(y, x\right) \geq e \ \Rightarrow \ x = y; \\ (\mathrm{P4}) \ \forall x, y \in X, \ P\left(x, y\right) = P^{*}\left(y, x\right). \end{array}$

Defining L-T<sub>1</sub> (1) using the dual of  $\leq_{\tau}$  would lead to the same axiom. On the other hand, for L a right-residuated complete po-groupoid, defining the L-T<sub>1</sub> (2) axiom is possible using  $P_{\tau}$  as in 5.5(2), but redefining such an axiom using  $Q_{\tau}$  may not be possible; conversely, for L a left-residuated complete po-groupoid, defining an  $L-T_1(2)$  axiom is possible using  $Q_{\tau}$ , dually to 5.5(2), but redefining such an axiom using  $P_{\tau}$  may not be possible. However, under the standing assumption of this Section that L is a unital quantale with IIA operator \*, it follows from Theorem 4.4(4) conjoined with Lemma 5.2(3) that the L- $T_1$  (2) axiom of 5.5(2) and its "dual" redefined using  $Q_{\tau}$  must be the same axiom. Hence there are essentially two distinct  $T_1$  axioms for many-valued topology coming from the four specialization orders  $\leq_{\tau}$  and  $P_{\tau}$  and their duals.

Corollary 5.6. Let L be a unital quantale equipped with an IIA operator \* and  $(X, \tau)$  be an L-topological space. The following hold:

- (1) L- $T_1$  (1) implies L- $T_0$ , but not conversely.
- (2) L- $T_1$  (2) implies L- $T_0$  axiom, but not conversely.

(3) L- $T_1$  (2) implies L- $T_1$  (1), but not conversely.

*Proof.* The first part of (1) follows from Definition 5.5(1) and Theorem 4.4(1); and the converse fails by Examples 5.4(2). The first part of (2) follows from Definition 5.5(2) and Theorem 4.4(2); and the converse fails by Examples 5.4(1). Finally, the first part of (3) follows by Theorem 5.3(1)(a); and the converse fails by Examples 5.4(1).

Examples for L- $T_1(1)$  and L- $T_1(2)$  spaces given or implied by Examples 5.4 and Corollary 5.6 are augmented by more "canonical" examples in Section 6.

Historical Discussion 5.7 (relationship to other  $T_1$  axioms for manyvalued topology). So far as the authors are aware, the L- $T_1$  (2) axiom of Definition 5.5(2) above is new to the literature for many-valued topology; and it is a matter of future work to carefully compare this axiom with many extant schemes of many-valued separation axioms. The authors are grateful to the referee for bringing to our attention that, on the other hand, the L- $T_1$  (1) axiom of 5.5(1) above is equivalent to the L- $T_1$  axiom proposed by Kubiak in Section 9 of [32] under the condition that L be a complete DeMorgan algebra. Leaving aside the differences between a semiquantale and a complete DeMorgan algebra (with  $\otimes = \wedge$ )—since the DeMorgan (quasi-)complementation plays no explicit role in the Kubiak definition and the tensor plays no explicit role in 5.5(1), we show that these definitions are equivalent under the condition that L be a semiquantale. Suppose  $(X, \tau)$  is an L-topological space; without loss of generality, it may be assumed that this space is L- $T_0$ .

(1) The L-T<sub>1</sub> axiom of Kubiak is temporarily denoted L-T<sub>1</sub> (K) and is defined as follows:  $(X, \tau)$  is L-T<sub>1</sub> (K) if

 $\forall x, y \in X, \ x \neq y \Rightarrow \exists u, v \in \tau, \ u(y) \nleq u(x) \text{ and } v(x) \nleq v(y).$ 

(2) To see that L- $T_1(K) \Rightarrow L$ - $T_1(1)$ , we first note that L- $T_1(K)$  can be rewritten as

 $\forall x, y \in X, \ x \neq y \Rightarrow x \nleq_{\tau} y \text{ and } y \nleq_{\tau} x.$  (K1)

Now let  $x, y \in X$  and suppose  $x \leq_{\tau} y$ . Then by contraposition of (K1), x = y, and reflexivity of  $\leq_{\tau}$  then yields  $y \leq_{\tau} x$ . So L- $T_1(1)$  holds. For L- $T_1(K) \Leftarrow L$ - $T_1(1)$ , we first note that L- $T_1(1)$  states

$$\forall x, y \in X, \ x \leq_{\tau} y \Leftrightarrow y \leq_{\tau} x,$$

which can rewritten as

$$\forall x, y \in X, \ x \nleq_{\tau} y \Leftrightarrow y \nleq_{\tau} x.$$
 (K2)

Now let  $x, y \in X$  and suppose  $x \neq y$ . Then by the antisymmetry of  $\leq_{\tau} (L-T_0 \text{ is assumed, so apply Theorem 4.4(1)})$ , it is the case that  $x \not\leq_{\tau} y$  or  $y \not\leq_{\tau} x$ . But in either case, (K2) yields  $x \not\leq_{\tau} y$  and  $y \not\leq_{\tau} x$ , and by (K1),  $L-T_1(K)$ .  $\Box$ 

- (3) From now on, the Kubiak axiom will also be referred to as  $L-T_1(1)$  and thereby distinguished from  $L-T_1(2)$ .
- (4) It is striking to see two different motivations leading to the same axiom L- $T_1$  (1): the motivation in [32] is to give a version of the L- $T_0$  separation axiom which is "symmetric" in its syntax, while the motivation in 5.5(1) above focuses on symmetry of the specialization ordering  $\leq_{\tau}$  (and its dual). See 7.1 of Section 7 for more discussion on the emerging place of the L- $T_1$  (1) axiom.

In order to bridge to the next section, it is noted by Corollary 5.6 that denying the L- $T_1$  (1) axiom is a strategy for showing asymmetry (as defined at the beginning of this Section) in many-valued topology and for many-valued spectra in particular: if a space or spectrum is asymmetric in the sense of not satisfying this axiom, then it is asymmetric as well in the sense of not satisfying the L- $T_1$  (2) axiom. This means the sequel especially focuses primarily on the L-specialization order  $\leq_{\tau}$ .

#### 6. Many-valued asymmetry and many-valued spectra: Examples

This section adduces and analyzes many example classes of spaces, with special emphasis on many-valued spectra as outlined in Section 3, with respect to the L- $T_1(1)$  axiom of Definition 5.5(1) as a means of evaluating whether such spaces or spectra are symmetric or asymmetric as judged by Definition 5.1.1 above. The primary focus is on the L- $T_1(1)$ axiom vis-a-vis the L- $T_1(2)$  axiom for reasons discussed at the end of section 5—whenever a space fails to be L- $T_1(1)$ , then it necessarily fails to be L- $T_1(2)$  because of Corollary 5.6(3); however, there are example classes dealing directly with the L- $T_1(2)$  axiom, e.g., Example 6.10 below. Canonical examples of spaces satisfying these axioms as well as canonical examples of spaces not satisfying these axioms are adduced and discussed below; these augment the examples given or implied by Examples 5.4 and Corollary 5.6 above.

It should be stressed that whenever an L-spectrum is not L- $T_1(1)$ , then immediately such a space is L-sober and not L- $T_1(1)$ ; recall from Section 3 above that L-sober spaces are precisely the L-spectra. Many of the example classes below are in fact of this character—L-sober and not L- $T_1(1)$ , even if the label L-sober is not mentioned. These results and classes include Theorem 6.5 and Examples 6.6, 6.7, 6.8, part of 6.9,

and part of 6.11. On the other hand, it should also be pointed out that examples are given below of spaces which are L- $T_1(1)$  and not L-sober—see Examples 6.11(8). Finally, there are examples below of spaces which are both L- $T_1(1)$  and L-sober—see Examples 6.9 in regard to the fuzzy real line  $\mathbb{R}(L)$  when L is a complete Boolean algebra.

When dealing with traditional spectra, prime elements of semilocales play an important role, and this notion, extended to semiquantales, plays an important role in this section. Another tensor-dependent notion which is also important in the sequel is tensor positivity. Both of these notions are given in the next definition.

**Definition 6.1**. Let A be a semiquantale.

(1) An element  $c \in A - \{\top\}$  is  $\otimes$ -prime if the following holds:

 $\forall a, b \in A, \ a \otimes b \leq c \Leftrightarrow a \leq c \text{ or } b \leq c.$ 

Note the defining predicate is explicitly a biconditional, a change from the definition of primes for, say, meet semilattices with respect to  $\otimes = \wedge$ ; cf. [5, 28].

- (2)  $Pr_{\otimes}(A)$  is the set of all  $\otimes$ -prime elements of A.
- (3) A or  $\otimes$  has positivity if  $\forall a, b \in A, a, b > \bot \Rightarrow a \otimes b > \bot$ .
- (4) A or  $\otimes$  has co-positivity if  $\forall a, b \in A, a, b < \top \Rightarrow a \otimes b < \top$ .

**Remark 6.1.1.** Useful observations used in the sequel, related to Definition 6.1 conjoined with notions of Section 2 above, include the following:

- (1) Suppose A is integral and  $\otimes$  is isotone. Then:
  - (a) Sufficiency in the predicate of 6.1(1)—the right-to-left implication—holds for each  $c \in A \{\top\}$ .
  - (b) The bottom element  $\perp$  of A is an *annihilator* of  $\otimes$  (i.e.,  $\forall a \in A, a \otimes \perp = \perp \otimes a = \perp$ ).
- (2) Suppose  $\otimes$  has positivity. Then necessity in the predicate of 6.1(1)—the left-to-right implication—holds for  $\perp$ .
- (3) Each semiquantale equipped with  $\otimes$ , given by  $a \otimes b = a$  or given by  $a \otimes b = b$ , has positivity; and each semiquantale having a prime bottom and equipped with  $\otimes = \wedge$  (binary) has positivity.
- (4) The unit interval [0, 1] equipped with the usual ordering and with  $\otimes$  as the arithmetic mean, or with  $\otimes$  given by  $a \otimes b = \frac{a+2b}{3}$ , or with  $\otimes$  as the harmonic mean (given by  $a \otimes b = \sqrt{ab}$ ), or with  $\otimes$  as multiplication has positivity.
- (5) Each complete lattice with  $\otimes = \wedge$  has co-positivity; the unit interval [0, 1] equipped with the usual ordering and with  $\otimes$  as multiplication or Łuksiewicz conjunction has co-positivity; and each of the examples of (4) above has co-positivity.

The following lemma characterizes tensor primes and, in so doing, somewhat mimics the proof of the traditional semilocale case and fixes notations used in the sequel.

**Lemma 6.2.**  $Pr_{\otimes}(A)$  is bijective with  $2pt(A) \equiv SQuant_{\top}(A, 2)$ ; and this bijection is an antitone order-isomorphism.

*Proof.* Let  $c \in Pr_{\otimes}(A)$ . Recall that the principal ideal determined by c, denoted  $\downarrow(c)$ , is defined by

$$\downarrow(c) = \{a \in A : a \le c\}.$$

Now to define a map p from A to 2, it suffices to define its *cokernel*, namely to specify

$$coker(p) := \{a \in A : p(a) = \bot\}.$$

Given  $c \in Pr_{\otimes}(A)$ , put  $p_c : A \to \mathbf{2}$  by

 $coker(p_c) = \downarrow(c).$ 

It needs to be checked that  $p_c$  preserves arbitrary  $\bigvee$  and each of  $\otimes$ ,  $\top$ . For the empty join case, it is immediate that  $p_c(\bot) = \bot$ . The proof of preservation of arbitrary, non-empty joins mimics that for the case when A is a frame [28] and is left to the reader. It now follows that  $p_c$  is isotone.

As for preservation of tensors, let  $a, b \in A$ . Then the condition  $p_c(a \otimes b) = p_c(a) \wedge p_c(b)$  is equivalent to the condition that  $p_c(a \otimes b) = \bot$  if and only if  $p_c(a) \wedge p_c(b) = \bot$ . Assume  $p_c(a \otimes b) = \bot$ . Then  $a \otimes b \leq c$ . But c is  $\otimes$ -prime, and thus  $a \leq c$  or  $b \leq c$ ; so W.L.O.G., let  $a \leq c$ . Then the isotonicity of  $p_c$  implies that

$$\perp \leq p_c(a) \leq p_c(c) = \perp,$$

so that  $p_c(a) \wedge p_c(b) = \bot$ . Conversely, assuming  $p_c(a) \wedge p_c(b) = \bot$ , we have  $p_c(a) = \bot$  or  $p_c(b) = \bot$ ; and W.L.O.G. letting  $p_c(a) = \bot$ , it is the case that  $a \leq c$ . Since c is  $\otimes$ -prime,  $a \otimes b \leq c$ , so that  $p_c(a \otimes b) = \bot$ .

Finally,  $p_c(\top) = \top$ , for if not, then  $p_c(\top) = \bot$ , forcing  $c = \top$ , a contradiction. Hence  $p_c(\top) = \top$ . This finishes the proof that  $p_c \in 2pt(A)$ .

The above paragraphs establish the well-definedness of a map  $\varphi$ :  $Pr_{\otimes}(A) \rightarrow \mathbf{2}pt(A)$  given by

$$\varphi\left(c\right) = p_c.$$

To establish the inverse map  $\psi : Pr_{\otimes}(A) \leftarrow 2pt(A)$ , let  $p \in 2pt(A)$  and put

$$c_p = \bigvee coker(p).$$

Now let  $a, b \in A$ . It is to be checked that  $a \otimes b \leq c_p \Leftrightarrow a \leq c_p$  or  $b \leq c_p$ . First we note that since p preserves arbitrary  $\bigvee$ ,

$$p(c_p) = p\left(\bigvee coker(p)\right) = \bigvee p^{\rightarrow}(coker(p)) = \bigvee \{\bot\} = \bot,$$

which means  $c_p \in coker(p)$ , and thus  $c_p$  is the largest member of coker(p). Now assume  $a \otimes b \leq c_p$ . Then

$$p(a) \wedge p(b) = p(a \otimes b) \leq p(c_p) = \bot,$$

so that  $p(a) = \bot$  or  $p(b) = \bot$ , and hence W.L.O.G.  $p(a) = \bot$ , forcing  $a \le c_p$ . And assuming  $a \le c_p$  or  $b \le c_p$ , say,  $a \le c_p$ , forces

$$p(a \otimes b) = p(a) \wedge p(b) \leq p(a) \leq p(c_p) = \bot$$

which says  $a \otimes b \in coker(p)$  and  $a \otimes b \leq c$ . Hence  $c_p \in Pr_{\otimes}(A)$ , and we have a well-defined map  $\psi : Pr_{\otimes}(A) \leftarrow 2pt(A)$  given by

$$\psi\left(p\right) = c_p.$$

It is left to the reader to check that  $\psi \circ \varphi = id_{Pr\otimes(A)}$  and  $\varphi \circ \psi = id_{2pt(A)}$ , namely that  $c = c_{p_c}$  and  $p_{c_p} = p$ . It follows that  $\varphi : Pr\otimes(A) \to 2pt(A)$  is a bijection.

To see that each of  $\varphi$  and  $\psi$  are antitone, we note that if  $c, d \in Pr_{\otimes}(A)$  with  $c \leq d$ , then  $\downarrow(c) \subset \downarrow(d)$ , so that

$$coker(p_c) \subset coker(p_d),$$

which implies

$$\varphi\left(d\right) = p_d \le p_c = \varphi\left(c\right)$$

On the other hand, if  $p, q \in 2pt(A)$  with  $p \leq q$  with the pointwise order, then

$$coker\left( q\right) \subset coker\left( p\right) ,$$

which forces

$$\psi(q) = c_q = \bigvee coker(q) \le \bigvee coker(p) = c_p = \psi(p).$$

This concludes the proof of the lemma.

For the remainder of this section, we fix a semiquantale L and consider any semiquantale A. Recall the L-spectrum  $LPt(A) = (Lpt(A), (\Phi_L)^{\rightarrow}(A))$  was constructed in Section 3. In keeping with remarks made at the close of Section 5, we focus primarily on the L-specialization order  $\leq_{\tau}$  or, in the case of the L-spectrum, the ordering  $\leq_{(\Phi_L)^{\rightarrow}(A)}$ . In the sequel, it is convenient to use a *reduction of notation*, namely, to denote  $\leq_{(\Phi_L)^{\rightarrow}(A)}$  by  $\leq_{\Phi_L}$  if there is no confusion.

Given  $p, q \in Lpt(A)$ , it is the case that  $p \leq_{\Phi_L} q$  if and only if

$$\forall a \in A, \ \Phi_L(a)(q) \le \Phi_L(a)(p),$$

which is equivalent to saying

$$\forall a \in A, \ q(a) \le p(a).$$

This means that  $p \leq_{\Phi_L} q$  if and only if  $q \leq p$  in the pointwise ordering of  $L^A$ .

**Lemma 6.3.** The L-spectrum  $(Lpt(A), (\Phi_L)^{\rightarrow}(A))$  of A is L-T<sub>1</sub>(1) if and only if

$$\forall p,q \in Lpt\left(A\right), \ q \leq p \Leftrightarrow p \leq q$$

in the pointwise ordering of  $L^A$ .

*Proof.* This follows from the previous paragraph, noting that the L-spectrum is L- $T_0$  (in fact, L-sober) from Theorem 3.5(5) above.

**Definition 6.4.** A semiquantale A is said to have two related distinct  $(\otimes)$ -primes if  $\exists a, b \in Pr_{\otimes}(A)$ ,  $a \leq b$  and  $a \neq b$ .

**Theorem 6.5**. The following statements hold, where L is assumed integral and  $\perp$  is assumed an annihilator for  $\otimes$  in each of L and A:

- (1) The L-spectrum  $(Lpt(A), (\Phi_L)^{\rightarrow}(A))$  of A fails to be L- $T_1(1)$  if L is consistent and A has two related distinct primes.
- (2) The 2-spectrum  $(2pt(A), (\Phi_2)^{\rightarrow}(A))$  of A fails to be 2-T<sub>1</sub>(1) if and only if A has two related distinct primes.

*Proof.* Ad (1). Assume L is consistent and A has two related distinct primes. The consistency of L implies that  $\bot < \top$  and that  $\{\bot, \top\}$  as a subset of L is order-isomorphic to 2; and hence we let 2 denote  $\{\bot, \top\}$ . It should be noted that 2 is closed with respect to the restriction of  $\otimes$  to 2, and that 2 with the relative ordering and the restriction of  $\otimes$  is an integral semiquantale, in fact a sub-(integral-)semiquantale of L. It should also be noted that the binary meet for 2, induced by the relative order, coincides with the tensor  $\otimes$  restricted to 2. Finally, it is the case from Lemma 6.2 that  $Pr_{\otimes}(A)$  is bijective with  $2pt(A) \equiv \mathbf{SQuant}_{\top}(A, \mathbf{2})$  via the notion of prime principal ideals.

The assumption that A has two related distinct primes means that  $\exists c, d \in Pr_{\otimes}(A)$  with  $c \leq d$  and  $c \neq d$ . Applying Lemma 6.2 and the bijection  $\varphi$  of its proof, put

$$p_{c} = \varphi(c), \quad p_{d} = \varphi(d).$$

It follows from the antitonicity of  $\varphi$  that  $p_d \leq p_c$ ; and it follows from the injectivity of  $\varphi$  that  $p_d \neq p_c$ . Now the inclusion map  $\hookrightarrow : \mathbf{2} \to L$ 

is a semiquantale morphism—this is a consequence of **2** being a subsemiquantale of L. Put  $p, q : A \to L$  by

$$p = \hookrightarrow \circ p_c, \quad q = \hookrightarrow \circ p_d.$$

Then it follows that  $p, q \in Lpt(A)$ , and with the pointwise ordering we have

$$q \leq p, \quad p \nleq q.$$

Lemma 6.3 now says the L-spectrum LPt(A) fails to be L- $T_1(1)$ .

Ad (2). The proof of sufficiency is included within the proof of (1) above. As for necessity, the assumption that 2Pt(A) fails to be  $2-T_1(1)$  means, from Lemma 6.3, that the condition

$$\forall p,q \in Lpt(A), q \leq p \Leftrightarrow p \leq q \text{ (pointwise ordering)}$$

fails. Hence, W.L.O.G.,  $\exists p, q \in Lpt(A)$ ,  $q \leq p$  and  $p \nleq q$ . Apply the inverse mapping  $\psi$  of  $\varphi$  from the proof of Lemma 6.2 to p, q to yield  $a_p, a_q \in Pr_{\otimes}(A)$ . The antitonicity and injectivity of  $\psi$  implies  $a_p \leq a_q$  and  $a_p \neq a_q$ . Hence A has two related distinct primes.

As will be demonstrated by corollaries and examples in the sequel, the condition of related distinct primes of Definition 6.4 and Theorem 6.5 can be convenient to apply. For the sequel, L is assumed consistent and integral, and  $\perp$  is assumed an annihilator in both L and A.

**Examples 6.6**. Let  $\mathfrak{T}_{cof}$  be the usual cofinite topology on  $\mathbb{R}$ . The following hold:

- (1) The traditional spectrum  $Pt(\mathfrak{T}_{cof})$  is not  $T_1$  (in the usual sense).
- (2) For L consistent and integral, the L-spectrum  $LPt(\mathfrak{T}_{cof})$  is not L- $T_1(1)$ .

Proof. Statement (1) is in [28], p. 44, as an example of a space which is sober but not  $T_1$ . As for statement (2), it is straightforward to show, using  $G_{\chi}$ , that  $Pt(\mathfrak{T}_{cof}) \equiv (pt(\mathfrak{T}_{cof}), \Phi^{\rightarrow}(\mathfrak{T}_{cof}))$  not being  $T_1$  implies the 2-spectrum  $2Pt(\mathfrak{T}_{cof}) \equiv (2pt(\mathfrak{T}_{cof}), (\Phi_2)^{\rightarrow}(\mathfrak{T}_{cof}))$  is not  $2-T_1(1)$ . It follows by Theorem 6.5(2) that  $\mathfrak{T}_{cof}$  has two related distinct primes (and this can also be checked directly). Hence 6.5(1) says the L-spectrum  $LPt(\mathfrak{T}_{cof})$  is not  $L-T_1(1)$ .

**Examples 6.7**. Let *L* be consistent and integral, and let *A* be a complete chain with  $\otimes$  isotone. The following hold:

- (1) If A has at least three elements and  $\otimes = \wedge$  (binary), then LPt(A) is not L- $T_1(1)$ . In particular this holds for A = [0, 1] with  $\otimes = \wedge$  (binary).
- (2) If A is integral, has positivity, and comprises exactly three elements, then LPt(A) is not  $L-T_1(1)$ .

*Proof.* For (1), there are  $\alpha < \beta < \gamma$  in A, so, in particular, we have  $\perp < \beta < \top$ , and thus, since  $\otimes = \wedge$ , both  $\perp$  and  $\beta$  are distinct related primes; so apply Theorem 6.5(1).

For (2), let  $A = \{\perp, \alpha, \top\}$ . Then by Remark 6.1.1((1)(a),(2)),  $\perp$  is  $\otimes$ -prime. It is claimed that  $\alpha$  is also  $\otimes$ -prime, and to see this it must be shown that

$$\forall a, b \in A, \ a \otimes b \leq \alpha \Leftrightarrow a \leq \alpha \text{ or } b \leq \alpha.$$

Now ( $\Leftarrow$ ) of the predicate always holds by Remark 6.1.1(1)(a). And to show ( $\Rightarrow$ ), there are nine, distinct, non-redundant cases which are generally trivial, with the possible exception of the case when  $a = \top = b$ , in which case the integrality of A implies that the inequality

$$\top \otimes \top \leq \alpha$$

is false; so that for all cases, the implication  $(\Rightarrow)$  is true. Thus  $\alpha$  is also  $\otimes$ -prime, and (2) now follows from Theorem 6.5(1).

**Examples 6.8.** As an extension of Corollary 6.7, let [0, 1] be equipped with the multiplication t-norm, i.e.,  $\otimes = \cdot$  and  $\top = 1$ ; and let  $X = \{x, y\}$  and  $\tau = \{\underline{0}, \underline{1}, u, u^2, u^3, \ldots\}$ , where, say,

$$u(x) = 2/3, u(y) = 1/3.$$

Then  $(X, \tau)$  is an *L*-topological space about which the following statements hold:

- (1)  $(X, \tau)$  is *L*-*T*<sub>0</sub>, but fails to be *L*-*T*<sub>1</sub>(1)—by inspection,  $x \leq_{\tau} y$ , but  $y \not\leq_{\tau} x$ .
- (2) The *L*-spectrum  $LPt(\tau)$  is  $L-T_0$ , but fails to be  $L-T_1(1)$ —by inspection,  $\tau$  has two related distinct  $\otimes$ -primes, and so Theorem 6.5(1) applies.
- (3) Another way to analyze the second claim of (2) is as follows: first,  $(X, \tau)$  is L- $T_0$ , so  $(X, \tau)$  is L-homeomorphic via  $\Psi_L$  to an L-subspace of  $LPt(\tau)$ —Theorem 3.5(3) above; second, L- $T_1(1)$ is hereditary with respect to the L-subspace topology (taken in the sense of subobjects of L-**Top**—see [55]); third, L- $T_1(1)$  is an L-topological invariant—Lemma 6.9.4 below;  $(X, \tau)$  fails to be L- $T_1(1)$  as noted in (1); and hence it follows that  $LPt(\tau)$  fails to be L- $T_1(1)$ .  $\Box$

**Examples 6.9.** This example class is based upon the *L*-fuzzy real line  $\mathbb{R}(L)$  and *L*-fuzzy unit interval  $\mathbb{I}(L)$  of [26, 16] and related *L*-topological spaces, for which [16, 23, 26, 30, 31, 32, 33, 34, 54, 56, 57, 58] may be used as general sources. The base *L* in this inventory of examples is a consistent, complete DeMorgan algebra— $\otimes = \wedge, e = \top, * = id_L$ , and  $': L \to L$  is an antitone involution. The first infinite distributive law

or Boolean negation is only assumed when explicitly needed and stated. Recall:

 $\mathbb{R}_L = \{\lambda : \mathbb{R} \to L \mid \lambda \text{ is antitone, } \lambda((-\infty) +) = \top, \ \lambda((+\infty) -) = \bot\};$ an equivalence relation is put on  $\mathbb{R}_L$  by

$$\lambda \approx \mu \iff \forall t \in \mathbb{R}, \ \lambda \left( t + \right) = \mu \left( t + \right),$$

or, equivalently,

$$\lambda \approx \mu \iff \forall t \in \mathbb{R}, \ \lambda\left(t-\right) = \mu\left(t-\right),$$

(see Claim 6.9.1 below); and the carrier set for the *L*-fuzzy real line is

$$\mathbb{R}(L) = \mathbb{R}_L / \approx L$$

Now for  $t \in \mathbb{R}$ , put  $L_t, R_t : \mathbb{R}(L) \to L$  by

$$L_t [\lambda] = (\lambda (t-))', \quad R_t [\lambda] = \lambda (t+).$$

These operators, respectively, determine a *left-hand* L-topology  $\tau_l(L)$  and a *right-hand* L-topology  $\tau_r(L)$  on  $\mathbb{R}(L)$ , namely

$$\tau_l\left(L\right) = \left\{L_t : t \in \mathbb{R}\right\} \cup \left\{\underline{\perp}, \underline{\top}\right\}, \quad \tau_r\left(L\right) = \left\{R_t : t \in \mathbb{R}\right\} \cup \left\{\underline{\perp}, \underline{\top}\right\},$$

and thereby determine the standard L-topology  $\tau(L)$  on  $\mathbb{R}(L)$ , given by

$$\tau\left(L\right) = \tau_{l}\left(L\right) \vee \tau_{r}\left(L\right)$$

using the fact that *L*-**Top** is a topological construct and, in particular, has complete fibres. Thus we have three spaces:  $\mathbb{R}_l(L) \equiv (\mathbb{R}(L), \tau_l(L))$ ,  $\mathbb{R}_r(L) \equiv (\mathbb{R}(L), \tau_r(L)), \mathbb{R}(L) \equiv (\mathbb{R}(L), \tau(L))$ , the latter called the *L*-fuzzy real line. We note that  $\mathbb{R}$  injects into  $\mathbb{R}(L)$  via the mapping jgiven by the correspondence  $r \mapsto [\lambda_r]$ , where

$$\lambda_r \left( t \right) = \begin{cases} \top, & t < r \\ \bot, & r < t \end{cases}$$

It can be seen that the notation " $L_t$ " and " $R_t$ ", as well as the associated monikers "left" and "right", are appropriate since

$$(L_t)_{|j^{\to}(\mathbb{R})} = \chi_{(-\infty,t)}, \quad (R_t)_{|j^{\to}(\mathbb{R})} = \chi_{(t,+\infty)}.$$

The *L*-fuzzy unit interval  $\mathbb{I}(L)$  has as carrier set that subset of  $\mathbb{R}(L)$ , also denoted  $\mathbb{I}(L)$ , satisfying certain boundary conditions—

$$\mathbb{I}(L) = \left\{ [\lambda] \in \mathbb{R}(L) : \bigwedge_{t < 0} \lambda(t) = \top, \bigvee_{t > 1} \lambda(t) = \bot \right\}$$

—and equipped with the  $L\text{-}subspace \ topology \ \tau \ (L)_{\mathbb{I}(L)} \ \text{from} \ \tau \ (L) \ \text{on} \ \mathbb{I} \ (L) \ ,$  namely

$$\tau\left(L\right)_{\mathbb{I}(L)} = \left\{ u \mid \mathbb{I}(L) : \mathbb{I}\left(L\right) \to L \mid u \in \tau\left(L\right) \right\}.$$

We also need an ordering on  $\mathbb{R}(L)$  which, in the sense of the embedding j above, extends that of  $\mathbb{R}$  to  $\mathbb{R}(L)$ . To construct this ordering, we need the following claim, whose statement and proof can be found in [30, 32]:

**Claim 6.9.1.** Let  $[\lambda], [\mu] \in \mathbb{R}(L)$ . Then  $[\forall t \in \mathbb{R}, \lambda(t+) \le \mu(t+)] \Leftrightarrow [\forall t \in \mathbb{R}, \lambda(t-) \le \mu(t-)]$ .

**Definition 6.9.2.** Put  $\leq$  on  $\mathbb{R}(L)$  by stipulating,  $\forall [\lambda], [\mu] \in \mathbb{R}(L)$ ,

 $[\lambda] \leq [\mu] \iff [\forall t \in \mathbb{R}, \ \lambda \ (t+) \leq \mu \ (t+)].$ 

Because of Claim 6.9.1,  $\leq$  may be equivalently defined on  $\mathbb{R}(L)$  by stipulating,  $\forall [\lambda], [\mu] \in \mathbb{R}(L)$ ,

$$[\lambda] \leq [\mu] \iff [\forall t \in \mathbb{R}, \ \lambda \ (t-) \leq \mu \ (t-)].$$

**Proposition 6.9.3.** The binary relation defined in Definition 6.9.2 is well-defined and a partial order on  $\mathbb{R}(L)$ . Further, for L = 2, it is a total order; slightly restated,  $\leq$  when restricted to  $j^{\rightarrow}(\mathbb{R})$  yields an orderisomorphism of  $j^{\rightarrow}(\mathbb{R})$  with  $\mathbb{R}$  and hence is a total order; similar statements hold for  $\mathbb{I}(L)$ .

**Extended Discussion 6.9.4**. With the above preparations, the following can now be said:

- (1) Each of  $\mathbb{R}_{l}(L)$ ,  $\mathbb{R}_{r}(L)$ ,  $\mathbb{R}(L)$  is L- $T_{0}$ —see [33, 54, 56], with similar statements holding for  $\mathbb{I}(L)$ .
- (2)  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are L- $T_1(1)$ . There are (at least) two proofs of this claim; only the case for  $\mathbb{R}(L)$  is discussed.
  - (a) The first proof is due to the referee and primarily based on [32]. For this proof we note that L is a complete DeMorgan algebra, precisely the setting of [32]; by Discussion 5.7 above, the L- $T_1$  (1) axiom is the same as the L- $T_1$  axiom given in [32]. Now [32] shows that the L- $T_2$  axiom of [32] implies L- $T_1$  (1) and that  $\mathbb{R}(L)$  is L- $T_2$ ; it is also shown in [23] that  $\mathbb{R}(L)$  satisfies the L- $T_2$  axiom of [22], which is stronger than the L- $T_2$  axiom of [32]. All of this is more than enough to imply that  $\mathbb{R}(L)$  is L- $T_1(1)$ .  $\Box$
  - (b) The second proof notes that  $\mathbb{R}(L)$  is already L- $T_0$  by (1) and proceeds to show directly that the crisp specialization ordering  $\leq_{\tau(L)}$  on  $\mathbb{R}(L)$  is symmetric. Let  $[\lambda], [\mu] \in \mathbb{R}(L)$ and suppose  $[\lambda] \leq_{\tau(L)} [\mu]$ . This means

$$\forall u \in \tau (L), \ u [\mu] \le u [\lambda] \quad (\bullet \bullet)$$

with respect to the ordering on L. Our first observation, letting the u's in  $(\bullet \bullet)$  be right-handed subbasic open sets, is that

$$\forall t \in \mathbb{R}, \ R_t \left[ \mu \right] \le R_t \left[ \lambda \right],$$

namely,

$$\forall t \in \mathbb{R}, \ \mu(t+) \le \lambda(t+).$$

It follows from Definition 6.9.2 that  $[\mu] \leq [\lambda]$  in  $\mathbb{R}(L)$ . Our second observation, letting the *u*'s in (••) be left-handed subbasic open sets, is that

$$\forall t \in \mathbb{R}, \ L_t\left[\mu\right] \leq L_t\left[\lambda\right],$$

namely,

$$\forall t \in \mathbb{R}, \ \left(\mu\left(t-\right)\right)' \leq \left(\lambda\left(t-\right)\right)', \quad \forall t \in \mathbb{R}, \ \lambda\left(t-\right) \leq \mu\left(t-\right).$$

It follows from Definition 6.9.2 that  $[\lambda] \leq [\mu]$  in  $\mathbb{R}(L)$  and Proposition 6.9.3 applies to say  $[\lambda] = [\mu]$  and hence  $[\mu] \leq_{\tau(L)} [\lambda]$ . A symmetric argument establishes that  $[\mu] \leq_{\tau(L)} [\lambda] \Rightarrow [\lambda] = [\mu] \Rightarrow [\lambda] \leq_{\tau(L)} [\mu]$ . It follows  $\leq_{\tau}$  is symmetric and that  $\mathbb{R}(L)$  is L- $T_1(1)$ .  $\Box$ 

(c) More is said about Hausdorff conditions in Section 7 below.

(3) Each of  $\mathbb{R}_l(L)$ ,  $\mathbb{R}_r(L)$  fails to be L- $T_1(1)$ . For the right-handed case, consider  $[\lambda_1]$ ,  $[\lambda_2]$  (the injection by j of the crisp numbers 1, 2). The claim is that  $[\lambda_2] \leq_{\tau_r(L)} [\lambda_1]$ . To verify this claim, it must be shown that

$$\forall u \in \tau_r (L), \ u [\lambda_1] \le u [\lambda_2].$$

This is trivial for the open sets  $\underline{\perp}, \underline{\top}$ . Now let  $R_t \in \tau_r(L)$  for  $t \in \mathbb{R}$ . Then

$$R_t [\lambda_1] = \lambda_1 (t+) = \begin{cases} \top, & t < 1 \\ \bot, & t \ge 1 \end{cases} \leq \begin{cases} \top, & t < 2 \\ \bot, & t \ge 2 \end{cases} = \lambda_2 (t+) = R_t [\lambda_2].$$

So  $[\lambda_2] \leq_{\tau(L)} [\lambda_1]$ . However, it is to be noted that

$$R_{3/2}[\lambda_2] = \top \nleq \bot = R_{3/2}[\lambda_1],$$

which implies that  $[\lambda_1] \not\leq_{\tau(L)} [\lambda_2]$ . It follows that  $\mathbb{R}_r(L)$  fails to be L- $T_1(1)$ . A similar verification establishes that  $\mathbb{R}_l(L)$  fails to be L- $T_1(1)$ .

(4) Let the traditional left-hand and right-hand topologies on  $\mathbb{R}$  be, respectively, denoted by  $\mathfrak{T}_l$  and  $\mathfrak{T}_r$ , where

$$\mathfrak{T}_{l} = \{(-\infty, t) : t \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}, \quad \mathfrak{T}_{r} = \{(t, +\infty) : t \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

Then (1) and (3) extend the fact that both  $(\mathbb{R}, \mathfrak{T}_l)$  and  $(\mathbb{R}, \mathfrak{T}_r)$ are  $T_0$  but not  $T_1$ . And noting that the standard topology  $\mathfrak{T}$  on  $\mathbb{R}$  is given by

$$\mathfrak{T} = \mathfrak{T}_l \vee \mathfrak{T}_r,$$

then (2) extends the fact that  $(\mathbb{R}, \mathfrak{T})$  is  $T_1$ .

(5) It is the case that  $\tau_l(L)$  is order-isomorphic to  $\mathfrak{T}_l$  and  $\tau_l(L)$  is order-isomorphic to  $\mathfrak{T}_r$ ; and it therefore follows by the functoriality of *LPt* that *LPt*( $\tau_l(L)$ ) is *L*-homeomorphic to *LPt*( $\mathfrak{T}_l$ ) and that *LPt*( $\tau_r(L)$ ) is *L*-homeomorphic to *LPt*( $\mathfrak{T}_r$ ).

To see the order-isomorphism in the left-handed case, put  $\varphi$ :  $\tau_l(L) \to \mathfrak{T}_l$  as follows:

$$\begin{split} \varphi\left(\underline{\bot}\right) &= \varnothing, \\ \varphi\left(\underline{\top}\right) &= \mathbb{R}, \\ \varphi\left(L_t\right) &= \left(-\infty, \, t\right). \end{split}$$

To see that  $\varphi$  is well-defined, suppose  $L_t = L_s$ . Then  $\forall [\lambda] \in \mathbb{R}(L)$ ,  $(\lambda(t-))' = (\lambda(s-))'$ , or

$$\forall [\lambda] \in \mathbb{R}(L), \ \lambda(t-) = \lambda(s-) \quad (\bullet \bullet \bullet)$$

Instantiating with the left-continuous representative  $\lambda_t$  corresponding to  $t \in \mathbb{R}$  yields

$$\lambda_t \left( s \right) = \lambda_t \left( t \right) = \top,$$

which forces  $s \leq t$ . Now instantiating  $(\bullet \bullet \bullet)$  with the left-continuous representative  $\lambda_s$  corresponding to  $s \in \mathbb{R}$  yields

$$\lambda_{s}(t) = \lambda_{s}(s) = \top,$$

which forces  $t \leq s$ . Hence t = s and  $(-\infty, t) = (-\infty, s)$ . Trivially,  $(-\infty, t) = (-\infty, s)$  implies t = s, which implies  $L_t = L_s$ , so  $\varphi$  is injective. The surjectivity of  $\varphi$  is clear by inspection, so  $\varphi$  is bijective. As for  $\varphi$  being isotone, suppose  $L_t \leq L_s$ . Then it follows that

$$\forall [\lambda] \in \mathbb{R}(L), \ \lambda(t-) \ge \lambda(s-) \quad (\bullet \bullet \bullet \bullet)$$

Instantiating  $(\bullet \bullet \bullet \bullet)$  with the left-continuous representative  $\lambda_s$  corresponding to  $s \in \mathbb{R}$  yields

$$\lambda_{s}(t) \geq \lambda_{s}(s) = \top,$$

forcing  $\lambda_s(t) = \top$ ,  $t \leq s$ , and  $(-\infty, t) \subset (-\infty, s)$ . As for the isotonicity of  $\varphi^{-1}$ , we note  $(-\infty, t) \subset (-\infty, s)$  implies  $t \leq s$ , and it can be checked that this implies  $L_t \leq L_s$  using the antitonicity of ' and of each representative  $\lambda$ . The order-isomorphism in the right-handed case is analogous and somewhat simpler and left

to the reader. These two order-isomorphisms underlie the orderisomorphism of the proof of Theorem 3 of [26] between  $\tau$  (*L*) and  $\mathfrak{T}$  when *L* is a complete Boolean algebra—see the proof of Lemma 6.9.5 below.

- (6) As a consequence of (5) and the functoriality of LPt, it follows that  $\tau_l(L)$  and  $\tau_r(L)$  are locales,  $LPt(\tau_l(L))$  is L-homeomorphic to  $LPt(\mathfrak{T}_l)$ , and  $LPt(\tau_r(L))$  is L-homeomorphic to  $LPt(\mathfrak{T}_r)$ .
- (7) Each of  $\tau_l(L)$ ,  $\mathfrak{T}_l$ ,  $\tau_r(L)$ ,  $\mathfrak{T}_r$  is a complete chain with at least three elements. Hence by Corollary 6.7 each of the *L*-spectra mentioned in (6) fails to be L- $T_1(1)$ . These four spectra touch on the issue of *L*-topological invariants presented in Corollary 4.8 above and utilized in (8) below.
- (8) In comparison with the left-handed and right-handed cases, what can be said about the *L*-spectrum of the standard topology  $\mathfrak{T}$ on  $\mathbb{R}$  or of the subspace topology  $\mathfrak{T}_{\mathbb{I}}$  on [0,1] in regard to being *L*-*T*<sub>1</sub> (1)? Note these particular *L*-spectra,

$$\mathbb{R}^{*}(L) \equiv LPt\left(\mathfrak{T}\right) = \left(Lpt\left(\mathfrak{T}\right), \left(\Phi_{L}\right)^{\rightarrow}\left(\mathfrak{T}\right)\right),$$

$$\mathbb{I}^{*}(L) \equiv LPt\left(\mathfrak{T}_{\mathbb{I}}\right) = \left(Lpt\left(\mathfrak{T}_{\mathbb{I}}\right), \left(\Phi_{L}\right)^{\rightarrow}\left(\mathfrak{T}_{\mathbb{I}}\right)\right),$$

are, respectively, referred to as the alternative L-fuzzy real line and alternative L-fuzzy unit interval. So, restated, what can be said about  $\mathbb{R}^*(L)$  and  $\mathbb{I}^*(L)$  in regard to being L- $T_1(1)$ ? A full resolution of this question is the subject of ongoing work by the authors, but a significant partial case can here be given. It follows from Lemma 6.9.5 below that  $\mathbb{R}^*(L)$  [ $\mathbb{I}^*(L)$ ] is L-homeomorphic to  $\mathbb{R}(L)$  [ $\mathbb{I}(L)$ , respectively] whenever L is a complete Boolean algebra. Now it follows from Corollary 4.8 above that L- $T_0$  and each of L- $T_1(1)$  and L- $T_1(2)$  (and their negations) are L-topological invariants. Also, from (2) above,  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are L- $T_1(1)$ . Hence for L a complete Boolean algebra,  $\mathbb{R}^*(L)$  and  $\mathbb{I}(L)$  are L- $T_1(1)$ .

**Lemma 6.9.5.** Let L be a DeMorgan frame. Then  $\mathbb{R}^*(L)$  is L-homeomorphic to  $\mathbb{R}(L)$  if and only if L is a complete Boolean algebra if and only if  $\mathbb{I}^*(L)$  is L-homeomorphic to  $\mathbb{I}(L)$ .

*Proof.* Only the case for the two real lines is proved; the unit intervals case is similar and left to the reader. It is known by the Meßner Lemma [43] that for L a DeMorgan frame,  $\mathbb{R}(L)$  is L-sober if and only if L is a complete Boolean algebra. Now if  $\mathbb{R}^*(L)$  is L-homeomorphic to  $\mathbb{R}(L)$ , then the L-sobriety of  $\mathbb{R}^*(L)$ —it is an L-spectrum and all L-spectra are sober (Section 3 above)—forces  $\mathbb{R}(L)$  to be L-sober since L-sobriety is an L-topological

invariant. So the Meßner Lemma says L is a complete Boolean algebra.

For the converse direction, Theorem 3 of [26] says that  $\tau(L)$  is order-isomorphic to  $\mathfrak{T}$ : this result of Hutton states that L is assumed to be a completely distributive Boolean algebra, but the proof only uses the first infinite distributive law (sometimes called complete distributivity in older writings). It follows by the functoriality of LPt (Section 3 above) that  $LPt(\tau(L))$  is L-homeomorphic to  $LPt(\mathfrak{T}) \equiv \mathbb{R}^*(L)$ . But since L is a complete Boolean algebra,  $\mathbb{R}(L)$  is L-homeomorphic to  $LPt(\tau(L))$ . Hence  $\mathbb{R}^*(L)$  is L-homeomorphic to  $\mathbb{R}(L)$ .  $\Box$ 

- (9) It follows from (2, 8) above that for L a complete Boolean algebra,  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are both L- $T_1(1)$  and L-sober. This is an important conjunction of properties, on which more is said in Section 7 below.
- (10) The converse direction of Lemma 6.9.5 was stated and proved as Application 2.15.8 in [56].  $\hfill \Box$

**Examples 6.10**. These examples are suggested by parallel to Example 6.11 below and concern the adjunction  $M_L \dashv G_{\chi}$  between *L*-**Top** and **Top**, where  $M_L$  is the *L*-indexed "Martin" functor stemming from [42] and  $G_{\chi}$  is the "characteristic" functor briefly introduced in the second to last paragraph of Section 3. We are concerned with the generation of L- $T_1(1)$ , non-L- $T_1(1)$ , L- $T_1(2)$ , and non-L- $T_1(2)$  spaces using this adjunction. When working with L- $T_1(1)$  issues, L is a semiquantale unless stated otherwise; when working with L- $T_1(2)$  issues, L is an integral IIA quantale unless stated otherwise. Some preparation is needed.

(1) The categorical functors  $M_L$ ,  $G_{\chi}$  are constructed by first setting up isotone mappings  $G_{\chi}$ ,  $M_L$  between fibres in **Top** and *L*-**Top** over the same carrier set. Let X be a set, let  $\mathbb{T}(X)$  and  $\mathbb{T}_L(X)$ be the respective complete fibres on X from **Top** and *L*-**Top**, and put

$$G_{\chi} : \mathbb{T}(X) \to \mathbb{T}_{L}(X) \quad \text{by} \quad G_{\chi}(\mathfrak{T}) = \{\chi_{U} : U \in \mathfrak{T}\},\$$
$$M_{L} : \mathbb{T}(X) \leftarrow \mathbb{T}_{L}(X) \quad \text{by} \quad M_{L}(\tau) = \{U \subset X : \chi_{U} \in \tau\},\$$

where in these definitions

$$\chi_U : X \to L$$
 by  $\chi_U (x) = \begin{cases} \top, x \in U \\ \bot, x \notin U \end{cases}$ 

Properties for the fibre maps  $G_{\chi}$ ,  $M_L$  are now listed which will be used in the sequel:

(a)  $G_{\chi}$ ,  $M_L$  are isotone as mappings between  $\mathbb{T}(X)$  and  $\mathbb{T}_L(X)$ .

- (b)  $G_{\chi}(M_L(\tau)) \subset \tau.$
- (c)  $M_L(G_{\chi}(\mathfrak{T})) = \mathfrak{T}.$
- (d)  $G_{\chi} \dashv M_L$  as fibre maps, and this adjunction is an isoreflection.
- (2) The categorical functors  $M_L : L$ -**Top**  $\to$  **Top** and  $G_{\chi} : L$ -**Top**  $\leftarrow$  **Top** can now be defined using the fibre maps  $G_{\chi}, M_L$  of (1):

$$M_L(X,\tau) = (X, M_L(\tau)), \quad M_L(f) = f_{\mathcal{I}}$$

$$G_{\chi}(X,\mathfrak{T}) = (X, G_{\chi}(\mathfrak{T})), \quad G_{\chi}(f) = f.$$

These correspondences have the following properties needed below:

- (a)  $M_L : L$ -**Top**  $\rightarrow$  **Top** and  $G_{\chi} : L$ -**Top**  $\leftarrow$  **Top** are concrete faithful functors.
- (b)  $G_{\chi}$  is an embedding; it is a categorical isomorphism if L = 2.
- (c)  $G_{\chi}$  reflects lifted morphisms, but  $M_L$  need not—this terminology comes from [8] and the notion comes from [52] and its references.
- (d)  $M_L \dashv G_{\chi}$ , and this adjunction is a monoreflection—in the sense that  $(G_{\chi} M_L)^{\rightarrow} (L\text{-}\mathbf{Top})$  is a monoreflective subcategory of *L*-**Top**, and an isocoreflection—in the sense that  $(M_L G_{\chi})^{\rightarrow}$  (**Top**) is a isocoreflective subcategory of **Top**.

With these preparations in hand, the issue of identifying or generating L- $T_1(1)$ , non-L- $T_1(1)$ , L- $T_1(2)$ , and non-L- $T_1(2)$  spaces via these functors can be addressed.

- (3) Concerning the behavior of  $G_{\chi}$  and  $M_L$  with respect to L- $T_0$  and L- $T_1(1)$ , the following hold:
  - (a)  $G_{\chi}$  both preserves  $T_0$  and reflects L- $T_0$  in the sense that  $(X, \mathfrak{T})$  is  $T_0$  if and only if  $G_{\chi}(X, \mathfrak{T})$  is L- $T_0$ .
  - (b) G<sub>χ</sub> both preserves T<sub>1</sub> and reflects L-T<sub>1</sub> (1) in the sense that (X, ℑ) is T<sub>1</sub> if and only if G<sub>χ</sub> (X, ℑ) is L-T<sub>1</sub> (1).
    Proof of (a, b) Both (a) and (b) follow immediately from

 $Proof \ of \ (a, b).$  Both (a) and (b) follow immediately from the observation that

 $\forall U \in \mathfrak{T}, \ [y \in U \Rightarrow x \in U] \Leftrightarrow [\chi_U(y) \le \chi_U(x)]. \quad \Box$ 

- (c)  $M_L$  preserves L- $T_0$ , in the sense that  $(X, \tau)$  is L- $T_0$  implies  $M_L(X, \tau)$  is  $T_0$ , if L = 2; the converse direction holds if L is consistent and co-positive (Definition 6.1(3) above).  $M_L$  reflects  $T_0$  in the sense that  $M_L(X, \tau)$  is  $T_0$  implies  $(X, \tau)$  is L- $T_0$ , so that  $M_L$  preserves the lack of  $T_0$  separation.
- (d)  $M_L$  preserves L- $T_1(1)$ , in the sense that  $(X, \tau)$  is L- $T_1(1)$  implies  $M_L(X, \tau)$  is  $T_1$ , if L = 2; the converse direction holds

if L is consistent and co-positive.  $M_L$  reflects  $T_1$  in the sense that  $M_L(X,\tau)$  is  $T_1$  implies  $(X,\tau)$  is L- $T_1(1)$ , so that  $M_L$ preserves asymmetry or the lack of L- $T_1(1)$  separation. *Proof of* (c, d). Clearly

 $\begin{bmatrix} (\forall U \in M_L(\tau), \ y \in U \Leftrightarrow x \in U) \Rightarrow x = y \end{bmatrix} \Rightarrow \\ \begin{bmatrix} (\forall u \in \tau, \ u(y) = u(x)) \Rightarrow x = y \end{bmatrix},$ 

so the second part of (c) holds. Now for the first part of (c), it is trivial that if L = 2, then the implication of the above display reverses. But if  $|L| \geq 3$ , this implication does not reverse: let  $\alpha \in L - \{\bot, \top\}$  and let  $X = \{x, y\}$  be equipped with  $\tau = \langle \langle \{\underline{\bot}, u, \underline{\top}\} \rangle \rangle$ , where  $u(x) = \alpha, u(y) = \bot$ ; then  $(X, \tau)$  is L- $T_0$ , but  $M_L(X, \tau)$  is not  $T_0$  by the copositivity of L; cf. proof of Proposition 6.10.3 below. Similar arguments establish (d), but cf. Proposition 6.10.3 in (5) below in combination with Corollary 5.6 above; and cf. the reflectivity claim of (d) with Proposition 6.10.4 in (5) below in combination with Corollary 5.6 above.

- (4) The behavior of  $G_{\chi}$  and  $M_L$  with respect to L- $T_1(2)$ , for L an integral IIA quantale, is suprisingly delicate. The issue is whether  $P_{\tau}$  is both L-antisymmetric and L-symmetric. In this part of Examples 6.10, we first consider whether  $G_{\chi}$  "strongly" preserves  $T_1$  in the sense that  $(X, \mathfrak{T})$  is  $T_1$  implies  $G_{\chi}(X, \mathfrak{T})$  is L- $T_1(2)$ , and then consider whether  $G_{\chi}$  reflects L- $T_1(2)$  in the sense that  $(X, \mathfrak{T})$  is  $T_1$  whenever  $G_{\chi}(X, \mathfrak{T})$  is L- $T_1(2)$ . We collect a few needed lattice-theoretic facts:
  - (a) For any IIA operator  $^*: L \to L$  the symmetry condition

$$P_{G_{\chi}(\mathfrak{T})}(x,y) = P^*_{G_{\chi}(\mathfrak{T})}(y,x)$$

simplifies to

$$P_{G_{\chi}(\mathfrak{T})}(x,y) = P_{G_{\chi}(\mathfrak{T})}(y,x)$$
:

this is because  $^* : L \to L$  is an order-isomorphism [9], and hence  $\bot^* = \bot$ ,  $\top^* = \top$ .

(b) In L, ⊥ acts as an annihilator for ⊗: this follows from the infinite distributivity of ⊗ over ∨ in the empty-indexed case.
(c) By the integrality of L, ⊤ is the identity for ⊗.

**Proposition 6.10.1.** For *L* an IIA integral quantale,  $G_{\chi}$  strongly preserves  $T_1$  in the sense that  $(X, \mathfrak{T})$  is  $T_1$  implies  $G_{\chi}(X, \mathfrak{T})$  is  $L-T_1(2)$ .

*Proof.* Let  $(X, \mathfrak{T})$  be a  $T_1$  topological space and let  $x, y \in X$ . Because of Theorem 4.4(4), the matter of *L*-antisymmetry of  $P_{G_{\chi}(\mathfrak{T})}$  has been dealt with in (3) since  $G_{\chi}$  preserves  $T_0$ , so only *L*-symmetry remains. Note

$$P_{G_{\chi}(\mathfrak{T})}(x,y) =$$

$$= \bigwedge_{u \in G_{\chi}(\mathfrak{T})} (u(x) \swarrow u(y)) =$$

$$= \bigwedge_{U \in \mathfrak{T}} (\chi_{U}(x) \swarrow \chi_{U}(y)) =$$

$$= \bigwedge_{U \in \mathfrak{T}} \left( \bigvee_{c \otimes \chi_{U}(y) \leq \chi_{U}(x)} c \right),$$

and, similarly,

$$P_{G_{\chi}(\mathfrak{T})}\left(y,x\right)=\bigwedge_{U\,\in\,\mathfrak{T}}\,\left(\bigvee_{c\,\otimes\,\chi_{U}(x)\,\leq\,\chi_{U}(y)}c\right).$$

Since  $(X, \mathfrak{T})$  is  $T_1$ , the specialization order  $\leq_{\mathfrak{T}}$  is symmetric, which implies that

$$\forall U \in \mathfrak{T}, \ y \in U \Leftrightarrow x \in U.$$

Hence

$$\forall U \in \mathfrak{T}, \ c \otimes \chi_{U}\left(y\right) \leq \chi_{U}\left(x\right) \Leftrightarrow c \otimes \chi_{U}\left(x\right) \leq \chi_{U}\left(y\right).$$

It now follows from the above displays that  $P_{G_{\chi}(\mathfrak{T})}(x,y) = P_{G_{\chi}(\mathfrak{T})}(y,x)$ , which says  $P_{G_{\chi}(\mathfrak{T})}$  is symmetric by (a) above.  $\Box$ 

**Proposition 6.10.2.** For L an IIA integral quantale,  $G_{\chi}$  reflects L- $T_1(2)$  in the sense that  $(X, \mathfrak{T})$  is  $T_1$  whenever  $G_{\chi}(X, \mathfrak{T})$  is L- $T_1(2)$ .

*Proof.* Let  $(X, \mathfrak{T})$  be a topological space and suppose  $G_{\chi}(X, \mathfrak{T})$  is L- $T_1(2)$ , i.e.,  $P_{G_{\chi}(\mathfrak{T})}$  is L-antisymmetric and L-symmetric. The L-antisymmetry of  $P_{G_{\chi}(\mathfrak{T})}$  is reflected by  $G_{\chi}$ , i.e.,  $\leq_{\mathfrak{T}}$  is antisymmetric, because of (3) above in light of Theorem 4.4(4). As for the symmetry of  $\leq_{\mathfrak{T}}$ , let  $x, y \in X$ . It is handy to put

 $\mathcal{U}_{\top}\left(y\right)\equiv\left\{U\in\mathfrak{T}:y\in U\right\},\quad\mathcal{U}_{\bot}\left(y\right)=\mathfrak{T}-\mathcal{U}_{\top}\left(y\right),$ 

$$\mathcal{U}_{\top}(x) \equiv \left\{ U \in \mathfrak{T} : x \in U \right\}, \quad \mathcal{U}_{\perp}(x) = \mathfrak{T} - \mathcal{U}_{\top}(x).$$

Referring to the proof of 6.10.1 above, we have that

$$P_{G_{\chi}(\mathfrak{T})}(x,y) = \bigwedge_{U \in \mathfrak{T}} \left( \bigvee_{c \otimes \chi_{U}(y) \leq \chi_{U}(x)} c \right).$$

Now for  $U \in \mathcal{U}_{\perp}(y)$ , it is the case using (b) above that

$$\bigvee_{c \otimes \chi_U(y) \le \chi_U(x)} c = \bigvee_{c \otimes \bot \le \chi_U(x)} c = \bigvee_{\bot \le \chi_U(x)} c = \top;$$

and for  $U \in \mathcal{U}_{\top}(y)$ , it is the case using (c) above that

$$\bigvee_{c \,\otimes\, \chi_{U}(y) \,\leq\, \chi_{U}(x)} c = \bigvee_{c \,\otimes\, \top \,\leq\, \chi_{U}(x)} c = \bigvee_{c \,\leq\, \chi_{U}(x)} c = \chi_{U}\left(x\right).$$

It follows that

$$P_{G_{\chi}(\mathfrak{T})}\left(x,y\right) = \bigwedge_{U \in \mathcal{U}_{\top}\left(y\right)} \chi_{U}\left(x\right) = \begin{cases} \top, \ \mathcal{U}_{\top}\left(y\right) \subset \mathcal{U}_{\top}\left(x\right) \\ \bot, \ \exists U \in \mathcal{U}_{\top}\left(y\right), \ x \notin U \end{cases},$$

and, similarly, that

$$P_{G_{\chi}(\mathfrak{T})}(y,x) = \bigwedge_{U \in \mathcal{U}_{\top}(x)} \chi_{U}(y) = \begin{cases} \top, \ \mathcal{U}_{\top}(x) \subset \mathcal{U}_{\top}(y) \\ \bot, \ \exists V \in \mathcal{U}_{\top}(x), \ y \notin V \end{cases}$$

These displays imply that  $P_{G_{\chi}(\mathfrak{T})}(x,y)$  is  $\top$  or  $\bot$ , and that  $P_{G_{\chi}(\mathfrak{T})}(y,x)$  is  $\top$  or  $\perp$ . Further, if either of  $P_{G_{\chi}(\mathfrak{T})}(x,y)$  or  $P_{G_{\chi}(\mathfrak{T})}(y,x)$  is  $\top$ , then, by L-symmetry of  $P_{G_{\chi}(\mathfrak{T})}$ , these displays yield  $\mathcal{U}_{\top}(x) = \mathcal{U}_{\top}(y)$ ; this says that  $x \leq_{\mathfrak{T}} y$  and  $y \leq_{\mathfrak{T}} x$ , and, since  $\leq_{\mathfrak{T}}$  is already known to be antisymmetric, this means x = y. Conversely, if x = y, each of  $P_{G_{\chi}(\mathfrak{T})}(x, y)$  and  $P_{G_{\chi}(\mathfrak{T})}(y, x)$  is  $\top$ . This means that

$$P_{G_{\chi}(\mathfrak{T})}\left(x,y\right)=\top\quad\Leftrightarrow\quad x=y\quad\Leftrightarrow\quad P_{G_{\chi}(\mathfrak{T})}\left(y,x\right)=\top.$$

Now suppose  $x \neq y$ . Then by the previous line  $P_{G_{\chi}(\mathfrak{T})}(x,y) = \bot$ and  $P_{G_{\chi}(\mathfrak{I})}(y,x) = \bot$ . The two previous displays now imply that

 $\exists U \in \mathcal{U}_{\top}(y), \ x \notin U, \quad \exists V \in \mathcal{U}_{\top}(x), \ y \notin V.$ 

This is equivalent to saying that  $(X, \mathfrak{T})$  is  $T_1$ . (5) The behavior of  $M_L$  with respect to L- $T_1(2)$  is now considered.

**Proposition 6.10.3.** For L a consistent, positive, and co-positive IIA integral quantale,  $M_L$  preserves L- $T_1(2)$ , in the sense that  $(X, \tau)$  is L-T<sub>1</sub>(2) implies  $M_L(X, \tau)$  is T<sub>1</sub>, if and only if L = 2. *Proof.* Under the assumption that L = 2, the preservation of L- $T_1(2)$  follows from the first part of (3)(d) above in combination with Corollary 5.6(3). For the reverse direction, suppose  $|L| \ge 3$ , let  $\{\perp, \alpha, \top\} \subset L$ , and let  $X = \{x, y\}$ . Consider the subbasis  $\{u, v\} \subset L^X$  for an L-topology on X, where

$$u(x) = \alpha, u(y) = \bot, \quad v(y) = \alpha, v(x) = \bot.$$

By inspection,  $(X, \tau)$  is L- $T_0$ . Now positivity of L implies that  $\perp \swarrow \alpha = \perp$ . From this and (4)(b, c) above, along with the fact, cited in (4)(a) above, that \* is an order-isomorphism, it follows

$$P_{\tau}(x,y) = \bigwedge_{w \in \tau} (w(x) \swarrow w(y))$$

$$\leq (v(x) \swarrow v(y))$$

$$= (\bot \swarrow \alpha)$$

$$= \bot$$

$$= \bot^{*}$$

$$= (\bot \swarrow \alpha)^{*}$$

$$= (u(y) \swarrow u(x))^{*}$$

$$\geq \left[\bigwedge_{w \in \tau} (w(y) \swarrow w(x))\right]^{*}$$

$$= P_{\tau}(y,x)^{*},$$

which implies that

$$P_{\tau}(x,y) = \bot = P_{\tau}^*(x,y).$$

It now follows that  $P_{\tau}$  is *L*-symmetric and hence that  $(X, \tau)$  is L- $T_1(2)$ . But co-positivity of *L* assures that  $\tau$  as generated from the subbasis  $\{u, v\}$  has no characteristic maps other than  $\perp$  and  $\perp$ , and this implies that  $M_L(X, \tau)$  is not  $T_0$  and hence not  $T_1$ .  $\square$ 

**Proposition 6.10.4**. For L an IIA integral quantale,  $M_L$  strongly reflects  $T_1$  in the sense that  $(X, \tau)$  is L- $T_1(2)$  whenever  $M_L(X, \tau)$  is  $T_1$ , so that  $M_L$  preserves L-asymmetry or the lack of L- $T_1(2)$  separation.

*Proof.* Suppose  $M_L(X, \tau)$  is  $T_1$ , where we recall that  $M_L(\tau) = \{U \subset X : \chi_U \in \tau\}$ . Since  $P_{\tau}(x, y) = P_{\tau}^*(y, x)$  whenever x = y (since  $\top^* = \top$ ), it suffices to assume that  $x \neq y$ . Since  $M_L(X, \tau)$  is  $T_1$ ,

 $\exists U \in M_L(\tau), x \in U, y \notin U, \quad \exists V \in M_L(\tau), y \in V, x \notin V.$ 

Now using (4)(b, c) above, along with the fact, cited in (4)(a) above, that \* is an order-isomorphism, it follows

$$P_{\tau}(x,y) = \bigwedge_{u \in \tau} (u(x) \swarrow u(y))$$

$$\leq (\chi_{U}(x) \swarrow \chi_{U}(y)) \land (\chi_{V}(x) \swarrow \chi_{V}(y))$$

$$= (\top \swarrow \bot) \land (\bot \swarrow \top)$$

$$= \top \land \bot$$

$$= \bot$$

$$= \bot \land \top$$

$$= \bot^{*} \land \top^{*}$$

$$= (\bot \swarrow \top)^{*} \land (\top \swarrow \bot)^{*}$$

$$= (\chi_{U}(y) \swarrow \chi_{U}(x))^{*} \land (\chi_{V}(y) \swarrow \chi_{V}(x))^{*}$$

$$\geq \left[ \bigwedge_{u \in \tau} (u(y) \swarrow u(x)) \right]^{*}$$

which implies that

$$P_{\tau}(x,y) = \bot = P_{\tau}^*(x,y).$$

Hence  $(X, \tau)$  is L- $T_1(2)$ .  $\Box$ 

- (6) Wrap-up for  $M_L \dashv G_{\chi}$  and Asymmetry. Under appropriate conditions on L, the following hold:
  - (a)  $(X, \mathfrak{T})$  is asymmetric (not  $T_1$ ) implies  $G_{\chi}(X, \mathfrak{T})$  is *L*-asymmetric both in the sense that it is not L- $T_1(1)$  and in the sense that it is not L- $T_1(2)$ .
  - (b)  $(X,\mathfrak{T})$  is symmetric  $(T_1)$  implies  $G_{\chi}(X,\mathfrak{T})$  is *L*-symmetric both in the sense of being *L*- $T_1(1)$  and in the sense of being *L*- $T_1(2)$ .
  - (c)  $(X, \tau)$  is *L*-asymmetric, either in the sense of not being *L*- $T_1(1)$  or in the sense of not being *L*- $T_1(2)$ , implies  $M_L(X, \tau)$  is asymmetric (not  $T_1$ ).

**Examples 6.11.** These example classes stem from a fruitful suggestion of the referee and concern  $L \cdot T_1(1)$  and non  $L \cdot T_1(1)$  spaces generated by the adjunction  $\omega_L \dashv \iota_L$ . Throughout these examples, unless stated otherwise, L is a semiquantale. The  $\omega_L, \iota_L$  functors are more nuanced than the  $M_L, G_{\chi}$  functors, and therefore somewhat more preparation is needed vis-a-vis  $M_L, G_{\chi}$  before addressing the issue of generating  $L \cdot T_1(1)$ and non  $L \cdot T_1(1)$  spaces.

- (1) The category L-Top given in Subsection 2.3 above is a topological construct. Closely linked to this topologicity is that each carrier set has a complete fibre of L-topologies and hence an associated notion of subbasis. Letting X be a set and {u<sub>γ</sub> : γ ∈ Γ} ⊂ L<sup>X</sup>, {u<sub>γ</sub> : γ ∈ Γ} is a subbasis of the smallest L-topology containing {u<sub>γ</sub> : γ ∈ Γ}, which L-topology exists because the fibre of L-topologies on X is a complete lattice. This smallest topology containing {u<sub>γ</sub> : γ ∈ Γ} is denoted ⟨⟨{u<sub>γ</sub> : γ ∈ Γ}⟩⟩. This L-topology can be described explicitly in terms of the subbasic open sets, as done in traditional topology, if L is a quantale. The same notation is used to denote traditional topologies generated by a subbasis of ordinary subsets.
- (2) The functors  $\omega_L$ ,  $\iota_L$  relate traditional topology and many-valued topology to each other and are originally due to Lowen [39] for the case L = [0, 1]. The most general extension of these functors to date is that of Kubiak [32] for L a complete lattice. The formal syntax of Kubiak's definitions also works in the more general setting of semiquantales, and it is these definitions which are presented and used.
- (3) Defining the functors  $\omega_L$ ,  $\iota_L$  is greatly simplified by the so-called "Halmos" notation for level sets. Let X be a set,  $a \in L^X$ , and let  $\alpha \in L \{\top\}$ . Then the (strict)  $\alpha$ -level set  $[u \nleq \alpha]$  is defined by

$$[u \nleq \alpha] = \{x \in X : u(x) \nleq \alpha\}.$$

If the ordering on L is linear, then  $\nleq$  may be replaced by >; Kubiak's use of  $\nleq$  in [31] paved the way for working with  $\omega_L, \iota_L$ in a context more general than that of complete chains. Clearly  $[u \nleq \alpha] = u^{\leftarrow}(\alpha, \top]$ , where  $(\alpha, \top]$  may be taken as  $\{\beta \in L : \beta \nleq \alpha\}$ .

(4) The categorical functors  $\omega_L$ ,  $\iota_L$  are constructed by first setting up isotone mappings  $\iota_L, \omega_L$  between fibres in *L*-**Top** and **Top** over the same carrier set. Let  $X \in |\mathbf{Set}|$  and let  $\mathbb{T}_L(X)$  and  $\mathbb{T}(X)$  be the respective complete fibres on X from *L*-**Top** and **Top**. Now put  $\iota_L : \mathbb{T}_L(X) \to \mathbb{T}(X), \ \omega_L : \mathbb{T}_L(X) \leftarrow \mathbb{T}(X)$  by

$$\iota_{L}(\tau) = \left\langle \left\langle \left\{ [u \nleq \alpha] : u \in \tau, \ \alpha \in L - \{\top\} \right\} \right\rangle \right\rangle,$$

$$\omega_L\left(\mathfrak{T}\right) = \left\langle \left\langle \left\{ u \in L^X : \forall \alpha \in L - \{\top\}, \left[ u \nleq \alpha \right] \in \mathfrak{T} \right\} \right\rangle \right\rangle.$$

It can be noted as in [31, 32] that if the upper topology is put on L with subbasis members of the form  $(\alpha, \top]$ , then  $\omega_L$  assigns to each ordinary topology on X that L-topology generated by the subbasis of all continuous maps from X to L; similarly,  $\iota_L$ assigns to each L-topology on X the smallest ordinary topology on X with respect to which all members of the L-topology are continuous maps from X to L. Properties needed below for the fibre maps  $\iota_L, \omega_L$  are now listed—additional properties can be found in [31, 32] under various restrictions on L:

- (a)  $\iota_L, \omega_L$  are isotone as mappings between  $\mathbb{T}(X)$  and  $\mathbb{T}_L(X)$ .
- (b)  $\forall \mathfrak{T} \in \mathbb{T}(X), \ \mathfrak{T} \subset \iota_L(\omega_L(\mathfrak{T})); \text{ and } \iota_L(\omega_L(\mathfrak{T})) \subset \mathfrak{T} \text{ if } (L, \leq)$  is completely distributive (with  $\otimes = \wedge$ ).
- (c)  $\forall \tau \in \mathbb{T}_L(X), \ \tau \subset \omega_L(\iota_L(\tau)).$
- (d)  $\iota_L \dashv \omega_L$  as fibre maps, provided  $(L, \leq)$  is completely distributive (with  $\otimes = \wedge$ ), and this adjunction is an isocoreflection.

Various weakenings of complete distributivity in (b) and (d) are considered in [31, 32]; e.g., (b) and (d) hold if the  $\wedge$ -prime elements of L are order-generating. Such improvements can also be applied below wherever complete distributivity is assumed.

(5) The categorical functors  $\omega_L : \mathbf{Top} \to L\text{-}\mathbf{Top}$  and  $\iota_L : \mathbf{Top} \leftarrow L\text{-}\mathbf{Top}$  can now be defined using the fibre maps  $\iota_L, \omega_L$  of (4):

$$\omega_L(X,\mathfrak{T}) = (X, \omega_L(\mathfrak{T})), \quad \omega_L(f) = f,$$

$$\iota_L(X,\tau) = (X,\iota_L(\tau)), \quad \iota_L(f) = f.$$

These correspondences have the following properties needed below:

- (a)  $\omega_L$ : **Top**  $\rightarrow$  *L*-**Top** and  $\iota_L$ : **Top**  $\leftarrow$  *L*-**Top** are concrete faithful functors.
- (b)  $\omega_L$  is an embedding if  $(L, \leq)$  is completely distributive (with  $\otimes = \wedge$ ); it is an isomorphism if L = 2.
- (c) Each of  $\omega_L$ ,  $\iota_L$  reflects lifted morphisms—see Example 6.10(2)(c) above for this terminology and citations.
- (d) If  $(L, \leq)$  is completely distributive (with  $\otimes = \wedge$ ), then  $\omega_L \dashv \iota_L$  as categorical functors; and this adjunction is an isoreflection and a monocoreflection; cf. Examples 6.10(2)(d) above.
- (6) With the above preparations in hand, the issue of identifying or generating L-T<sub>1</sub>(1) and non-L-T<sub>1</sub>(1) spaces via these functors can now be addressed. In [32] it is proved that for L a complete DeMorgan algebra and for L-topological space (X,τ), (X,τ) is L-T<sub>1</sub>(1) if and only if ι<sub>L</sub>(X,τ) is T<sub>1</sub> as an ordinary topological space. Since the proof in [32] makes explicit use of the DeMorgan (quasi-)complementation, a proof for L a semiquantale in the following lemma is given.

**Lemma 6.11.1.** Let L be a semiquantale and  $(X, \tau) \in |L\text{-Top}|$ . Then  $(X, \tau)$  is L-T<sub>1</sub>(1) if and only if  $\iota_L(X, \tau)$  is T<sub>1</sub>. *Proof.* Let  $x, y \in X$  with  $x \neq y$ . For necessity, we assume  $(X, \tau)$  is L-T<sub>1</sub>(1), which implies  $\exists u, v \in \tau$  with  $u(x) \nleq u(y)$  and  $v(y) \nleq v(x)$ . Set  $U = [u \nleq \alpha], V = [v \nleq \beta]$ , where  $\alpha = u(y), \beta = v(x)$ . Then  $x \in U$  and  $y \notin U$ , and  $y \in V$  and  $x \notin V$ . Since U, V are subbasic open sets in  $\iota_L(\tau)$ , it follows that  $\iota_L(X, \tau)$  is T<sub>1</sub>. Now for sufficiency, we assume  $\iota_L(X, \tau)$  is T<sub>1</sub>, which implies  $\exists U, V \in \iota_L(\tau)$  with  $x \in U$  and  $y \notin U$ , and  $y \in V$  and  $x \notin V$ . Since  $\iota_L(\tau)$  is an ordinary topology, each of its open sets can be written as a union of finite intersections of subbasic open sets. In the case of U, we can write

$$U = \bigcup_{\gamma \in \Gamma} \bigcap_{i=1}^{n\gamma} [u_i^{\gamma} \nleq \alpha_i^{\gamma}]$$

for some  $u_i^{\gamma} \in \tau$  and some  $\alpha_i^{\gamma} \in L - \{\top\}$ . Since  $x \in U$  and  $y \notin U$ , it follows  $\exists \delta \in \Gamma, \exists j \in \{1, ..., n_{\delta}\}$  such that

 $x \in \left[u_j^{\delta} \nleq \alpha_j^{\delta}\right], \quad y \notin \left[u_j^{\delta} \nleq \alpha_j^{\delta}\right],$ 

which implies  $u_j^{\delta}(x) \nleq \alpha_j^{\delta}$  and  $u_j^{\delta}(y) \le \alpha_j^{\delta}$ . This means  $u_j^{\delta}(x) \nleq u_j^{\delta}(y)$  by transitivity. Summarizing,  $\exists u \in \tau$  with  $u(x) \nleq u(y)$ . By a similar argument,  $\exists v \in \tau$  with  $v(y) \nleq v(x)$ . It follows  $(X, \tau)$  is L- $T_1(1)$ .

We note that the appropriate modification of the proof of 6.11.1 shows that for L a semiquantale and  $(X, \tau) \in |L\text{-}\mathbf{Top}|$ ,  $(X, \tau)$  is  $L\text{-}T_0$  if and only if  $\iota_L(X, \tau)$  is  $T_0$ : this extends the result first given in [32] for complete DeMorgan algebras and proved using the DeMorgan involution.

From Lemma 6.11.1, we have that the  $\iota_L$  functor characterizes both those *L*-topological spaces which are  $L-T_1(1)$  as well as those which are not  $L-T_1(1)$ . Given an *L*-topological space, we construct its  $\iota_L$  modification and check whether that ordinary topological space is  $T_1$ . It would also be convenient to know how to produce  $L-T_1(1)$  or non  $L-T_1(1)$  *L*-topological spaces, which brings us to our next example class.

(7) Example classes are now given which show how to produce L-topological spaces which are L- $T_1(1)$  as well as L-topological spaces which are not L- $T_1(1)$ .

**Lemma 6.11.2**. Let L be a semiquantale and  $(X, \mathfrak{T}) \in |\mathbf{Top}|$ . Then the following statements hold:

- (a) Let  $(L, \leq)$  be completely distributive (with  $\otimes = \wedge$ ). If  $(X, \mathfrak{T})$  is not  $T_1$ , then  $\omega_L(X, \mathfrak{T})$  is not L- $T_1(1)$ .
- (b) If  $(X, \mathfrak{T})$  is  $T_1$ , then  $\omega_L(X, \mathfrak{T})$  is L- $T_1(1)$ .

Proof. For (a), we recall from (4)(b) above that  $\iota_L(\omega_L(\mathfrak{T})) \subset \mathfrak{T}$ , from which it follows that if  $(X,\mathfrak{T})$  is not  $T_1$ , then  $\iota_L\omega_L(X,\mathfrak{T})$  is not  $T_1$ . From this and Lemma 6.11.1, it follows that  $\omega_L(X,\mathfrak{T}) = (X, \omega_L(\mathfrak{T}))$  is not L- $T_1(1)$ . Now for (b), we have from (4)(b) that  $\mathfrak{T} \subset \iota_L(\omega_L(\mathfrak{T}))$ , in which case if  $(X,\mathfrak{T})$  is  $T_1$ , then so is  $\iota_L\omega_L(X,\mathfrak{T}) = (X, \iota_L(\omega_L(\mathfrak{T})))$ . Lemma 6.11.1 now applies to say that  $\omega_L(X,\mathfrak{T})$  is L- $T_1(1)$ .

(8) Asymmetry and L-spectra. As a final twist on examples, and to relate the above example classes back to (L-)spectra, it is wellknown [52, 53, 61] that if L is a semiframe ( $\otimes = \wedge$ ) which admits even one (semiframe) endomorphism other than the identity, then  $\omega_L$  "destroys" sobriety: given any sober topological space  $(X, \mathfrak{T})$ ,  $\omega_L (X, \mathfrak{T})$  is not L-sober; now invoking Lemma 6.11.2(b), for any Hausdorff topological space  $(X, \mathfrak{T})$ ,  $\omega_L (X, \mathfrak{T})$  is L- $T_1$  (1) but not L-sober. Finally, note that there are many completely distributive semiframes with an endomorphism other than the identity [52, 53, 61] and assume L is such a semiframe which is consistent: if  $(X, \mathfrak{T})$ is not  $T_1$ , then  $\omega_L (X, \mathfrak{T})$  is not L- $T_1$  (1) by Lemma 6.11.2(a); and, further, if  $\omega_L (\mathfrak{T})$  has two related distinct ( $\wedge$ -)primes, then  $LPt(\omega_L (X, \mathfrak{T}))$  is an L-sober space which fails to be L- $T_1$  (1).  $\Box$ 

#### 7. Summary and open questions for many-valued asymmetry

Issues relating to symmetry naturally arise in traditional topology via the symmetry axiom of a metric space, related to Hausdorff spaces, and via the symmetry condition satisfied by specialization orders of  $T_1$  topological spaces.

This paper couches the issue of symmetry vis-a-vis asymmetry for L-topological spaces—under appropriate requirements on L—via two "standard" specialization orders (and their duals) associated with such spaces, orders giving rise to two related L- $T_1$  separation axioms—L- $T_1$  (1) and L- $T_1$  (2)—and hence two different senses of symmetry for L-topological spaces, and, via their negations, two different senses of asymmetry for such spaces. In this context, it is also important to note that for L-topological spaces and under fairly general conditions on L, the antisymmetry of each of these specialization orders is equivalent to the well-known L- $T_0$  separation axiom and the L- $T_1$  (1) axiom is equivalent to the Kubiak L- $T_1$ separation axiom. Open questions below are presented in light of the linkage of these specialization orders for many-valued topology to low-order separation axioms, a linkage which behooves us to briefly summarize the historiography of such axioms for many-valued topology, summaries which make use of previous sections of this paper.

**Overview of** L- $T_0$  **Axiom 7.0.** For almost two decades after the initiation of many-valued topology in 1968 in [6], scholars searched unsuccessfully for an appropriate  $T_0$  axiom as part of the overall attempt to generalize separation to this larger topological canvas: see the surveys included in [52, 31]. One of the main stumbling blocks to all of these early candidates was that each failed to be satisfied by the fuzzy unit intervals  $\mathbb{I}(L)$  and fuzzy real lines  $\mathbb{R}(L)$ . Then *zeitgeist*-like, the current L- $T_0$  axiom appeared independently and simultaneously in 1986 from three different sources each with their own motivation, namely [51], [37, 38], and [66]. Subsequently, the L- $T_0$  axiom has continued to reveal itself in a variety of important contexts and developments. The main points of the last 30 years for this axiom are now summarized:

- (1) The L- $T_0$  axiom is satisfied by  $\mathbb{I}(L)$  and  $\mathbb{R}(L)$  for each complete DeMorgan algebra L. This is proved in [51] and noted in subsequent papers. Further, the alternative L-fuzzy real line  $\mathbb{R}^*(L)$ and the alternative L-fuzzy unit interval  $\mathbb{I}^*(L)$  are L-sober and hence L- $T_0$  for all semiquantales L.
- (2) The L- $T_0$  axiom appears in the theory of many-valued spectra and characterizes when the L-extension  $\Psi_L$  of the second Stone comparison map is injective, an important aspect of many-valued spectra and various schema of representation and compactification theorems in [51, 52, 53, 54, 56, 3, 9, 28, 47, 48, 49, 50]. The injectivity of  $\Psi_L$  is an independent motivation for this axiom.
- (3) The L- $T_0$  axiom appears in the compactification work of [37, 38] and subsequent papers, where it is named the (L- $)subT_0$  axiom out of deference to the traditional  $T_0$  axiom, and these compactifications are an independent motivation for this axiom.
- (4) The L- $T_0$  axiom appears in [66] and is subsequently used to characterize the epireflective hull of Sierpinski objects in S[0, 1]-**Top**—the category of stratified L-topological spaces with L = [0, 1], and this categorical behavior is an independent motivation for this axiom.
- (5) The L- $T_0$  axiom is both the preservation and the reflection of the traditional  $T_0$  axiom by the  $G_{\chi}$  functors, as well as the reflection of the  $T_0$  axiom by the  $M_L$  functors (Examples 6.10(3) above).

- (6) The L- $T_0$  axiom is shown in [32] to be both the preservation and reflection of the traditional  $T_0$  axiom by the  $\iota_L$  functors; and this result is extended to semiquantale bases by appropriate modification of the proof of Lemma 6.11.1 above.
- (7) The L- $T_0$  axiom characterizes antisymmetry for both crisp specialization orders for many-valued topological spaces and many-valued specialization orders for many-valued topological spaces. This appears in [9] and is summarized in Theorem 4.4 above.  $\Box$

The above summary would seem to establish the "canonicity" of the  $L-T_0$  axiom in many-valued topology. The  $L-T_1(1)$  and  $L-T_1(2)$  axioms are now overviewed, beginning with the  $L-T_1(1)$  axiom.

**Overview of** L- $T_1(1)$  **Axiom 7.1.1**. The main points of the last 20 years for this axiom are now summarized:

- (1) The L- $T_1$  (1) axiom was first proposed in [32] as part of a suggested scheme of separation axioms, beginning with the L- $T_0$  axiom, and motivated by making the L- $T_0$  axiom symmetric in its predicate. More precisely, the L- $T_0$  axiom is equivalent to saying:  $\forall x, \forall y, \text{ with } x \neq y, \exists u \in \tau \text{ with } u(x) \nleq u(y) \text{ or } \exists v \in \tau \text{ with } v(y) \nleq v(x)$ . Replacing or with and makes a "symmetric axiom" and gives the L- $T_1$  (1) axiom as proposed in [32]. "Symmetrization" of L- $T_0$  is the first motivation for L- $T_1$  (1).
- (2) The L- $T_1$  (1) axiom proposed in this paper is part of the overall development of the crisp and many-valued specialization orders for many-valued spaces, a development in which the L- $T_0$  axiom is equivalent to traditional antisymmetry of the crisp specialization order as well as equivalent to many-valued antisymmetry [9] of the many-valued specialization order (when L is appropriately residuated), extending the situation for specialization orders of traditional topological spaces and the traditional  $T_0$  axiom. Given that in the traditional setting, the  $T_1$  axiom is equivalent to symmetry of the traditional specialization order, the motivation in this paper is define L- $T_1$  (1) axiom by requiring both antisymmetry (or L- $T_0$ ) and symmetry of these orders is the second (and independent) motivation for L- $T_1$  (1).
- (3) The L- $T_1(1)$  axiom is satisfied by  $\mathbb{I}(L)$  and  $\mathbb{R}(L)$  for complete DeMorgan algebras. This is stated and proved in [32] and cited in Examples 6.9.4(2) above. Additionally, the alternative *L*-fuzzy real line  $\mathbb{R}^*(L)$  and the alternative *L*-fuzzy unit interval  $\mathbb{I}^*(L)$  are L- $T_1(1)$  if *L* is a complete Boolean algebra.

- (4) Many example classes are given in the above sections, especially Section 6, of many-valued spaces which are not  $L-T_1(1)$  and Lsober, L- $T_1(1)$  and L-sober, and L- $T_1(1)$  and not L-sober, under appropriate conditions on L. These examples not only show that L- $T_1(1)$  and L-sobriety are generally independent, yet sometimes coincide, but reflect and in some cases extend the very examples used to show that  $T_1$  and sobriety are independent, yet sometimes coincide, in traditional topology. To expand on these observations, we note that the spectrum in traditional topology often generates both  $T_1$  (symmetric) and non- $T_1$  (asymmetric) sober spaces. Hence a special tool in this paper was the notion of the L-spectrum. It was found that for consistent, integral semiquantale L with  $\perp$  as annihilator, the L-spectrum of an integral semiquantale with annihilator  $\perp$  and at least two related distinct tensor-primes is asymmetric; and, in particular, for L a complete DeMorgan algebra, the left and right topologies of  $\mathbb{R}$  and the left and right L-topologies of  $\mathbb{R}(L)$  are all asymmetric, and the Lspectrum of each of these topologies is asymmetric, with similar results for the related L-subspace topologies of  $\mathbb{I}(L)$ .
- (5) The L- $T_1(1)$  axiom is both the preservation and the reflection of the traditional  $T_1$  axiom by the  $G_{\chi}$  functors, as well as the reflection of the  $T_1$  axiom by the  $M_L$  functors (Examples 6.10(3) above).
- (6) The L- $T_1(1)$  axiom is shown in [32], for L a complete DeMorgan algebra, to be both the preservation and reflection of the traditional  $T_1$  axiom by the  $\iota_L$  functors; and this result is extended to semiquantale bases by Lemma 6.11.1 above.
- (7) Under appropriate conditions on L, the  $\omega_L$  functor can both produce L- $T_1(1)$  many-valued spaces from  $T_1$  traditional spaces and non-L- $T_1(1)$  many-valued spaces from non- $T_1$  traditional spaces.

The above summary seems to indicate an important role, perhaps a "canonical" role, for the L- $T_1(1)$  axiom.

**Overview of** L- $T_1$  (2) **Axiom 7.1.2**. Turning to the L- $T_1$  (2) axiom, it is naturally motivated by the fact that in the traditional setting, the  $T_1$ axiom is equivalent to symmetry of the traditional specialization order. Hence, this paper defines the L- $T_1$  (2) axiom by requiring both manyvalued antisymmetry (or L- $T_0$ ) and many-valued symmetry of the manyvalued specialization order of a many-valued space. Though this paper furnishes example classes of L- $T_1$  (2) spaces and many example classes of non-L- $T_1$  (2) spaces (since not L- $T_1$  (1) implies not L- $T_1$  (2)), this axiom needs more study and example classes of spaces satisfying the L- $T_1(2)$  axiom which explore the full generality of IIA unital quantale. Currently, the following can be said under the appropriate lattice-theoretic assumptions:

- (1) Each of  $\mathbb{R}_l(L)$ ,  $\mathbb{R}_r(L)$  fails to be L- $T_1(2)$ —see Examples 6.9.4(3).
- (2) Each of the *L*-spectra  $Lpt(\tau_l(L))$ ,  $Lpt(\mathfrak{T}_l)$ ,  $Lpt(\tau_r(L))$ ,  $Lpt(\mathfrak{T}_r)$  is *L*-sober and not L- $T_1(2)$ .
- (3) Ordinary  $T_1$  topological spaces produce L- $T_1$  (2) spaces via  $G_{\chi}$ ; and non- $T_1$  spaces produce non-L- $T_1$  (2) spaces via  $G_{\chi}$ .
- (4) Non- $T_1$  topological spaces indicate non-L- $T_1$  (2) spaces via  $M_L$ .

**Open Questions 7.2**. In light of the above summaries, there are a number of open questions to be explored:

- (1) This paper allows a reframing or revisiting of open questions in many-valued topology surrounding sobriety and Hausdorff separation and their relationship to  $T_1$  conditions:
  - (a) It is well known for traditional topological spaces that Hausdorff ⇒ T<sub>1</sub> ⇒ T<sub>0</sub>, and Hausdorff ⇒ sober ⇒ T<sub>0</sub>, with sobriety and T<sub>1</sub> unrelated; indeed it is known that that the implication Hausdorff ⇒ sober + T<sub>1</sub> is strict—see a proof in [13]. Referring to the L-sobriety of Section 3, we note it is the case that the "weak" L-T<sub>2</sub> axiom of [32] and the "strong" L-T<sub>2</sub> axiom of [22] both imply L-T<sub>1</sub> (1) [23]. This current paper gives examples showing L-T<sub>1</sub> (1) and L-sobriety are unrelated, and both L-T<sub>1</sub> (1) and L-sobriety imply L-T<sub>0</sub>; but it is not the case that either of these two L-T<sub>2</sub> axioms imply L-sobriety.

**Proposition 7.2.1.** For any L a complete DeMorgan frame which is not a Boolean algebra (e.g., L = [0,1] or complete chain with  $|L| \ge 3$ ), both  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are strong (and weak) L- $T_2$  but not L-sober.

*Proof.* For such L both  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are strong (and weak) L- $T_2$  by [32] and [23], but by the Meßner Lemma, neither of these spaces is L-sober—see Lemma 6.9.5 above.  $\Box$ 

**Open Question**. Does either of these L- $T_2$  axioms imply L-sobriety if L is a complete Boolean algebra or L is a complete DeMorgan algebra which is not a frame?

**Open Question**. While it is the case that each of these L- $T_2$  axioms imply L- $T_1(1)$ , it is not known if either of these L- $T_2$  axioms imply L- $T_1(2)$  (where an appropriate residuation structure is assumed on L).

(b) The L-sobriety of Section 3 is not the only sobriety axiom for many-valued topology, and the L- $T_2$  axioms of [32] and [23] are not the only Hausdorff axioms for many-valued topology. See Section 1.1 of [48] for a survey of several current sobriety axioms in many-valued topology; and see [31, 32, 52, 54, 56] and their references for other Hausdorff axioms. One of these sobriety axioms and one of these Hausdorff axioms merit attention. The axiom of  $\iota_L$ -sobriety arises in the representation theorems of [47] based upon many-valued frames as the characterization of the bijectivity of the units of the underlying adjunctions (these results assume L a complete chain), exactly parallel to the origin of the L-sobriety of Section 3 above, and it is dubbed  $\iota_L$ -sobriety (among its other names) precisely because it is the reflection of traditional sober spaces by the  $\iota_L$  functor. This sobriety axiom is defined for arbitrary complete lattices or even semiquantales L by saying that L-topological space  $(X, \tau)$  is  $\iota_L$ -sober if and only if  $\iota_L(X,\tau)$  is a traditional sober topological space. Similarly, the  $\iota_L$ - $T_2$  axiom [31] is defined by saying that L-topological space  $(X, \tau)$  is  $\iota_L$ -Hausdorff if and only if  $\iota_L(X, \tau)$  is a traditional Hausdorff topological space.

**Proposition 7.2.2.** Let *L* be a semiquantale. Then  $\iota_L$ -Hausdorff  $\Rightarrow L$ - $T_1(1) \Rightarrow L$ - $T_0$ , and  $\iota_L$ -Hausdorff  $\Rightarrow \iota_L$ -sober  $\Rightarrow L$ - $T_0$ , with  $\iota_L$ -sobriety and L- $T_1(1)$  unrelated. Further, for *L* a completely distributive DeMorgan algebra,  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are  $\iota_L$ -Hausdorff and hence  $\iota_L$ -sober, as well as L- $T_1(1)$ ; if *L* is further non-Boolean, then  $\mathbb{R}(L)$  and  $\mathbb{I}(L)$  are also not *L*-sober, so that  $\iota_L$ -Hausdorff  $\Rightarrow$  *L*-sober.

*Proof.* The first sentence holds because of the first sentence of (a) above and because all many-valued properties in question are reflected by  $\iota_L$  to their traditional counterparts by the results of [32] and Lemma 6.11.1 above. The second claim holds because of Proposition 6.11 of [31] and Examples for Goal 1.2(8) of [48]; and the third claim is, again, the Meßner Lemma (Lemma 6.9.5 above).

**Open Question**. Does  $\iota_L$ -Hausdorff  $\Rightarrow L$ - $T_1(2)$  if an appropriate residuation structure on L is assumed?

(2) While general tensors are allowed in the example classes of Examples 6.6, 6.7, 6.8, this is not always the case in Examples 6.9 and 6.11.

**Open Question**. For what tensor  $\otimes$  other than the binary meet  $\wedge$  can the example classes of 6.9 and 6.11 be constructed with the same or similar properties as those based upon  $\otimes = \wedge$  and  $* = id_L$ ?

(3) The relationship of the L- $T_1(1)$  and L- $T_1(2)$  axioms have been discussed relative to other separation axiom schemes in manyvalued topology in (1)(a,b) above; but that discussion is clearly space-intensive. Rather different questions emerge when categorical perspectives are considered. For example, let L be a complete, distributive lattice, and let *L*-**KRegSobTop** be the full subcategory of L-Top comprising all L-compact, L-regular, Lsober spaces: L-compact refers to the compactness axiom introduced in [6] and studied systematically in [17], L-regular refers to the regularity axiom introduced and studied in [51, 54, 56]for L a frame, this axiom is equivalent to reflection of regular locales by the  $L\Omega$  functor of Section 3 above, and L-sobriety is that of Section 3 above. Then it is known [56] that **KHausTop** both embeds into *L*-KRegSobTop and is categorically equivalent to *L*-**KRegSobTop**, justifying viewing the latter as a valid categorical extension of traditional compact Hausdorff spaces.

**Open Question**. Is it the case for such underlying L that L-compact, L-regular, L-sober spaces are L- $T_1(1)$  or L- $T_1(2)$  (where in the latter an appropriate residuation structure is assumed)?

**Open Question**. Can compactness be dropped in the previous question?

(4) It is shown in Example 6.9 for L a complete DeMorgan algebra that  $\mathbb{R}(L)$  is L- $T_1(1)$ .

**Open Question**. If *L* is a DeMorgan frame or DeMorgan quantale, will  $\mathbb{R}(L)$  be *L*-*T*<sub>1</sub>(2)?

(5) The alternative *L*-fuzzy real line  $\mathbb{R}^{*}(L)$  is *L*-*T*<sub>1</sub>(1) if *L* is a complete Boolean algebra.

**Open Question**. Under what conditions on L will  $\mathbb{R}^{*}(L)$  be L- $T_{1}(1)$ ?

**Open Question**. Under what conditions on L will  $\mathbb{R}^*(L)$  be L- $T_1(2)$ ?

(6) Concerning Theorem 6.5:

**Open Question**. Under what conditions, if any are needed, does the converse of Theorem 6.5(1) hold?

**Open Question**. How close can we come to a characterization like that in Theorem 6.5(2) for the crisp case?

(7) It is an interesting question of what can be said if  $\mathbb{R}_l(L)$  or  $\mathbb{R}_r(L)$  is *L*-sober. For example, by Example 6.9(5) it follows that  $\mathbb{R}_l(L)$  is *L*-sober if and only if  $\mathbb{R}_l(L)$  is *L*-homeomorphic to each of  $LPt(\tau_r(L))$  and  $LPt(\mathfrak{T}_r)$ ; similarly for the right-handed case.

**Open Question**. Is the *L*-sobriety of either of  $\mathbb{R}_l(L)$  or  $\mathbb{R}_r(L)$  related to the structure of *L*? Cf. Lemma 6.9.5 above.

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Department of Mathematical Sciences, Kent State University, Kent, Ohio, USA 44242

*E-mail address*, Denniston: jdennist@kent.edu *E-mail address*, Melton: amelton@kent.edu

Department of Mathematics & Statistics, College of Science, Technology, Engineering, Mathematics (STEM), Youngstown State University, Youngstown, OH, USA 44555-3347

E-mail address: rodabaug@math.ysu.edu