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REVERSIBLE SPACES AND PRODUCTS

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Abstract. We demonstrate that there are connected reversible spaces such that their product is not reversible. This is an answer to a question posed by Chatyrko and Hattori.

1. Introduction

In [3] Rajagopalan and Wilansky introduced and studied reversible spaces, i.e. spaces X such that each continuous bijection $f\colon X \to X$ of X onto itself is a homeomorphism. Simple examples of reversible spaces are Hausdorff compacta and discrete spaces D_τ of cardinality $\tau \geq 1$. Other examples of reversible spaces are locally Euclidean spaces (in particular, n-dimensional manifolds without boundary). Even some manifolds with a non-empty boundary can be reversible, for example, the half-open interal $[0, 1)$. Recall that there are many non-reversible manifolds (see [2]).

In [1] Chatyrko and Hattori observed that any product of topological spaces is non-reversible whenever at least one of the spaces is non-reversible. However, a product of two reversible spaces can be reversible and non-reversible as well. For example, $\mathbb{R} \times \mathbb{R}$ is reversible, while $D_{\aleph_0} \times [0, 1)$ is not reversible. They asked if the topological product of two connected reversible spaces is reversible.

In this paper we suggest two connected reversible spaces such that their product is not reversible. Moreover, one of theses spaces is a 2-dimensional non-compact connected manifold in $\mathbb{R}^3$ without boundary, and the other is the closed interval $[0, 1]$.

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2. Example

For a subspace A of a topological space by $\mathrm{Bd}A$ and $\mathrm{Int}A$ we denote the boundary and interior of A, respectively. As usual, $[0,1]$ denotes the unit interval. Let X be a plane with countably many disjoint closed disks removed from it and countably many handles attached. More precisely, let D_n be the closed disk of radius $1/4$ centered at $(n,0)$, and D'_n be the closed disk of radius $1/4$ centered at $(n,1)$, $n = 0,1,2,\dots$. The space X is obtained from $\mathbb{R}^2 \setminus \bigcup_{n=0}^{\infty}(D_n \cup \mathrm{Int}D'_n)$ by attaching a handle to each pair of circles $\mathrm{Bd}D'_{2i}$ and $\mathrm{Bd}D'_{2i+1}$, $i = 0,1,2,\dots$ (see the picture below). Let $X' = X \cup \mathrm{Bd}D_0$.

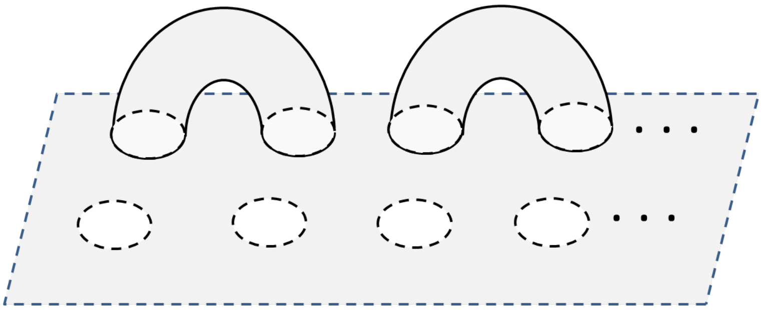

Figure 1. The space X

Let $Y = X \times [0,1]$, $Y' = X' \times [0,1]$ (see the picture below).

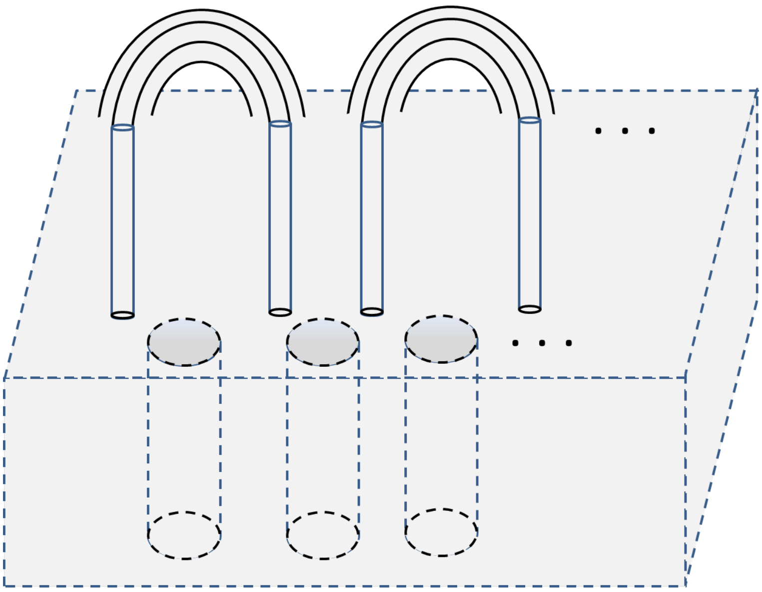

Figure 2. The space Y

Let $T = (S^1 \times [0,1]) \times [0,1]$ be a cylindrical "shell" (we interpret $S^1 \times [0,1]$ as a ring in the plane). Let $Z = (X \cap ([-1/2, 1/2] \times [-1/2, 1/2])) \times [0,1]$. Note that Z is homeomorphic to a "half-open shell" $T' = (S^1 \times (0,1]) \times [0,1]$.

Lemma 2.1. *There exists a continuous bijection of T' onto T which is the identity on $S^1 \times \{1\} \times [0,1]$.*

Proof. Note that T' is homeomorphic to the space $T \setminus (S^1 \times [1/4, 3/4] \times \{1\})$. Now the proof is illustrated by a sequence of diagrams below.

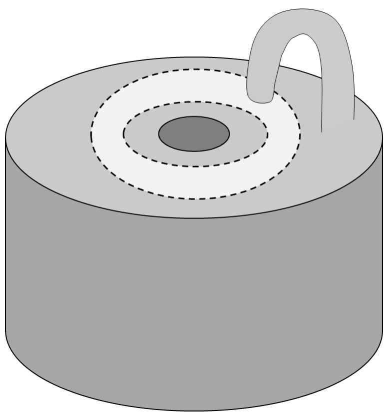

FIGURE 3. Forming a solid handle.

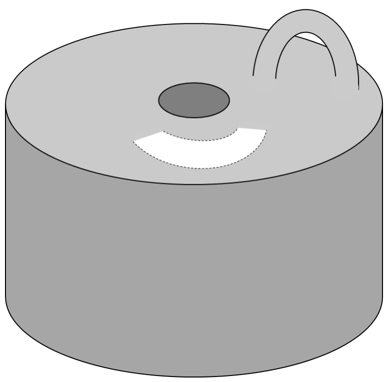

FIGURE 4. Moving the strip.

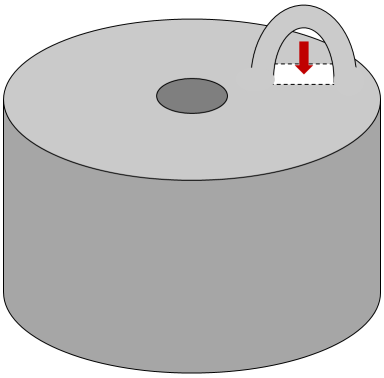

FIGURE 5. Pushing the solid handle down. □

Lemma 2.2. *There exists a continuous bijection of Y onto Y'.*

Proof. Apply Lemma 2.1 to the subspace Z so that the map described in the Lemma fixes $(\mathrm{Bd}[-1/2, 1/2] \times [-1/2, 1/2]) \times [0, 1]$. □

Lemma 2.3. *There exists a continuous bijection of Y onto itself which is not a homeomorphism.*

Proof. According to Lemma 2.2 there exists a continuous bijection of Y to Y'. To finish the proof, use the map illustrated below.

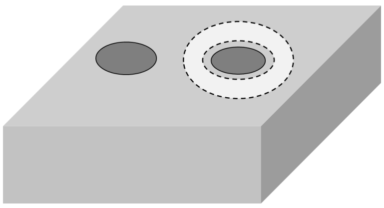

FIGURE 6. Part of the space Y'.

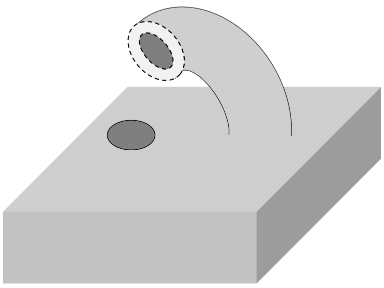

FIGURE 7. The map of Y' onto Y. □

Theorem 2.4. *There exists a reversible space X such that $X \times [0,1]$ is not reversible. Moreover, X can be chosen to be a connected 2-manifold in $\mathbb{R}^3$ without boundary.*

Proof. Let X be the space described in the beginning of the section. The reversibility of X follows from the fact that it is a manifold without boundary. Lemma 2.3 shows that $X \times [0,1]$ is not reversible. $\square$

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