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PRESERVATION OF COUNTABLE COMPACTNESS AND PSEUDOCOMPACTNESS BY FORCING

AKIRA IWASA

Abstract. We study conditions under which countable compactness and pseudocompactness are preserved by forcing that satisfies the countable covering property.

1. Introduction

Let $\mathbf{V}$ be a ground model and let $\mathbb{P}$ be a forcing notion. Let $\mathbf{V}^{\mathbb{P}}$ denote the forcing extension of $\mathbf{V}$ by $\mathbb{P}$. For a topological space $\langle X, \tau \rangle$ in $\mathbf{V}$, we define a topological space $\langle X, \tau^{\mathbb{P}} \rangle$ in $\mathbf{V}^{\mathbb{P}}$ such that $\tau^{\mathbb{P}}$ is the topology generated by τ in $\mathbf{V}^{\mathbb{P}}$. Note that we have in general $\tau \subsetneq \tau^{\mathbb{P}}$ because new open sets are introduced by $\mathbb{P}$. Also note that by definition τ is a base for $\tau^{\mathbb{P}}$.

We say that a forcing $\mathbb{P}$ preserves a topological property φ if, whenever $\langle X, \tau \rangle$ satisfies φ, $\langle X, \tau^{\mathbb{P}} \rangle$ satisfies φ as well. (In other words, we say that $\mathbb{P}$ preserves φ if, whenever X satisfies φ in $\mathbf{V}$, X satisfies φ in $\mathbf{V}^{\mathbb{P}}$.) Note that Hausdorffness, regularity and Tychonoffness are preserved by any forcing ([3] Lemma 22).

The following result is important for our study and it was noticed independently by several people (see, for example, [8, Lemma 7] and [1, Proposition 5.5]).

Theorem 1.1. *For a compact Hausdorff space X, the following are equivalent:*

(1) *The compactness of X is preserved by any forcing.*
(2) *The compactness of X is preserved by adjoining a Cohen real.*
(3) *X is scattered.*

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In this note, we replace "compactness" in Theorem 1.1 by "countable" compactness (Theorem 2.11) and by "pseudocompactness" (Theorem 3.8), and we focus on forcings that satisfy the countable covering property (Definition 2.3).

In section 2, we study preservation of countable compactness. In section 3, we study preservation of pseudocompactness. In section 4, we give some examples.

2. Preservation of Countable Compactness

Recall that a topological space is *countably compact* if every countable open cover of the space has a finite subcover. We often use the fact that a T_1 space X is countably compact if and only if every infinite subset of X has an accumulation point ([12, 17F.2]).

The following proposition shows that a forcing can destroy the countable compactness of any space if the space is not compact.

Proposition 2.1. *Suppose that a topological space $\langle X, \tau \rangle$ is not compact. Then there is a forcing $\mathbb{P}$ such that $\langle X, \tau^{\mathbb{P}} \rangle$ is not countably compact.*

Proof. Assume that $\langle X, \tau \rangle$ is not compact. Then there is an open cover $\mathcal{U}$ of X with no finite subcover. Let $|\mathcal{U}| = \kappa$. If $\kappa = \aleph_0$, then $\langle X, \tau \rangle$ is not countably compact, and $\mathcal{U}$ witnesses the fact that $\langle X, \tau^{\mathbb{P}} \rangle$ is not countably compact for any forcing $\mathbb{P}$. So assume that $\kappa > \aleph_0$. Let $\mathbb{P} = Fn(\omega, \kappa)$, which collapses the uncountable cardinal κ to a countable ordinal ([9, VII, Lemma 5.2]). Then, in $\mathbf{V}^{\mathbb{P}}$, $\mathcal{U}$ is a countable open cover of X with no finite subcover. Hence, $\langle X, \tau^{\mathbb{P}} \rangle$ is not countably compact. □

Using Proposition 2.1, we obtain the following "countably compact" version of Theorem 1.1.

Proposition 2.2. *For a countably compact Hausdorff space X, the following are equivalent:*

(1) *The countable compactness of X is preserved by any forcing.*
(2) *X is compact and scattered.*

Proof. (1) $\Longrightarrow$ (2). If X is not compact, then, by Proposition 2.1, X is not countably compact in $\mathbf{V}^{\mathbb{P}}$ for some forcing $\mathbb{P}$. So assume that X is compact and is not scattered. By Theorem 1.1, X is not compact in $\mathbf{V}^{Fn(\omega,2)}$, where $Fn(\omega, 2)$ is the forcing that adjoins a Cohen real ([9, VII, Definition 5.1]). Then, again by Proposition 2.1, there is an $Fn(\omega, 2)$-name $\dot{\mathbb{P}}$ for a forcing such that X is not countably compact in $\mathbf{V}^{Fn(\omega,2)*\dot{\mathbb{P}}}$.
(2) $\Longrightarrow$ (1). By Theorem 1.1, X remains compact in any forcing extension. □

Proposition 2.2 is similar to Theorem 1.1 and is not very interesting. Therefore, we decided to restrict ourselves to a subclass of forcings. In Proposition 2.1, κ becomes a countable set in $\mathbf{V}^{\mathbb{P}}$ such that for every countable set $S \in \mathbf{V}$, we have $\kappa \nsubseteq S$. In other words, the forcing $Fn(\omega, \kappa)$ produces a countable set that is not covered by any countable set in the ground model. We consider this situation extreme and exclude it from our study. We focus on forcings that satisfy the countable covering property.

Definition 2.3. A forcing $\mathbb{P}$ is said to satisfy the *countable covering property* if, for every countable set $A \in \mathbf{V}^{\mathbb{P}}$ such that $A \subseteq \mathbf{V}$, there is a countable set $B \in \mathbf{V}$ such that $A \subseteq B$.

Fact 2.4. Proper forcings satisfy the countable covering property ([6, Fact 3.13]). In particular, a forcing which has the countable chain condition satisfies the countable covering property.

We use a forcing defined by David Booth.

Definition 2.5 ([4]; [11, p. 20])**.** Let D be a countable infinite set and let λ be a cardinal. Assume that $\mathcal{F} = \{F_\xi \subseteq D : \xi < \lambda\}$ is a free filter on D. Define a Booth forcing $\mathbb{B}(\mathcal{F})$ such that

$$\mathbb{B}(\mathcal{F}) = \{\langle s, E\rangle : s \in [D]^{<\omega} \text{ and } E \in [\lambda]^{<\omega}\}.$$

Order $\mathbb{B}(\mathcal{F})$ by $\langle s, E\rangle \leq \langle s', E'\rangle$ if and only if $s \supseteq s', E \supseteq E'$, and $s - s' \subseteq \bigcap_{\xi \in E'} F_\xi$.

Fact 2.6. $\mathbb{B}(\mathcal{F})$ is σ-centered and so it satisfies the countable chain condition. Forcing with $\mathbb{B}(\mathcal{F})$ adds an infinite set $E \subseteq D$ such that $E \setminus F$ is finite for all $F \in \mathcal{F}$.

Here is a crucial lemma.

Lemma 2.7. *Suppose that X is a separable non-compact regular space. Let D be a countable dense subset of X. Then there is a free filter $\mathcal{F}$ on D such that in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$ there exists an infinite set $E \subseteq D$ such that E is a closed discrete subset of X.*

Proof. Since X is a non-compact regular space, there is an open cover $\mathcal{U}$ of X such that for every finite subset $\mathcal{U}'$ of $\mathcal{U}$, $\bigcup \mathcal{U}'$ is not dense in X. For each $x \in X$, take an open neighborhood U_x of x such that $\overline{U_x} \subseteq U$ for some $U \in \mathcal{U}$. Then $\{U_x : x \in X\}$ is an open cover of X such that for every finite subset S of X, $\bigcup\{\overline{U_x} : x \in S\}$ is not dense in X. Let D be a countable dense subset of X and let

$$\mathcal{F} = \left\{D \setminus \bigcup\{\overline{U_x} : x \in S\} : S \in [X]^{<\omega}\right\}.$$

Then $\mathcal{F}$ is a free filter base defined on D. As in Fact 2.6, forcing with $\mathbb{B}(\mathcal{F})$ adds an infinite set $E \subseteq D$ such that $E \setminus F$ is finite for all $F \in \mathcal{F}$, and so $E \cap U_x$ is finite for all $x \in X$. Thus, E is a closed discrete subset of X. □

A Booth forcing can destroy countable compactness.

Corollary 2.8. *Suppose that $\langle X, \tau \rangle$ is a separable, non-compact, countably compact regular space. Then there is a Booth forcing $\mathbb{B}(\mathcal{F})$ such that $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ is not countably compact.*

Proof. Let D be a countable dense subset of $\langle X, \tau \rangle$. By Lemma 2.7, there is a free filter $\mathcal{F}$ on D such that in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$ D contains an infinite closed discrete subset of X. Thus, $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ is not countably compact. □

We note the following fact.

Fact 2.9 ([5, 1.7.10])**.** A topological space is called *perfect* if it has no isolated points. A topological space is called *scattered* if its every nonempty subspace has an isolated point. Every topological space X can be uniquely represented as $X = P \cup S$, where P is perfect, S is scattered, and $P \cap S = \emptyset$. P is a closed subset of X and it is called the *perfect kernel* of X.

In the proof of Lemma 4.1 in [7], only regularity (rather than Tychonoffness) is used, and so we can replace "Tychonoff" by "regular" in the following proposition.

Proposition 2.10 ([7, Lemma 4.1])**.** *Suppose that $\langle X, \tau \rangle$ is a regular space and that forcing with $\mathbb{P}$ adjoins a real. If $\langle X, \tau \rangle$ is perfect, then $\langle X, \tau^{\mathbb{P}} \rangle$ is neither countably compact nor pseudocompact.*

Here is the main theorem of this section.

Theorem 2.11. *For a countably compact regular space X, the following are equivalent:*

(1) *The countable compactness of X is preserved by any forcing that satisfies the countable covering property.*
(2) *The countable compactness of X is preserved by Booth forcing $\mathbb{B}(\mathcal{F})$ for any free filter $\mathcal{F}$.*
(3) *X is scattered and every countable subset of X has a compact closure.*

Proof. (1) $\Longrightarrow$ (2). $\mathbb{B}(\mathcal{F})$ satisfies the countable covering property (Fact 2.4, Fact 2.6).

(2) $\Longrightarrow$ (3). First assume to the contrary that X is not scattered. Let P be the perfect kernel of X as in Fact 2.9; then P is nonempty. Let $\mathcal{F}$ be any free ultrafilter on ω. Then $\mathbb{B}(\mathcal{F})$ adjoins a real and in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$ P is not countably compact by Proposition 2.10. Since P is a closed subset of X

and countable compactness is hereditary with respect to closed subsets, we can conclude that X is not countably compact in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$.

Next assume to the contrary that there exists a countable set $D \subseteq X$ such that $\overline{D}$ is not compact. Then $\overline{D}$ is a separable, non-compact, countably compact regular space. By Corollary 2.8, there is a Booth forcing $\mathbb{B}(\mathcal{F})$ such that, in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$, $\overline{D}$ is not countably compact. Since $\overline{D}$ remains a closed subset of X in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$, we can conclude that X is not countably compact in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$.

(3) $\Longrightarrow$ (1). Suppose that a forcing $\mathbb{P}$ satisfies the countable covering property. In $\mathbf{V}^{\mathbb{P}}$, take a countable infinite subset A of X. We will show that A has an accumulation point. Take a countable set $B \in \mathbf{V}$ such that $A \subseteq B$. By the assumption, $\overline{B}$ is compact (and scattered), so $\overline{B}$ remains compact in $\mathbf{V}^{\mathbb{P}}$ by Theorem 1.1. Since A is an infinite subset of $\overline{B}$, A has an accumulation point. □

Remark 2.12. Condition (3) of Theorem 2.11 implies that X is countably compact (assuming that X is T_1).

Recall that a topological space is *sequentially compact* if every sequence in the space has a convergent subsequence. Below is a proposition regarding sequential compactness, which says that a forcing satisfying the countable covering property cannot destroy sequential compactness while preserving countable compactness.

Proposition 2.13. *Let $\langle X, \tau\rangle$ be a sequentially compact regular space and let $\mathbb{P}$ be a forcing that satisfies the countable covering property. Then the following are equivalent:*

(1) *$\langle X, \tau^{\mathbb{P}}\rangle$ is sequentially compact.*
(2) *$\langle X, \tau^{\mathbb{P}}\rangle$ is countably compact.*

Proof. (1) $\Longrightarrow$ (2). Sequentially compact spaces are countably compact ([12, 17G.2]).

(2) $\Longrightarrow$ (1). Suppose that $\langle X, \tau^{\mathbb{P}}\rangle$ is countably compact. Note that scatteredness is preserved by any forcing ([8, Lemma 5]) and scattered countably compact regular spaces are sequentially compact ([2, Proposition 1]). Therefore, if $\langle X, \tau\rangle$ is scattered, then $\langle X, \tau^{\mathbb{P}}\rangle$ remains scattered (and is countably compact), and so $\langle X, \tau^{\mathbb{P}}\rangle$ is sequentially compact. So assume that $\langle X, \tau\rangle$ is not scattered. In $\mathbf{V}^{\mathbb{P}}$, take a sequence $A = \{a_n : n \in \omega\} \subseteq X$. We will show that A contains a convergent subsequence. Let P be the perfect kernel of $\langle X, \tau\rangle$ as in Fact 2.9; then P is nonempty. In $\mathbf{V}^{\mathbb{P}}$, P is a closed subset of the countably compact space $\langle X, \tau^{\mathbb{P}}\rangle$ so P is countably compact in $\mathbf{V}^{\mathbb{P}}$, which means that the countable compactness of P is preserved by the forcing $\mathbb{P}$. By Proposition 2.10, $\mathbb{P}$

does not adjoin a real. Since $\mathbb{P}$ satisfies the countable covering property, there is a countable set $B \in \mathbf{V}$ such that $A \subseteq B$. Since $\mathbb{P}$ does not adjoin a real, it does not add a new subset of B; therefore, A must be in $\mathbf{V}$. Since $\langle X, \tau\rangle$ is sequentially compact, A contains a convergent subsequence. $\square$

3. Preservation of Pseudocompactness

Now let us study preservation of pseudocompactness. Recall that a topological space X is pseudocompact if X is a Tychonoff space and every continuous real-valued function defined on X is bounded. Note that countably compact Tychonoff spaces are pseudocompact ([5, Theorem 3.10.20]). Using [5, Theorem 3.10.22], it is not difficult to prove the following two lemmas.

Lemma 3.1. *Let X be a pseudocompact space. If A is an infinite set of isolated points of X, then A is not a closed subset of X.*

Lemma 3.2. *Let X be a Tychonoff space such that the set of all isolated points is dense in X. If every infinite set of isolated points of X has an accumulation point, then X is pseudocompact.*

We prove the following lemma.

Lemma 3.3. *Let X be a Tychonoff space such that the set of all isolated points is dense in X. If every countable set of isolated points of X has a scattered compact closure, then X is scattered.*

Proof. Let X^* be a compactification of X; that is, X^* is compact and X is dense in X^*. It suffices to show that X^* is scattered. Assume to the contrary that X^* is not scattered. The Main Theorem [10, p. 214]states that a compact Hausdorff space is scattered if and only if it cannot be continuously mapped onto $[0,1]$. So there is a continuous onto map $f : X^* \to [0,1]$. Observe that by Lemma 3.2, X is pseudocompact. Since the continuous image of a pseudocompact space is pseudocompact ([5, 3.10.24]) and pseudocompactness is equivalent to compactness for subsets of the real line ([5, 3.10.21; 3.10.1]), we can conclude that $f(X)$ is compact. Since X is dense in X^*, $f(X)$ is dense in $f(X^*) = [0,1]$, and so $f(X) = [0,1]$. Let I_0 be the set of all isolated points in X. By the assumption, we have $\overline{I_0} = X$. Since $f(I_0)$ is a subset of $[0,1]$, $f(I_0)$ is separable, so there is a countable set $A \subseteq I_0$ such that $\overline{f(A)} = \overline{f(I_0)}$. We have $[0,1] = f(X) = f(\overline{I_0}) \subseteq \overline{f(I_0)}$ so $\overline{f(A)} = [0,1]$. By the assumption, A has a (scattered) compact closure, and so it is easy to see that $f(\overline{A}) = \overline{f(A)}$. Hence, we have $f(\overline{A}) = [0,1]$. This is a contradiction because, as mentioned before, the scattered compact Hausdorff space $\overline{A}$ cannot be continuously mapped onto $[0,1]$ ([10, p. 214]). $\square$

We define a forcing which we use in the main theorem of this section. This forcing is essentially the same as the one in the proof of Lemma 4.2 in [7], except that a sequence of discrete points $\{x_i\}_i$ is replaced by a countable discrete set C.

Definition 3.4. Let X be a regular space and let C be a countable discrete subset of X. Suppose that there is a countable closed discrete subset D of X such that

$$D \subseteq \overline{C} \setminus C.$$

Let $D = \{d_n : n \in \omega\}$. For each $n \in \omega$, let $\mathcal{U}_n$ be a neighborhood base of d_n. Define a forcing $\mathbb{Q}(C, D)$ as follows. A condition $p = (s_p, U_p)$ in $\mathbb{Q}(C, D)$ has the form

- $s_p \in \bigcup_{n\in\omega} {}^nC$, where ${}^nC$ is the set of all sequences in C of length n.
- $U_p \in \prod_{n\in\omega} \mathcal{U}_n$.

For $p = (s_p, U_p)$ and $q = (s_q, U_q)$, define $p \leq_{\mathbb{Q}(C,D)} q$ if

- $s_p \supseteq s_q$,
- $(\forall n \in \omega)(U_p(n) \subseteq U_q(n))$, and
- $(\forall n \in dom(s_p) \setminus dom(s_q))(s_p(n) \in U_q(n))$.

Here are properties of $\mathbb{Q}(C, D)$.

Proposition 3.5. *Let $\mathbb{Q}(C, D)$ be as in Definition 3.4. $\mathbb{Q}(C, D)$ is σ-centered so it satisfies the countable chain condition, and, in $\mathbf{V}^{\mathbb{Q}(C,D)}$, C contains an infinite closed (discrete) subset of X.*

Proof. (The proof is essentially the same as that of Lemma 4.2 in [7].) It is easy to see that $\mathbb{Q}(C, D)$ is σ-centered. Let G be a $\mathbb{Q}(C, D)$-generic filter over $\mathbf{V}$. Let $S = \bigcup\{s_p : p \in G\}$ and let $\tilde{S} = \{S(n) : n \in \omega\}$. For each $n \in \omega$, $\{p \in \mathbb{Q}(C, D) : |s_p| \geq n\}$ is a dense subset of $\mathbb{Q}(C, D)$, so $\tilde{S}$ is an infinite subset of the discrete set C. To show that $\tilde{S}$ is a closed subset of X, fix $x \in X \setminus \tilde{S}$. We will find an open set containing x and missing $\tilde{S}$. Since D is a closed discrete subset of X, there is an open set V such that $x \in V$ and $|\overline{V} \cap D| \leq 1$. For each $n \in \omega$ with $d_n \notin \overline{V}$, pick $H_n \in \mathcal{U}_n$ such that $H_n \cap V = \emptyset$; if $d_n \in \overline{V}$, then choose any $H_n \in \mathcal{U}_n$. The set $\{p \in \mathbb{Q}(C, D) : (\forall n \in \omega)(U_p(n) \subseteq H_n)\}$ is a dense subset of $\mathbb{Q}(C, D)$. So there is $p \in G$ such that $U_p(n) \subseteq H_n$ for all $n \in \omega$, which implies that $S(n) \in H_n$ for all but finitely many $n \in \omega$. Thus, $\tilde{S} \cap V$ is finite. Let $V' = V \setminus \tilde{S}$; then V' is an open set such that $x \in V'$ and $V' \cap \tilde{S} = \emptyset$. $\square$

$Fn(\omega, 2)$ is the forcing that adjoins a Cohen real ([9, VII, Definition 5.1]). We use the lemma below to prove the main theorem of this section.

Lemma 3.6. *Suppose that C is a countable discrete subset of a regular space X such that $\overline{C}$ is not scattered. Then there is a two-stage iterated forcing*

$$\mathbb{P} := Fn(\omega, 2) * \dot{\mathbb{Q}}(C, \dot{D})$$

such that $\mathbb{P}$ satisfies the countable chain condition and, in $\mathbf{V}^{\mathbb{P}}$, C contains an infinite closed (discrete) subset of X.

Proof. Let P be the perfect kernel of $\overline{C}$ as in Fact 2.9; then $P \subseteq \overline{C} \setminus C$. By Lemma 2.10, P is not countably compact in $\mathbf{V}^{Fn(\omega,2)}$, and so there is a countable closed discrete subset D of P in $\mathbf{V}^{Fn(\omega,2)}$. Since P is a closed subset of $\overline{C}$, P is closed in X, and so D is a closed (and discrete) subset of X. In $\mathbf{V}^{Fn(\omega,2)}$, force with $\mathbb{Q}(C, D)$; by Proposition 3.5, C gets an infinite closed discrete subset of X. Since $1 \Vdash_{Fn(\omega,2)}$ "$\dot{\mathbb{Q}}(C, \dot{D})$ satisfies the countable chain condition," $\mathbb{P}$ satisfies the countable chain condition ([9, VIII, Lemma 5.7]). $\square$

Remark 3.7. In Lemma 3.6, $Fn(\omega, 2)$ can be replaced by any forcing that adjoins a real and satisfies the countable chain condition.

Here is a preservation theorem for pseudocompact spaces.

Theorem 3.8. *For a pseudocompact space X, the following are equivalent:*

(1) *The pseudocompactness of X is preserved by any forcing that satisfies the countable covering property.*
(2) *The pseudocompactness of X is preserved both by Booth forcing $\mathbb{B}(\mathcal{F})$ for any free filter $\mathcal{F}$ and by $Fn(\omega, 2) * \dot{\mathbb{Q}}(C, \dot{D})$ for any countable discrete set $C \in \mathbf{V}$ and any $Fn(\omega, 2)$-name $\dot{D}$.*
(3) *X is scattered and every countable set of isolated points of X has a compact closure.*

Proof. (1) $\Longrightarrow$ (2). By Fact 2.6 and Lemma 3.6, both $\mathbb{B}(\mathcal{F})$ and $Fn(\omega, 2) * \dot{\mathbb{Q}}(C, \dot{D})$ satisfy the countable covering property.

(2) $\Longrightarrow$ (3). Let I_0 be the set of all isolated points of X. First we show the following claim:

Claim 1. I_0 is dense in X.

Proof: Assume to the contrary that I_0 is not dense in X; then X is not scattered. Let $X = P \cup S$, where P is perfect and S is scattered as in Fact 2.9. Since $I_0 \subseteq S$ and I_0 is dense in S, we have $S \subseteq \overline{I_0}$. Since $X \setminus \overline{I_0} \neq \emptyset$ and $X \setminus \overline{I_0} \subseteq P$, we have $int(P) \neq \emptyset$, where $int(P)$ is the interior of P. It is easy to see that $\overline{int(P)}$ is a perfect set. Let $\mathcal{F}$ be any free ultrafilter on ω. In $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$, $\overline{int(P)}$ is not pseudocompact by Proposition 2.10. The closure of the interior of a set is called a closed domain and pseudocompactness is hereditary with respect to closed domains ([5, 3.10.F.(d)]). Therefore, X is not pseudocompact in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$. This completes the proof of Claim 1.

Here is another claim

Claim 2. Every countable subset of I_0 has a scattered compact closure.

Proof: First assume to the contrary that there is a countable set $C \subseteq I_0$ such that $\overline{C}$ is not compact. By Lemma 2.7, a Booth forcing $\mathbb{B}(\mathcal{F})$ for some filter $\mathcal{F}$ adds an infinite set $E \subseteq C$ such that E is a closed subset of $\overline{C}$. Since $\overline{C}$ is closed in X, E is a closed subset of X. By Lemma 3.1, X is not pseudocompact in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$.

Next assume to the contrary that there is a countable set $C \subseteq I_0$ such that $\overline{C}$ is not scattered. Note that C is a discrete set because so is I_0. By Lemma 3.6, we can define a forcing $\mathbb{P} := Fn(\omega, 2) * \dot{\mathbb{Q}}(C, \dot{D})$ such that in $\mathbf{V}^{\mathbb{P}}$ C contains an infinite closed subset of X. By Lemma 3.1, X is not pseudocompact in $\mathbf{V}^{\mathbb{P}}$. This completes the proof of Claim 2.

By Claim 1, Claim 2, and Lemma 3.3, we can conclude that X is scattered.

(3) $\implies$ (1). Let $\mathbb{P}$ be a forcing which satisfies the countable covering property. We use Lemma 3.2 to show that X is pseudocompact in $\mathbf{V}^{\mathbb{P}}$. Since X is scattered, the set of all isolated points of X is dense in X. In $\mathbf{V}^{\mathbb{P}}$, take an infinite set A of isolated points of X; we will show that A has an accumulation point. Since $\mathbb{P}$ satisfies the countable covering property, there exists a countable set $B \in \mathbf{V}$ such that $A \subseteq B$. We may assume that B consists of isolated points of X. By the assumption, $\overline{B}$ is compact (and scattered), and so it remains compact in $\mathbf{V}^{\mathbb{P}}$ by Theorem 1.1. Since A is an infinite subset of $\overline{B}$, A has an accumulation point. □

Remark 3.9. Condition (3) in Theorem 3.8 implies that X is pseudocompact by Lemma 3.2 (assuming that X is Tychonoff).

4. Examples

We give an example of a space whose countable compactness is destroyed by a Booth forcing.

Example 4.1. *There is a countably compact space $\langle Y, \tau \rangle$ such that $\langle Y, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ is not countably compact for some Booth forcing $\mathbb{B}(\mathcal{F})$.*

Proof. Let $Y = \gamma'\mathbb{N} \setminus \{\delta\}$ be the space in [5, 3.12.17.(d)], which is a scattered, separable, non-compact, countably compact, locally compact (regular) space. By Corollary 2.8, Y is not countably compact in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$ for some Booth forcing $\mathbb{B}(\mathcal{F})$. □

Here is a well-known space.

Example 4.2 ([5, 3.12.20.(a)]). *There is a non-countably compact pseudocompact space $\langle T, \tau \rangle$ such that $\langle T, \tau^{\mathbb{P}} \rangle$ remains pseudocompact for any forcing $\mathbb{P}$ that satisfies the countable covering property.*

Proof. The Tychonoff Plank $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ is a non-countably compact pseudocompact space and it satisfies condition (3) in Theorem 3.8. Therefore, T remains pseudocompact in $\mathbf{V}^{\mathbb{P}}$ for any forcing $\mathbb{P}$ that satisfies the countable covering property. $\{(\omega_1, n) : n \in \omega\}$ is an infinite closed discrete subspace of T, which witnesses the fact that T is not countably compact. □

Combining Example 4.1 and Example 4.2, we can construct a countably compact space whose countable compactness is destroyed by forcing but pseudocompactness is preserved.

Example 4.3. *There is a countably compact Tychonoff space $\langle X, \tau \rangle$ such that for some Booth forcing $\mathbb{B}(\mathcal{F})$, $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ is not countably compact but is pseudocompact.*

Proof. Let Y be the space in the proof of Example 4.1 and let T be the Tychonoff Plank in the proof of Example 4.2. Since Y is separable, the set of all isolated points in Y is countable; let $D = \{d_n : n \in \omega\}$ be the set of all isolated points in Y. Since the set of all isolated points in a scattered space is dense, we have $\overline{D} = Y$. Let $\langle X, \tau \rangle$ be the quotient space of the topological sum of Y and T obtained by identifying $d_n \in Y$ and $(\omega_1, n) \in T$ for each $n \in \omega$. It is easy to see that X is still scattered and locally compact; in particular, it is Tychonoff ([5, 3.3.1]). X is countably compact because every infinite subset of $\{(\omega_1, n) : n \in \omega\} = \{d_n : n \in \omega\}$ now has an accumulation point in X. Since $\overline{D}^X = Y$ (here we regard D and Y as subspaces of X) and Y is not compact, X does not satisfy condition (3) of Theorem 2.11. Therefore, $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ is not countably compact for some Booth forcing $\mathbb{B}(\mathcal{F})$. Let I_0 be the set of all isolated points in X. Then $I_0 \subseteq T$ and every countable subset of I_0 has a compact closure (which is a subset of T). So X satisfies condition (3) of Theorem 3.8. Hence, $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$ remains pseudocompact. □

References

[1] Leandro F. Aurichi, *D-spaces, topological games, and selection principles*, Topology Proc. **36** (2010), 107–122.

[2] John Warren Baker, *Ordinal subspaces of topological spaces*, General Topology and Appl. **3** (1973), 85–91.

[3] James E. Baumgartner and Franklin D. Tall, *Reflecting Lindelöfness*, Topology Appl. **122** (2002), no. 1-2, 35–49.

[4] David Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970/1971), no. 1, 1–24.

[5] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[6] Martin Goldstern, *Tools for your forcing construction* in Set Theory of the Reals (Ramat Gan, 1991). Ed. Haim Judah. Israel Math. Conf. Proc., 6. Ramat Gan: Bar-Ilan Univ., 1993. 305–360.

[7] Akira Iwasa, Masaru Kada, and Shizuo Kamo, *Preservation of convergence of a sequence to a set*, Topology Proc. **44** (2014), 97–105.

[8] I. Juhász and W. Weiss, *Omitting the cardinality of the continuum in scattered spaces*, Topology Appl. **31** (1989), no. 1, 19–27.

[9] Kenneth Kunen, *Set Theory. An Introduction to Independence Proofs.* Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York: North-Holland Publishing Co., 1980.

[10] A. Pelczyński and Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222.

[11] Mary Ellen Rudin, *Lectures on Set Theoretic Topology.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 23. Providence, R.I.: American Mathematical Society, 1975.

[12] Stephen Willard, *General Topology.* Reading, Mass.-London-Don Mills, Ont.: Addison-Wesley Publishing Co., 1970.

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