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ON SEPARATORS OF THE SPACE OF NONNEGATIVELY CURVED PLANES

A. K. AMARASINGHE

Abstract. We shall prove that the Hilbert cube cannot be separated by a weakly infinite dimensional subset. As a corollary we obtain that the complement of a weakly infinite dimensional subset of the space of complete nonnegatively curved metrics is continuum connected. We can extend this result to the associated moduli space when the set removed is a Hausdorff space with Haver's property $\mathcal{C}$. These results are refinements of theorems proven by Igor Belegradek and Jing Hu [3].

The spaces of Riemannian metrics with positive scalar curvature are subjects of intensive study [10]. The connectedness properties of such spaces on $\mathbb{R}^2$ were studied recently by Igor Belegradek and Jing Hu [3]. They proved that in the space $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ of complete Riemannian metrics of nonnegative curvature on the plane equipped with the topology of C^k uniform convergence on compact sets, the complement $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2) \setminus X$ is connected for every finite dimensional X. Note that the space $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ is separable metric [3]. In this note we extend Belegradek–Hu's result to the case of infinite dimensional spaces X. We recall that infinite dimensional spaces split in two disjoint classes: strongly infinite dimensional (like the Hilbert cube) and weakly infinite dimensional (like the union $\cup_n I^n$). We prove Belegradek–Hu's theorem for weakly infinite dimensional X. This extension is final since strongly infinite dimensional spaces can separate the Hilbert cube.

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We note that in [3] there is a similar connectedness result with finite dimensional X for the moduli space $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$, the quotient space of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ by the $\mathrm{Diff}(\mathbb{R}^2)$-action via pullback. In the case of the moduli space we manage to extend its connectedness result to the subsets $X \subset \mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$ with Haver's property $\mathcal{C}$ (called $\mathcal{C}$-spaces in [5]). It is known that the property $\mathcal{C}$ implies the weak infinite dimensionality [5]. There is an old open problem that asks if every weakly infinite dimensional compact metric space has property $\mathcal{C}$. For general spaces these two classes are different [1].

1. Infinite Dimensional Spaces

We denote the *Hilbert cube* by $Q = [-1, 1]^\infty = \Pi_{n=1}^\infty \mathbb{I}_n$. The *pseudo interior* of Q is the set $s = (-1, 1)^\infty$ and the *pseudo boundary* of Q is the set $B(Q) = Q \setminus s$. The *faces* of Q are the sets $W_i^- = \{x \in Q | x_i = -1\}$ and $W_i^+ = \{x \in Q | x_i = 1\}$. Every space under consideration is a separable metric space.

A space $S \subseteq X$ is said to *separate* X if $X \setminus S$ is disconnected. Let X be a space and let A and B be two disjoint closed subsets of X; a *separator between A and B* is a closed subset $S \subseteq X$ such that $X \setminus S$ can be written as the disjoint union of open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Definition 1.1. Let X be a space and Γ be an index set. A family of pairs of disjoint closed sets $\tau = \{(A_i, B_i) : i \in \Gamma\}$ of X is said to be *essential* if, for every family $\{L_i : i \in \Gamma\}$ where L_i is a separator between A_i and B_i, we have $\bigcap_{i \in \Gamma} L_i \neq \emptyset$.

If τ is not essential, then it is called *inessential*.

We recall that the classical covering dimension can be defined in terms of essential families.

Definition 1.2. For a space X, we define $\mathrm{dim}X \in \{-1, 0, 1, \cdots\} \cup \{\infty\}$ by

$\mathrm{dim}X = -1$ if and only if $X = \emptyset$;
$\mathrm{dim}X \leq n$ if and only if every family of $n + 1$ pairs of disjoint closed subsets is inessential;
$\mathrm{dim}X = n$ if and only if $\mathrm{dim}X \leq n$ and $\mathrm{dim}X \nleq n - 1$;
$\mathrm{dim}X = \infty$ if and only if $\mathrm{dim}X \neq n$ for all $n \geq -1$.

A space X is called *strongly infinite dimensional* if there exists an infinite essential family of pairs of disjoint closed subsets of X. X is called *weakly infinite dimensional* if X is not strongly infinite dimensional.

We recall that a space X is *continuum connected* if every two points $x, y \in X$ are contained in a connected compact subset. Every continuum connected space is connected, but the converse is not true.

The following well-known theorem about subspaces of Q is the infinite dimensional counterpart of [5, Lemma 1.8.16] and a proof can be found in [5, Remark 1.8.17].

Proposition 1.3. *Let $S \subseteq Q$ such that S meets every continuum from W_1^+ to W_1^-. Then S is strongly infinite dimensional.* □

The next theorem and the subsequent corollary are extensions of a theorem of Mazurkiewicz (see [5, Theorem 1.8.18]) to weakly infinite dimensional subspaces of Q.

Theorem 1.4. *Let $x, y \in Q \setminus S$ where $S \subseteq Q$ is such that it intersects every continuum from x to y. Then S is strongly infinite dimensional.*

Proof. The quotient space $Q/(W_1^- \cup W_1^+) \cong (Q/W_1^-)/W_1^+$ is the suspension of Q; hence, it is a compact, convex, infinite dimensional subspace of ℓ^2. By Ott-Heinrich Keller [7], it is homeomorphic to the Hilbert cube Q. Denote this homeomorphism by $h : Q \to Q/(W_1^- \cup W_1^+)$, and let $h(W_1^-) = w_1^-$ and $h(W_1^+) = w_1^+$.

Since the Hilbert cube is n-homogeneous, there is a homeomorphism $g : Q \to Q$ such that $g(x) = w_1^-$ and $g(y) = w_1^+$. The map $h^{-1}g = f : Q \to Q$ is a homeomorphism such that $f(x) = W_1^-$ and $f(y) = W_1^+$.

Note that for $S \subset Q \setminus \{x, y\}$, we have $f(S)$ homeomorphic to S and $f(S) \subset Q \setminus \{W_1^- \cup W_1^+\}$. If C is any continuum from W_1^- to W_1^+ in Q, then $f^{-1}(C)$ is a continuum from x to y, and by hypothesis, $f^{-1}(C) \cap S \neq \emptyset$. We have $f(f^{-1}(C) \cap S) = C \cap f(S) \neq \emptyset$ so $f(S)$ intersects every continuum from W_1^- to W_1^+. Hence, by Proposition 1.3, $f(S)$ is strongly infinite dimensional, and so is S. □

Corollary 1.5. *If $S \subset Q$ is a weakly infinite dimensional subspace, then $Q \setminus S$ is continuum connected, and further, if S is closed, then $Q \setminus S$ is path connected.*

Proof. The second statement follows from the fact that any open connected subspace of a locally path connected space is path connected. □

Clearly, some strongly infinite dimensional compacta can separate the Hilbert cube. One such example is the subspace $Q \times \{0\}$ which separates the Hilbert cube $Q \times [-1, 1]$.

Definition 1.6. A topological space X has property $\mathcal{C}$ (is a $\mathcal{C}$-space) if, for every sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X, there exists a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ of families of pairwise disjoint open subsets of X such that, for $i = 1, 2, \dots$, each member of $\mathcal{H}_i$ is contained in a member of $\mathcal{G}_i$ and the union $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ is a cover of X.

The following is a theorem on dimension lowering mappings, the proof can be found in [5, Theorem 6.3.9].

Theorem 1.7. *If $f : X \to Y$ is a closed mapping of space X to $\mathcal{C}$-space Y such that, for every $y \in Y$, the fiber $f^{-1}(y)$ is weakly infinite dimensional, then X is weakly infinite dimensional.*

If one uses weakly infinite dimensional spaces instead of $\mathcal{C}$-spaces, the situation is less clear even in the case of compact Y.

Problem 1.8. Suppose that a Lie group G admits a free action by isometries on a metric space X with compact metric weakly infinite dimensional orbit space X/G. Does it follow that X is weakly infinite dimensional?

This is true for compact Lie groups in view of the slice theorem [2]. It also true for countable discrete groups [9, Theorem 3.9].

2. Applications

Now we proceed to generalize two theorems by Belegradek and Hu. We use the following result proven in [3, Theorem 1.3].

Theorem 2.1. *If K is a countable subset of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ and X is a separable metric space, then, for any distinct points $x_1, x_2 \in X$ and any distinct metrics $g_1, g_2 \in \mathcal{R}^k_{\geq 0}(\mathbb{R}^2)\backslash K$, there is an embedding of X into $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)\backslash K$ that takes x_1 and x_2 to g_1 and g_2, respectively.*

Here is our extension of the first Belegradek–Hu theorem.

Theorem 2.2. *The complement of every weakly infinite dimensional subspace S of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ is continuum connected. If S is closed, $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is path connected.*

Proof. Let S be a weakly infinite dimensional subspace of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$. Fix two metrics $g_1, g_2 \in \mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$. Theorem 2.1 implies that g_1 and g_2 lie in a subspace $\hat{Q}$ of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ that is homeomorphic to Q. Since $S \cap \hat{Q}$ is at most weakly infinite dimensional, $\hat{Q} \setminus S$ is continuum connected by Theorem 1.4. Then g_1 and g_2 lie in a continuum in $\hat{Q}$ that is disjoint from S. Hence, $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is continuum connected. If S is closed, from Corollary 1.5, we can conclude that $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is path connected. $\square$

In view of Theorem 1.4, we can state that if a subset S separates an open subset of the separable Hilbert space ℓ^2, then S is strongly infinite dimensional. From this fact we derive the following theorem.

Theorem 2.3. *The complement of every weakly infinite dimensional subspace S of $\mathcal{R}^\infty_{\geq 0}(\mathbb{R}^2)$ is locally connected. If S is closed, $\mathcal{R}^\infty_{\geq 0}(\mathbb{R}^2) \setminus S$ is locally path connected.*

Proof. Let $\mathcal{R}^\infty_{\geq 0}(\mathbb{R}^2)$ be $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ re-topologized with the C^∞ topology, the space $\mathcal{R}^\infty_{\geq 0}(\mathbb{R}^2)$ is homeomorphic to ℓ^2, the separable Hilbert space [3]. Let $x \in \ell^2$; then there is a neighborhood U of x homeomorphic to ℓ^2, and the set $U \setminus S$ is connected and path connected if S is closed. □

We do not know if the space $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ is locally path connected for $k < \infty$.

We prove a result similar to Theorem 2.2 for the associated moduli space $\mathcal{M}^{k,c}_{\geq 0}(\mathbb{R}^2)$, when the subspace removed is a Hausdorff space having property $\mathcal{C}$. This is a generalization of another Belegradek–Hu theorem.

Theorem 2.4. *If $S \subset \mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$ is a closed Hausdorff space with property $\mathcal{C}$, then $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is path connected.*

Proof. Denote by S_1 the set of smooth subharmonic functions with $\alpha(u) \leq 1$, where

$$\alpha(u) = \lim_{r\to\infty} \frac{\sup\{u(z) : |z| = r\}}{\log r}.$$

Note that S_1 is closed in the Frechét space $C^\infty(\mathbb{R}^2)$, it is not locally compact, and it is equal to the set of smooth subharmonic functions u such that the metric $e^{-2u}g_0$ is complete where g_0 is the standard Euclidean metric [3]. Let $q : S_1 \to \mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$ denote the continuous surjection sending u to the isometry class of $e^{-2u}g_0$. Let $\hat{S} = q^{-1}(S)$. Fix two points $g_1, g_2 \in \mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$, which are q images of u_1 and u_2 in S, respectively. By Theorem 2.1, we may assume that u_1 and u_2 lie in an embedded copy $\hat{Q}$ of Hilbert cube. It suffices to show that $\hat{Q} \setminus \hat{S}$ is path connected.

The set $\hat{Q} \cap \hat{S}$ is compact; hence, $\hat{q}$, the restriction of q to $\hat{Q} \cap \hat{S}$, is a continuous surjection. The map $\hat{q} : \hat{Q} \cap \hat{S} \to q(\hat{Q}) \cap S$ is a map between compact spaces and, in particular, it is a closed map. The set $\hat{q}(\hat{Q}\cap\hat{S})$ has property $\mathcal{C}$. We have each fiber $q^{-1}(y)$ to be finite dimensional [3], and hence $\hat{Q} \cap \hat{S}$ is weakly infinite dimensional by Theorem 1.7. Therefore, $\hat{Q} \setminus \hat{S}$ is path connected by Corollary 1.5, and so is $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$. □

It should be noted that the Hausdorff condition is essential in Theorem 2.4. If S is not Hausdorff, the map $\hat{q}$ above ceases to be a map between compact metric spaces. In [3] the authors did not require the Hausdorff condition in the formulation of their Theorem 1.6 when S is finite dimensional.

Suppose that Problem 1.8 has an affirmative answer for the Lie group $G = \mathrm{conf}(\mathbb{R}^2)$, the groups of conformal transformations on the plane. Then in Theorem 2.4 one can replace the property-$\mathcal{C}$ condition with the weak infinite dimensionality of S. Assuming that Problem 1.8 has an affirmative answer for the Lie group $G = \mathbb{C}^* \rtimes \mathbb{C}$, we can say that Problem 1.8 has an affirmative answer for the Lie group $\mathrm{conf}(\mathbb{R}^2)$.

Proposition 2.5. *If S is a closed Hausdorff weakly infinite dimensional subspace of $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$ and Problem 1.8 has an affirmative answer in the case of $G = \mathit{conf}(\mathbb{R}^2)$, then $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is path connected.*

Proof. We use the same setting as in the proof of Theorem 2.4. As stated in the proof of Theorem 1.6 in [3], two functions u and v of S_1 lie in the same isometry class if and only if $v = u \circ \psi - \log |a|$ for some $\psi \in \mathrm{conf}(\mathbb{R}^2)$; i.e., they lie in the same orbit under the action of $\mathrm{conf}(\mathbb{R}^2)$ on the space $C^\infty(\mathbb{R}^2)$ given by $(u, \psi) \mapsto u \circ \psi - \log |a|$. The subspace S_1 of $C^\infty(\mathbb{R}^2)$ is invariant under this action. Let $\pi : S_1 \to S_1/\mathrm{conf}(\mathbb{R}^2)$ be the projection onto the orbit space of this action. Also we note that the action of $\mathrm{conf}(\mathbb{R}^2)$ on S_1 is a free action by isometries.

Let S be a closed, weakly infinite dimensional Hausdorff subset of $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2)$. Let f and g be two elements in the complement of S. Then there are functions u and v mapping to f and g, respectively, by q. As noted above, u and v land in the same class if and only if $v = u \circ \psi - \log |a|$ for some $\psi \in \mathrm{conf}(\mathbb{R}^2)$. As shown in [3, Theorem 1.4], u and v lie in an embedded copy Q of Hilbert cube. Denote $q^{-1}(S) = \hat{S}$. It suffices to prove that $Q \cap \hat{S}$ is weakly infinite dimensional, so we would have a path joining u and v in $Q \setminus \hat{S}$, which transforms to a path joining g and f in $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$.

The set $\hat{S}$ is closed; hence, $Q \cap \hat{S}$ is compact. So the restriction of q to $Q \cap \hat{S}$ is a continuous surjection $\hat{q} : Q \cap \hat{S} \to q(Q) \cap S$ of compact separable metric spaces. Define the map $\eta : S_1/\mathrm{conf}(\mathbb{R}^2)$ by $uG \mapsto u^*$, the isometry class of $e^{-2u} g_0$. This map is injective by definition, and the diagram

$$\begin{tikzcd} S_1 \arrow[rr, "\pi"] \arrow[dr, "q"'] & & S_1/\mathrm{conf}(\mathbb{R}^2) \arrow[dl, "\eta"] \\ & \mathcal{M}^k_{\geq 0} & \end{tikzcd}$$

commutes. Let Y be the η preimage of $q(Q) \cap S$ in $S_1/\mathrm{conf}(\mathbb{R}^2)$. The action restricted to the preimage $\pi^{-1}(Y)$ of S_1 is an action of $\mathrm{conf}(\mathbb{R}^2)$ on $\pi^{-1}(Y)$ with orbit space Y, $Q \cap \hat{S} \subseteq \pi^{-1}(Y)$, and the set Y is weakly infinite dimensional. By the hypothesis, we can conclude that $\pi^{-1}(Y)$ is weakly infinite dimensional. Hence, $q(Q) \cap \hat{S}$ is weakly infinite dimensional; therefore, $\mathcal{M}^k_{\geq 0}(\mathbb{R}^2) \setminus S$ is path connected. $\square$

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