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FURTHER STUDY OF SIMPLE SMALE FLOWS USING FOUR BAND TEMPLATES

KAMAL M. ADHIKARI AND MICHAEL C. SULLIVAN

Abstract. In this paper, we discuss how to realize a nonsingular Smale flow with a four band template on a 3-sphere. This extends the work done by Michael C. Sullivan on Lorenz Smale flows and by Bin Yu on realizing Lorenz like Smale flows on 3-manifolds, and continues the work of Elizabeth L. Haynes and Sullivan on realizing simple Smale flows with a different four band template on a 3-sphere.

1. Introduction

A nonsingular Smale flow on a 3-manifold M is a structurally stable flow with a 1-dimensional chain recurrent set. A chain recurrent set consists of a finite number of disjoint basic sets which are compact and transitive. A basic set may be an attractor, a repeller, or a saddle set. We study the realizations of a nonsingular Smale flow when the saddle set is modeled by a four band template and this extends the work done in [13]. A template is a compact branched 2-manifold with boundary which has a smooth semiflow and is built locally from two types of charts, joining and splitting. The most popular template is a Lorenz template which was introduced by R. F. Williams [20] to study the Lorenz attractor. Joan S. Birman and Williams [2] proved the template theorem which says that in Smale flow, the chaotic saddle set can be represented by a template and any knot type of the periodic orbits can be studied within a template.

In the past, much work has been done to realize Smale flows using templates. Michal C. Sullivan studied a special type of nonsingular Smale

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flow (NSF) on S^3 by using a Lorenz template [17]. Bin Yu [22] discussed the realizations of a nonsingular Smale flow by using Lorenz-like templates and extended the work done by Sullivan in [17]. Elizabeth L. Haynes and Sullivan studied the Smale flows on S^3 modeled by a four band template [13]. Here, we discuss how to realize nonsingular Smale flows on S^3 when the saddle set of the flow is modeled by a four band template different from the template used in [13]. This makes a further extension of [13] and we hope that this work will add one more point for the detailed study of NSFs on 3-manifolds.

2. Background

Definition 2.1. A *flow* on a manifold M is a continuous function ϕ_t:$M \times \mathbb{R} \to M$ such that $\phi_t(p, 0) = p$, for all $p \in M$ and $\phi_t(\phi_t(p, s), t) = \phi_t(p, s+t)$, for all $p \in M, t \in \mathbb{R}$.

An *orbit* of a point $p \in M$ is given by $O(p) = \{q \in M| \ q = \phi_t \ (p, t), t \in \mathbb{R}\}$ where ϕ_t is a flow map. A set $\Lambda \subset M$ is called an *invariant set* for a flow ϕ_t if $\phi_t(\Lambda, t) = \Lambda$ for all $t \in \mathbb{R}$. An invariant set $\Lambda \subset M$ is said to be *hyperbolic* or to have a hyperbolic structure if the tangent bundle of M restricted to Λ splits into three sub-bundles, namely stable bundles, unstable bundles, and center of the flow, each of which is invariant under $D\phi_t$ for all t.

Definition 2.2. Let X be a subset of a hyperbolic invariant set Λ of a flow ϕ_t on M. Then the *stable* and *unstable manifolds* of X in M are given by
$W^s(X) = \{y \in M| \lim_{t\to\infty} ||\phi_t(x) - \phi(y)|| = 0\}$
$W^u(X) = \{y \in M| \lim_{t\to-\infty} ||\phi_t(x) - \phi(y)|| = 0\}$ for all $x \in X$.

Definition 2.3. A point $x \in M$ is chain recurrent for a flow ϕ_t if, for any $\epsilon > 0$, there exists a sequence of points $\{x = x_1, x_2, ..., x_n = x\}$ and real numbers$\{t_1, t_2, ..., t_n - 1\}$ such that $t_i > 1$ and $||\phi_{t_i}(x_i) - x_{i+1}|| < \epsilon$ for all $1 \le i \le n-1$. The chain recurrent set is the set of all chain recurrent points on M.

According to Smale's theorem, if the flow is hyperbolic on its chain recurrent set, the chain recurrent set is the disjoint union of basic sets where each basic set is closed, invariant, contains a dense orbit, and the periodic orbits form a dense subset. A basic set may be an attractor, repeller, or saddle set. For a nonsingular Smale flow, attractors and repellers are necessarily isolated closed orbits. A basic saddle set may be an isolated closed orbit or the suspension of a non-trivial shift of finite type [3], [4]. For the latter case, we say the saddle set is chaotic. A chaotic saddle set can be modeled by a template.

Definition 2.4. A given flow ϕ_t on a manifold M is called a *Morse–Smale flow* if

(1) the chain recurrent set is hyperbolic,
(2) each basic set consists of a single closed orbit or fixed point, and
(3) the stable and unstable manifolds of basic sets meet transversally.

Definition 2.5. A given flow ϕ_t on a manifold M is called a *Smale flow* if

(1) the chain recurrent set is hyperbolic,
(2) the stable and unstable manifolds of any two orbits in the chain recurrent set meet transversally, and
(3) each basic set is zero or one dimensional.

A *Lorenz Smale flow* is a Smale flow with three basic sets: an attracting closed orbit, a repelling closed orbit, and a nontrivial saddle set modeled by a Lorenz template. A Lorenz-like Smale flow is a Smale flow with an attracting closed orbit, a repelling closed orbit, and a nontrivial saddle set modeled by Lorenz-like templates. Similarly, we can study any Smale flow by taking a template model of its saddle set.

Next we review some useful concepts of knot theory. Detail can be found in [10] and [9]. Our intention is to study the knot type within a template and to get the linking structure of the attractor and the repeller for the flow. A *knot* is an embedding of S^1 into S^3. We can say it is a curve in three-dimensional Euclidean space, $\mathbb{R}^3$, homeomorphic to a circle S^1. Two knots are said to be equivalent if there is an isotopy of S^3 taking one into another. All isotopic knots are of the same knot type. A knot group is the fundamental group of complement of the knot in S^3. The core of a solid torus can be considered as an unknot and the knot group of an unknot is infinite cyclic. A link of n components is an embedding of n-disjoint copies of S^1 into S^3. A knot can be given an orientation whenever it is necessary. For the link, we can assign a linking number observing the orientations of the two knots at the crossing. A Hopf link always has the linking number ± 1.

For any Smale flow with single attracting and repelling orbits and with a saddle set Λ, the linking number of the attractor-repeller link can be determined by using a structure matrix of the saddle set [6] where the structure matrix can be determined by using a Markov partition of the saddle set Λ.

Theorem 2.6 ([17, Theorem 9])**.** *For a Lorenz-Smale flow in S^3, the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case, the saddle set is modeled by a standardly embedded Lorenz template; i.e.,* *both bands are unknotted, untwisted, and unlinked, with the core of each band a meridian of the trefoil component of* $a \cup r$*. In the former case there are three possibilities:* (1) *The saddle set is standardly embedded.* (2) *One band is twisted with* n *full-twists for any* n *but remains unknotted and unlinked to the other band, which must be unknotted and untwisted.* (3) *One band is a* (p, q) *torus knot for any pair of coprime integers with twist* $p + q - 1$*. The other band is unknotted, untwisted, and unlinked to the knotted one.*

The proof for the above can be found in [17] and for the below in [22].

Theorem 2.7 ([22, Theorem 1])**.** *For an* $L(0, 1)$ *Lorenz-like Smale flow on* S^3 *the following and only the following configurations are realizable. The link* $a \cup r$ *is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a standard embedded* $L(0, 1)$ *Lorenz-like template; i.e., the saddle set is modeled by embedded* $L(0, 1)$ *and the cores of both bands are unknotted and unlinked to each other. In the former case, there are three possibilities:* (1) *The saddle set is standardly embedded.* (2) *The saddle set is modeled by embedded* $L(2p + 2q - 2, 2p + 2q - 1)$*. The cores of two bands are two parallel* (p, q) *torus knots where* p *and* q *are any coprime integers.* (3) *The saddle set is modeled by embedded* $L(0, 2p + 2q - 1)$*. The core of the twisted band is a* (p, q) *torus knot; the core of the other band is unknotted and unlinked with the former one.*

Theorem 2.8 ([22, Theorem 2])**.** *For an* $L(1, 1)$ *Lorenz-like Smale flow on* S^3 *the following and only the following configurations are realizable. The link* $a \cup r$ *is a link which is composed of a trivial knot and a* $(p, 3)$ *torus knot in the boundary of a solid torus neighborhood of the trivial knot where* p *is any integer such that* p *and* 3 *are coprime. The saddle set is modeled by embedded* $L(2n + 1, 4n + 1)$ *for any* n*. The linking number of the cores of these two bands is* $2n$*, the core of one knot is unknotted, and the core of the other band is a* $(2, 2n + 1)$ *torus knot.*

The Lorenz template and the Lorenz-like templates in theorems 2.6, 2.7, and 2.8 are shown in Figure 1.

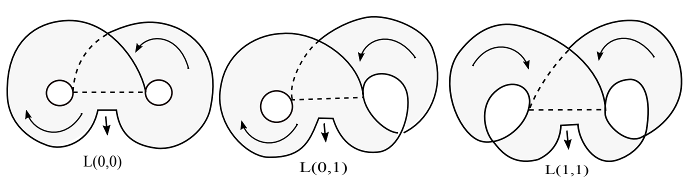


FIGURE 1. Lorenz template $L(0, 0)$ and Lorenz-like templates $L(0, 1)$ and $L(1, 1)$.

Theorem 2.9 ([13, Theorem 4.1])**.** *For a simple Smale flow on S^3 with saddle set modeled by U, the link $a \cup r$ is either a Hopf link or a figure-8 knot and meridian. In the latter case the bands are untwisted, unknotted, and unlinked. In the Hopf link case, one or two bands may form* (p, q) *torus knots about a or r; however, the two looped bands in the template U cannot both be knotted, twisted, or linked.*

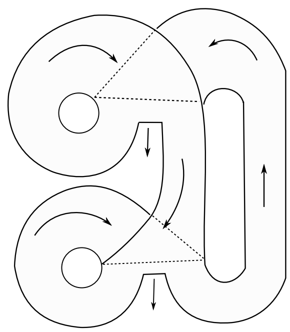


FIGURE 2. Template U.

The proof of the above theorem is given in [13].

Observing the above theorems, there is an obvious question to ask if we could use some more template models to study the Smale flow and extend the existing theorems: What could the pair $a \cup r$ be for the simple Smale flow if we could model the saddle set with the thickened version of some other possible templates? We can see in the proofs of all of the above theorems that the isolating neighborhood of a saddle set is represented by a thickened version of its respective templates. The exit set and entrance set of the thickened template are glued, respectively, to the attractor and repeller to obtain S^3. To prove the following theorems we use a similar concept which extends the previous theorems one step further.

3. Realizing an NSF with a Four Band Template

Theorem 3.1. *For a simple Smale flow on S^3 with the saddle set modeled by H, the link $a \cup r$ is either a Hopf link or a figure-8 knot and its meridian. In the latter case the saddle set is modeled by a standardly embedded template H (see Figure 3) where the bands are untwisted, unlinked, and unknotted. In the Hopf link case, either* (a) *the saddle set is a standardly embedded template or* (b) *one or two bands may form a* (p, q) *torus knot about a or r but none of the bands can be twisted, knotted, or linked.*

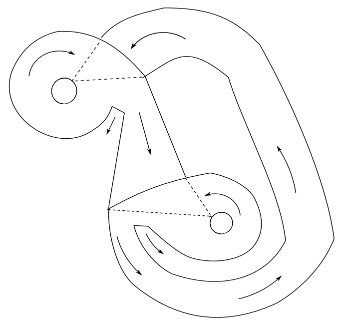


Figure 3. Template H

Proof. The thickened template is a genus 3 handlebody as shown in Figure 4. We still call this H throughout the proof. The exit set is shown in Figure 4.

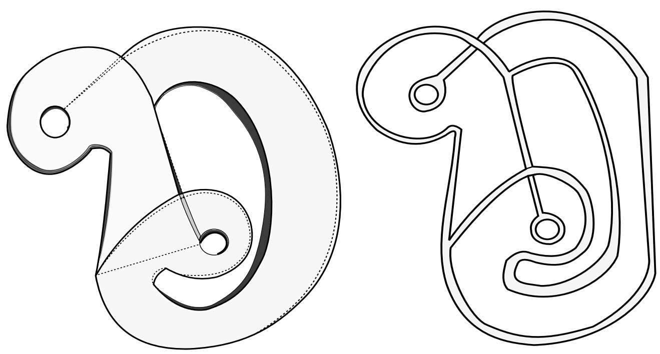


Figure 4. Thickened template and the exit set of H

We denote the exit set by E_x; from Figure 5, we can see that the exit set is divided into three annuli and two rectangular strips. Let the annuli be C_1, C_2, and C_3 and the rectangular strips L_1 and L_2. Thus, the core of the exit set is partitioned into three loops c_1, c_2, and c_3 and two line segments l_1 and l_2.

If we denote A as the tubular neighborhood of an attractor and R as the tubular neighborhood of a repeller, the boundary of A is glued into the exit set E_x and then the boundary ∂R is attached to the boundary of $A \cup H$ to get $A \cup H \cup R$ which is S^3.

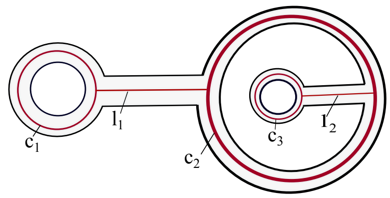
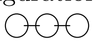


FIGURE 5. Exit set configuration

If we look at the exit set in Figure 5, we can see that the only allowed configuration, l_1 and l_2, must attach to opposite sides of c_2 and the structure  is not allowed. We look at various cases based on how many c_i's are essential in ∂A. We can exchange the roles of c_1 and c_3 without loss of generality. For the discussion now, we assume that c_3 lies inside the disk bounded by c_2 and that c_1 lies outside that disk.

Case 1: When all c_i's are inessential in ∂A

We can further divide this case into two subcases depending on whether c_2 and c_3 both lie inside the disk bounded by c_1 or not.

Case 1(a). If they don't lie inside the disk bounded by c_1, then they lie on ∂A as shown in Figure 6. The loop c_1 in this case bounds a different disk in ∂A as does c_3. We can slightly push the disks inside A to create solid balls B_1 and B_3, respectively, such that if we take these balls out, the closure of $A - (B_1 \cup B_3)$ still remains a solid torus.

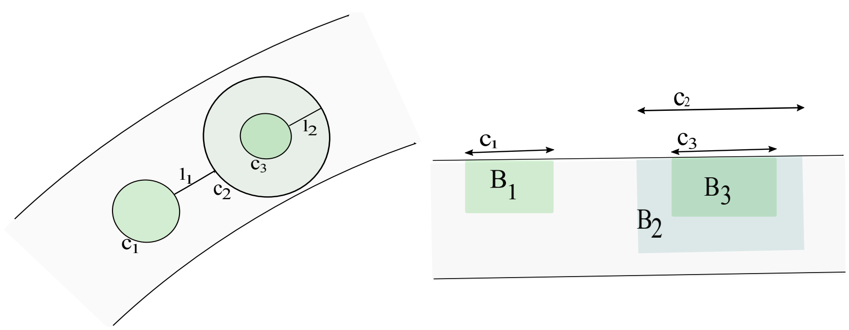


FIGURE 6. Inessential curves on the surface of A

Let us denote $A' = Cl(A - (B_1 \cup B_3))$ and $H' = H \cup B_3$ where the gluing is done in the annulus C_3 and denote $H'' = H' \cup B_1$ where the gluing is done in C_1. Now H'' is a solid torus. We can further push down the disk bounded by c_2 to create a ball B_2 by pushing down deeper than

B_3 such that $Cl(A' - B_2)$ still remains a solid torus. Then if we glue B_2 to C_2 to get H''', we will get H''' as a solid 3 ball, the result of which makes $A \cup H$ a single solid torus where a can be taken as its core. Thus, we get $a \cup r$, a Hopf link.

Case 1(b). If c_2 and c_3 lie inside the disk bounded by c_1, then they lie in ∂A as shown in Figure 7.

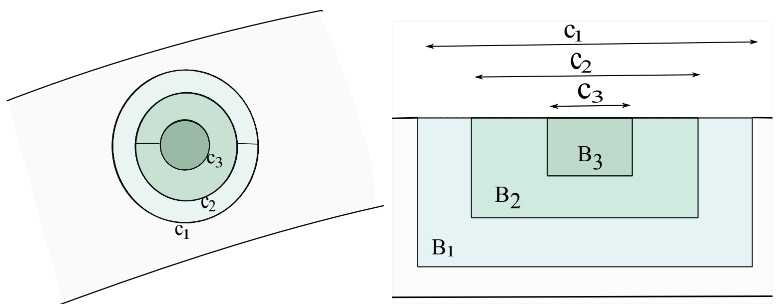


FIGURE 7. When c_2 and c_3 lie inside the disk bounded by c_1 on ∂A.

As above, c_3 bounds a disk in ∂A. Push this disk slightly inside A to get a ball B_3. Attach B_3 to H at C_3. While creating B_3, care is taken that the closure of $A - B_3$ still remains a solid torus. Once we attach B_3 to C_3, denote $H' = H \cup B_3$ which is a genus 2 handlebody and denote $A' = Cl(A - B_3)$ which is a solid torus. Now we choose c_2. There is a disk bounded by the inner half of c_2, l_2, and the region outside of c_3. Push this disk inside A slightly deeper than B_3 to get another 3 ball, B_2, such that $A'' = Cl(A' - B_2)$ still remains a solid torus. Then we glue B_2 to H' to get a solid torus $H'' = H' \cup B_2$.

Now in a similar manner, dig B_1 deeper than B_2 so that $A''' = Cl(A'' - B_1)$ remains a solid torus and glue B_1 to H'' to get H''' as a solid 3 ball. Thus, these attachments make $H''' \cup A'''$ a single solid torus where a can be considered as its core. Thus, when A and R are glued together to form S^3, $a \cup r$ forms a Hopf link.

Case 1(c). In Case 1, we can always switch the roles of c_1 and c_3. Thus, if c_2 and c_1 are both inside the disk bounded by c_3, we will get the same result as in 1(b).

Case 2: When one c_i is essential and the others inessential

When one c_i is an essential curve on ∂A, the essential loop c_i may be any (p, q) curve on the surface of A. In this situation we further get the following subcases depending on which c_i is essential and which c_i's are inessential. In this case, too, we can switch the roles of c_1 and c_3 without affecting the result.

Case 2(a). Suppose c_1 is essential and c_2 and c_3 inessential. In this case, c_3 bounds a disk in ∂A. As before, we create a small ball, B_3, pushing down the disk a little bit such that the closure of $A - B_3$ is still a solid torus. We glue this ball to H to get H' which will be a genus 2 handlebody. Now push the disk bounded by c_2 slightly deeper than B_3 and get another ball, B_2. We will create this ball taking care that the closure of $A' - B_2$ is still a solid torus A'' where $A' = Cl(A - B_3)$. Then we glue B_2 to H' in the boundary of C_2 to get H'' which will become a solid torus now. Since c_1 in this solid torus is a (p, q) curve on ∂A, we create a tubular neighborhood B_1 of c_1 in A'' such that $Cl(A'' - B_1)$ still remains a solid torus. Then we can glue two solid tori, (H'' and $A''' = cl(A'' - B_1)$), together along a longitudinal annulus in their boundaries to get another solid torus. Thus, in this new solid torus, we can take a as its core and hence we get $a \cup r$, a Hopf link.

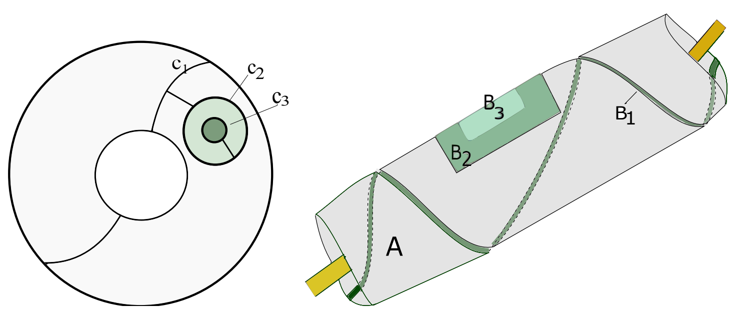


FIGURE 8. When c_1 is an essential curve on ∂A

Case 2(b). Suppose c_2 is essential and c_1 and c_3 inessential. Then both c_1 and c_3 bound a disk in ∂A. We create balls B_1 and B_3 inside A from c_1 and c_3, respectively, through these disks and attach them to H one by one in C_1 and C_3, respectively. Then we get $H'' = H \cup B_1 \cup B_3$ which will be a solid torus with c_2 as its longitude and $A'' = Cl(A - (B_1 \cup B_3))$ a solid torus if we create B_1 and B_3 thin enough to leave A as a solid torus after we take them out.

Now creating a small tubular neighborhood of c_2 in A'' and gluing it to H'' through this neighborhood (which is a solid torus), we see that $A \cup H$ is a solid torus because the gluing of two solid tori together along their longitudes always gives a solid torus and the attractor a can be considered as its core. Therefore, $a \cup r$ is a Hopf link.

Case 2(c). Suppose c_3 is essential and c_1 and c_2 are inessential, but c_3 lies inside the disk bounded by c_2. This case cannot happen.

Case 3: When two c_i's are essential and one inessential

We consider the following three subcases here.

Case 3(a). If c_1 and c_2 are essential and c_3 inessential, then c_3 bounds a disk. We push the disk inside A to make a ball, B_3, and attach this ball to the annulus, C_3, as we did in the previous cases. Then $H \cup B_3$ becomes a genus 2 handlebody, whereas $A - B_3$ still remains a solid torus. Also c_1, c_2, and l_1 on the two sides form (bound) a disk in ∂A. Now we will take this disk and shrink it a little bit away from l_1 such that its closure remains a disk. Push this new disk slightly inside A and create a thin solid ball, B_1, such that $A - (B_1 \cup B_3)$ still remains a solid torus. Then we glue B_1 to $H \cup B_3$ and get $H'' = H \cup B_3 \cup B_1$, a solid torus. Now from the (p, q) curve c_2 in ∂A, we take out the tubular neighborhood of c_2 which is a solid torus such that $A''= A - (B_1 \cup B_2 \cup B_3)$ still remains a solid torus. Here B_2 is the tubular neighborhood of c_2. Then we glue A'' and H'' to get $A'' \cup H''$, a solid torus. Thus, we get $A \cup H$, a solid torus where a can be considered as its core. Therefore, $a \cup r$ is a Hopf link.

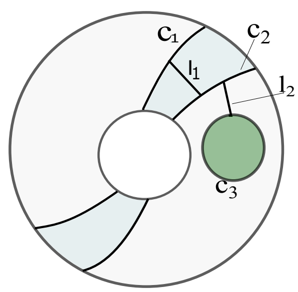


FIGURE 9. When c_1 and c_2 are essential curves on ∂A

Case 3(b). If c_1 and c_3 are essential and c_2 inessential, then the essential curve c_3 cannot be placed inside the disk bounded by c_2. Thus, this case cannot happen.

Case 3(c). If c_2 and c_3 are essential and c_1 inessential, we can switch the roles of c_1 and c_3 in 3(a). Therefore, the result will be same as in case 3(a).

Case 4: When all c_i's are essential

If all c_i's are essential, they must be the parallel (p, q) curves. In this case we will try to find the fundamental group of $A \cup H$ by using the Seifert–van Kampen theorem (see [15]). Since the gluing work is done between ∂A and the exit set of H, we will use the generators of the exit set and H to compute the fundamental group. Figure 10 gives the generators for the exit set and for H.

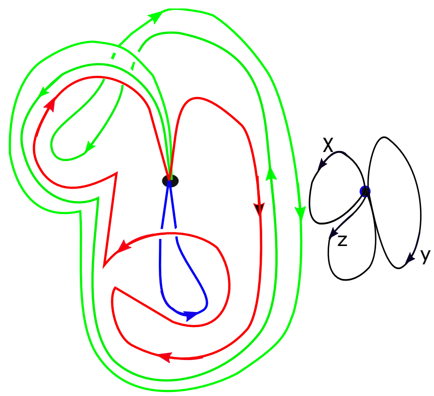


FIGURE 10. Generators of the exit set and H

By using the Seifert–van Kampen theorem, the fundamental group of $A \cup H$ is given by

$$\pi_1(A \cup H) = \{a, x, y, z \mid a^p = z, a^p = yzx^{-1}, a^p = xy^{-1}x^{-1}yx^{-1}\}.$$

Using a Tietze transformation (see [10]), we can get

$$\pi_1(A \cup U) = \{a, x \mid a^p x a^p x^{-1} a^{-p} x a^p x a^{-p} x^{-1} = 1\}. \tag{1}$$

Now we use R. H. Fox's free differential calculus [5] (see also [10]) and find the Alexander polynomial of (1) which is given by $\Delta(t) = 2t^p - t^{2p} - 1 + t^{-1}$.

But this can only be the Alexander polynomial of a knot when $p = 0$. Thus, $\pi_1(A \cup H)$ is infinite cyclic. Hence, the repeller r is an unknot and $A \cup H$ is a solid torus.

Next we see what we will get if we glue H to the tubular neighborhood R of the repeller through the entrance set of H. This will give us an idea about the attractor.

The generators of the entrance set are shown in Figure 11.

Using the Seifert–van Kampen theorem we can find the fundamental group of $R \cup H$ as follows:

$$\pi_1(R \cup H) = \{r, x, y, z \mid r = x, r = z^{-1}yz, r = z^{-1}y^{-1}x^{-1}yz^{-1}y^{-1}xyz\}.$$

A calculation shows that $\pi_1(H \cup R)$ is isomorphic to the knot group of a figure-8 knot.

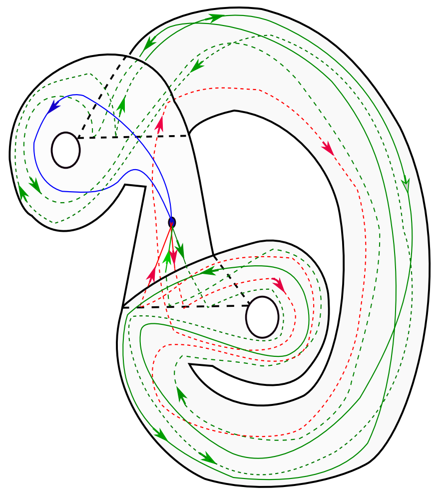


FIGURE 11. Generators of entrance set E_n

In Figure 12 we construct a realization where indeed the attractor a is a figure-8 knot. By the Gordon–Luecke theorem [11] this is the only possibility for a. Since $p = 0$, r is a meridian of A. It follows that the three boundary orbits in the saddle set are unknotted and unlinked. Of course the roles of a and r can be switched by flow reversal. □

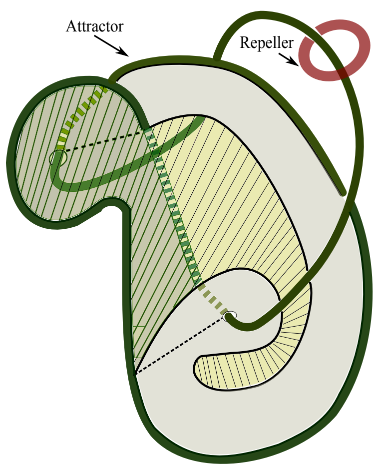


FIGURE 12. Realization of Case 4 with figure-8 knot attractor

Theorem 3.2. *For a simple Smale flow on S^3 with the saddle set modeled by H^+, the link $a \cup r$ is a Hopf link.*

Proof. Let H^+ denote the template shown in Figure 13. Let us also denote by H^+ the thickened template as we did in the previous theorem. In this case, too, the exit set E_x is divided into three annuli and two rectangular strips. The thickened template is shown in Figure 13 and the exit sets are shown in Figure 14.

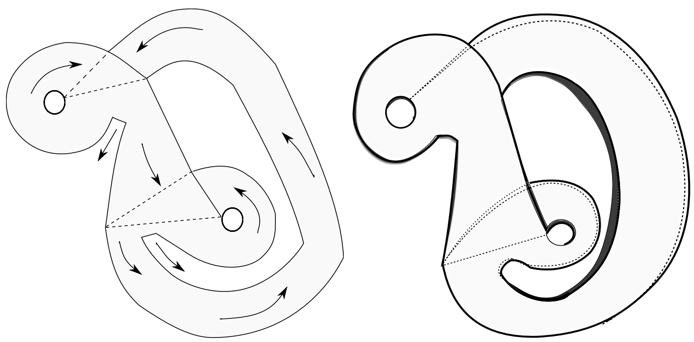


FIGURE 13. Template H^+ and thickened template

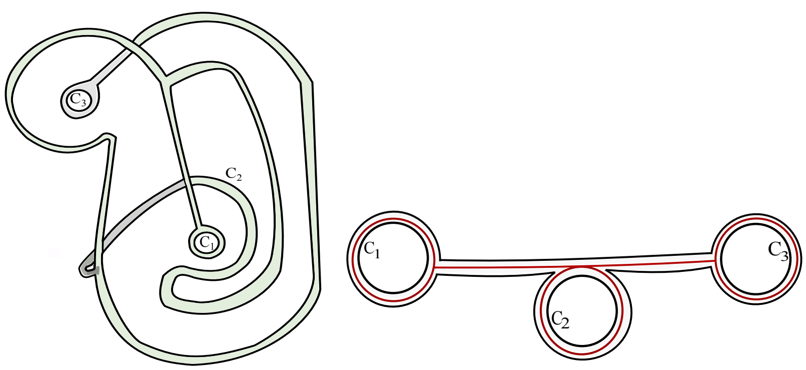
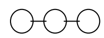


FIGURE 14. Exit set and exit set configuration

Let C_1, C_2, and C_3 denote the three annuli and L_1 and L_2 the two rectangular strips. The core of the exit set is thus partitioned into three loops c_1, c_2, and c_3 and two line segments l_1 and l_2. Let A be the tubular neighborhood of the attractor a and let R be the tubular neighborhood of the repeller r. As in the previous case, we glue the boundary of A into the exit set E_x and get $A \cup H^+$. Then we glue it to R to get S^3. The only configuration for the exit set is l_1, and l_2 must attach to the same side of c_2 and the only allowed structure is .

Exactly the way we did in the previous theorem, we divide the proof into different cases depending on how many c_i's are essential in ∂A.

Case 1: When all c_i's are inessential in ∂A

If we look at the exit set of H^+, we can see that c_2 in ∂A cannot bound a disk in ∂A because it links a closed orbit of the saddle set. Hence, we cannot attach any disk or ball to it. Thus, this case cannot happen.

Case 2: When one c_i is essential and the other two inessential

If only one c_i is essential, it must be c_2 because only c_2 cannot bound a disk and thus c_1 and c_3 must be inessential curves on ∂A. So c_1 and c_3 bound a disk in ∂A. Push these disks inside A a little bit to get two balls, B_1 and B_3, such that $A-(B_1 \cup B_3)$ still remains a solid torus. Then attach these balls to annuli C_1 and C_3, respectively. Then $H^{+\prime\prime} = (H^+ \cup B_1) \cup B_3$ becomes a solid torus and $A'' = Cl(A - (B_1 \cup B_3))$ also becomes a solid torus. Then we can take the tubular neighborhood of c_2 in $H^{+\prime\prime}$ which is a solid torus and attach this solid torus to A'' along the (p, q) annulus C_2. This will give us $A'' \cup H^{+\prime\prime}$, a solid torus where a can be considered as its core. Thus, $a \cup r$ is a Hopf link.

Case 3: When two c_i's are essential and one inessential

Suppose two c_i's are essential. Then we will have the following subcases.

Case 3(a). Suppose c_1 and c_2 are essential and c_3 inessential. Since c_3 is inessential, create a thin solid ball, B_3, as before and attach it to C_3. Thus, $H^{+\prime} = H^+ \cup B_3$ becomes a genus 2 handlebody. Now choose the disk bounded by the curves c_1, l_1, and c_2 and shrink it a little bit so that its closure is still a disk. Push this disk inside $A' = A - B_3$ slightly to create a thin solid ball, B_1. Attach this solid ball to C_1 such that the resulting $H^{+\prime\prime} = H^+ \cup B_3 \cup B_1$ becomes a solid torus. Now since c_2 is a (p, q) curve, take the tubular neighborhood of c_2 and attach it to the annulus C_2 to make $A \cup H^+$ a solid torus at the end. Thus, we get $a \cup r$, a Hopf link.

Case 3(b). Suppose c_2 and c_3 are essential and c_1 inessential. We can exchange the roles of c_1 and c_3 in 3(a). This does not make any difference. We will get $a \cup r$, a Hopf link, in this case, too. The proof exactly follows the same process as in 3(a).

Case 3(c). Suppose c_1 and c_3 are essential and c_2 inessential. But c_2 cannot bound a disk. So this case cannot happen.

Case 4: When all c_i's are essential

If all c_i's are essential on ∂A, then all of these are parallel (p, q) curves on ∂A. We find the fundamental group of $A \cup H^+$ using the Seifert–van Kampen theorem as we did in Theorem 3.1. Since the gluing of the exit set is done with ∂A, we find the generators of the exit set and then glue H^+ and A together through the exit set. The generators of H^+ and the exit set are given in Figure 15.

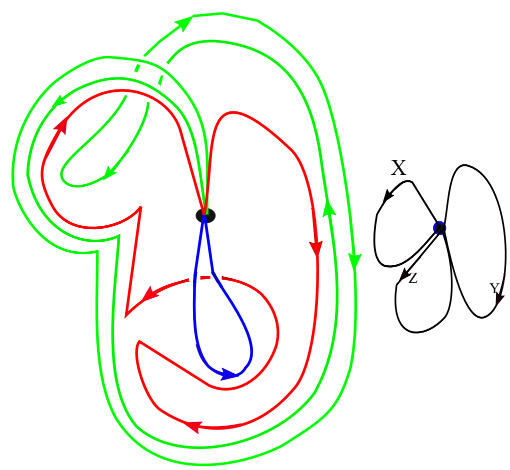


Figure 15. Generators of the exit set and H^+

The diagram of the exit set is the same as in Theorem 3.1 except for the crossing. So we get the same set of generators for H^+ and for the exit set. This gives us the same fundamental group of $A \cup H^+$ when we apply the Seifert–van Kampen theorem. As before, using Fox's free differential calculus, the Alexander polynomial of $A \cup H^+$ is $\Delta(t) = 2t^p - t^{2p} - 1 + t^{-1}$ which gives a knot structure only when $p = 0$.

But when $p = 0$, we get c_2 an $(0, q)$ curve and, similarly, two other essential curves. Since c_2 cannot bound a disk, c_2 cannot be a $(0, q)$ curve. Because of this, the case cannot exist. Thus, the only possible cases for the Smale flows on S^3 with the saddle set modeled by H^+ are cases 2 and 3 where the only possible structure for $a \cup r$ is a Hopf link. $\square$

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