http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 50, 2017

Pages 39–48

http://topology.nipissingu.ca/tp/

Inverse Limits of Iterates of Set-valued Functions

by

James P. Kelly

Electronically published on June 4, 2016

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 50 (2017)
Pages 39-48

E-Published on June 4, 2016

INVERSE LIMITS OF ITERATES OF SET-VALUED FUNCTIONS

JAMES P. KELLY

Abstract. We present a function $F\colon [0,1] \to C([0,1])$ that is upper semi-continuous, and we show that if $n, m \in \mathbb{N}$ with $n \neq m$, then $\varprojlim F^n$ and $\varprojlim F^m$ are not homeomorphic. This answers a question posed by Matevž Črepnjak (2015). Additionally, we compare F to two other functions: a continuous function $g\colon [0,1] \to [0,1]$ and an upper semi-continuous function $H\colon [0,1] \to 2^{[0,1]}$. We apply known results to state that $\varprojlim F$, $\varprojlim g$, and $\varprojlim H$ are all homeomorphic. We show, however, that the inverse limits of iterates of these functions are not homeomorphic to one another.

1. Introduction

The study of inverse limits of upper semi-continuous, set-valued functions is introduced by Williams S. Mahavier in [7] and further developed by W. T. Ingram and Mahavier in [4]. In these foundational papers, the authors demonstrate that many of the well-known properties which hold for inverse limits of continuous, single-valued functions do not always hold in the more general context of set-valued functions.

One such property is known as the subsequence property. This property implies that for a continuous function f on a compact Hausdorff space X, the inverse limit of f is homeomorphic to the inverse limit of f^n for every natural number n. Ingram and Mahavier give two examples illustrating that the subsequence property does not always hold for upper semi-continuous, set-valued functions [4, Examples 3 & 4]. In [3, Problem 6.51], Ingram asks if there exists an upper semi-continuous, set-valued

2010 *Mathematics Subject Classification.* 54F15, 54C60, 54D80.
Key words and phrases. full projection property, inverse limit, subsequence property.

function F such that for each $n, m \in \mathbb{N}$ with $n \neq m$, the inverse limits of F^n and F^m are not homeomorphic. Matevž Črepnjak constructs such a function in [1]. The graph of this function is not connected, so Črepnjak poses the following question [1, Problem 3.14].

Question 1.1. Is there an upper semi-continuous function $F\colon [0,1] \to 2^{[0,1]}$ with connected graph such that for $n, m \in \mathbb{N}$ with $n \neq m$, the inverse limits of F^n and F^m are not homeomorphic?

We define an upper semi-continuous function $F\colon [0,1] \to C([0,1])$ in §3, and we demonstrate that this provides a positive answer to Question 1.1. Then, in §4, we discuss two other functions, g and H, and we compare their inverse limits and the inverse limits of their iterates with those of F.

2. Preliminary Definitions and Results

A set X is a *continuum* if it is a non-empty, compact, connected subset of a metric space. A subset of a continuum X which is itself a continuum is called a *subcontinuum* of X. A continuum is called *decomposable* if it is the union of two proper subcontinua. A non-degenerate continuum which is not decomposable is called *indecomposable*.

Given a compact Hausdorff space X, we define the following *hyperspaces* of X,

$$\begin{aligned} 2^X &= \{A \subseteq X : A \text{ is closed in } X\} \\ C(X) &= \left\{A \in 2^X : A \text{ is connected}\right\}. \end{aligned}$$

If X and Y are compact Hausdorff spaces, a function $F\colon X \to 2^Y$ is called *upper semi-continuous* if, for each open set $V \subseteq Y$, the set $\{x \in X : F(x) \subseteq V\}$ is open in X. The *graph* of a function $F\colon X \to 2^Y$ is the set

$$\Gamma(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

In [4, Theorem 2.1], it is shown that F is upper semi-continuous if and only if its graph is closed in $X \times Y$.

For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space and $F_i\colon X_{i+1} \to 2^{X_i}$ be upper semi-continuous. Then the *inverse limit* of the sequence $(F_i)_{i=1}^{\infty}$ is the set

$$\varprojlim (F_i)_{i=1}^{\infty} = \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : x_i \in F_i\left(x_{i+1}\right) \text{ for all } i \in \mathbb{N} \right\}.$$

In this paper, we will primarily be concerned with inverse limits where, for each $i \in \mathbb{N}$, $X_i = [0,1]$ and each F_i is the same function F. In this case, we write $\varprojlim F$ rather than $\varprojlim (F_i)_{i=1}^{\infty}$.

If $F\colon X \to 2^X$, we define the composition $F \circ F\colon X \to 2^X$ by

$$(F \circ F)(x) = \bigcup_{y \in F(x)} F(y).$$

We define $F^1 = F$, and, for each $n \geq 2$, we define $F^n = F \circ F^{n-1}$. We also define the *inverse* of F to be the function $F^{-1}\colon X \to 2^X$ where $x \in F^{-1}(y)$ if and only if $y \in F(x)$.

In [5], a class of set-valued functions called irreducible functions is defined. These functions are defined in terms of their inverse which is the union of continuous functions. The definition in [5] is more general than we need in this paper, so we give the following simplified definition.

Definition 2.1. A function $F\colon [0,1] \to 2^{[0,1]}$ is called *irreducible* if $F(0) = \{0\}$, $F(1) = \{1\}$, and there exist distinct continuous functions

$$f_1, \ldots, f_n\colon [0,1] \to [0,1],\ n \geq 2,$$

such that

$$\Gamma\left(F^{-1}\right) = \bigcup_{i=1}^{n} \Gamma\left(f_i\right),$$

and $\Gamma(F)$ is an arc from $(0,0)$ to $(1,1)$.

The following results appear in [5, Theorem 3.10] and [5, Theorem 4.16] respectively.

Theorem 2.2. *For each $i \in \mathbb{N}$, let $F_i\colon [0,1] \to 2^{[0,1]}$ be an irreducible function. Then $\varprojlim F_i$ is an indecomposable continuum.*

Theorem 2.3. *Let $n, m \in \mathbb{N} \setminus \{1\}$. Suppose $F\colon [0,1] \to 2^{[0,1]}$ is an irreducible function whose inverse is the union of n continuous functions, and $G\colon [0,1] \to 2^{[0,1]}$ is an irreducible function whose inverse is the union of m continuous functions. Then $\varprojlim F$ is homeomorphic to $\varprojlim G$ if and only if n and m have the same prime factors.*

In particular, if F is an irreducible function on $[0,1]$, then its inverse limit is homeomorphic to the inverse limit of an open mapping on $[0,1]$. The inverse limits of open mappings on $[0,1]$ form a class of continua called *Knaster continua*. These are indecomposable continua with the property that every proper subcontinuum is an arc.

3. An Upper Semi-Continuous Function for Which All Iterates Have Distinct Inverse Limits

In this section we define an upper semi-continuous function $F\colon [0,1] \to C([0,1])$, and we show that if $n, m \in \mathbb{N}$ with $n \neq m$, then $\varprojlim F^n$ is not homeomorphic to $\varprojlim F^m$.

Definition 3.1. We define $F\colon [0,1] \to C([0,1])$ as follows:

$$F(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ [0,1], & x = \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x \le 1. \end{cases}$$

The graph of F is pictured in Figure 1, and the graph of F^2 is pictured in Figure 2.

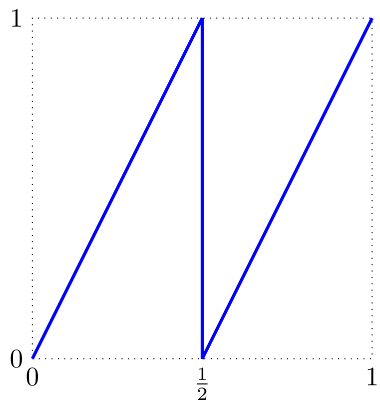

Figure 1. F

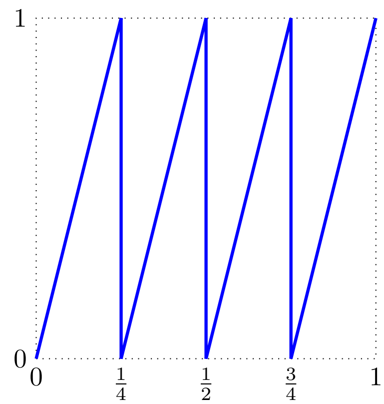

Figure 2. F^2

This function, F, has been studied by Ingram [2] and Scott Varagona [11]. Since F and its iterates are irreducible functions we may use Theorem 2.3 to compare their inverse limits.

Remark 3.2. We begin by making some observations about the graph of F^n for each $n \in \mathbb{N}$. Fix a number $n \in \mathbb{N}$, and for each $j = 0, \ldots, 2^n$, let $a_j = j/2^n$.

(1) For each $j = 1, \ldots, 2^n - 1$, the graph of F^n includes the vertical line segment $\{a_j\} \times [0,1]$.
(2) For each $j = 0, \ldots, 2^n - 1$, the graph of F^n includes the straight line segment with endpoints $(a_j, 0)$ and $(a_{j+1}, 1)$.

Hence, the graph of F^n consists of $2^n - 1$ vertical line segments and 2^n diagonal line segments for a total of $2^{n+1} - 1$ line segments. Moreover, the union of these line segments is an arc from $(0,0)$ to $(1,1)$, so F^n is an irreducible function whose inverse is the union of $2^{n+1} - 1$ continuous functions.

In order to show that each iterate of F yields a distinct inverse limit, we must show that for each $n, m \in \mathbb{N}$ with $n \neq m$, the numbers $2^n - 1$ and $2^m - 1$ do not have the same prime factors. The following theorem is known as Zsigmondy's theorem. We state it as it appears in [9, Theorem 3].

Theorem 3.3 (Zsigmondy's Theorem)**.** *Let a and n be integers greater than 1. There exists a prime divisor q of $a^n - 1$ such that q does not divide $a^j - 1$ for all $0 < j < n$, except exactly in the following cases:*

(1) *$n = 2$, $a = 2^s - 1$, where $s \geq 2$.*
(2) *$n = 6$, $a = 2$.*

This allows us to prove our main result of this section.

Theorem 3.4. *If $n, m \in \mathbb{N}$ with $n \neq m$, then $\varprojlim F^n$ and $\varprojlim F^m$ are not homeomorphic.*

Proof. For each $n \in \mathbb{N}$, let $P(n)$ be the set of prime factors of $2^n - 1$. From Theorem 2.3 and Remark 3.2, it suffices to show that for each $n, m \in \mathbb{N}$, if $n \neq m$, then $P(n+1) \neq P(m+1)$.

By Zsigmondy's theorem, for every $n \geq 2$ other than 6, $P(n)$ contains an element which is not an element of $P(j)$ for any $0 < j < n$. Then by inspection, we see that no two of $P(2), P(3), P(4), P(5)$, and $P(6)$ are equal:

$$\begin{aligned} P(2) &= \{3\}, \\ P(3) &= \{7\}, \\ P(4) &= \{3, 5\}, \\ P(5) &= \{31\}, \\ P(6) &= \{3, 7\}. \end{aligned}$$

□

4. Comparing the Inverse Limits of Three Upper Semi-Continuous Functions

We now define two other functions and compare their inverse limits and the inverse limits of their iterates with those of F.

Definition 4.1. We define $g\colon [0,1] \to [0,1]$ by

$$g(x) = \begin{cases} 3x, & 0 \leq x \leq \frac{1}{3} \\ 2 - 3x, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x - 2, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

We also define $H\colon [0,1] \to 2^{[0,1]}$ to be the upper semi-continuous function whose graph is the union of three straight line segments: one from $(0,0)$ to $(2/3,1)$, one from $(2/3,1)$ to $(1/3,0)$, and one from $(1/3,0)$ to $(1,1)$.

The graph of g is pictured in Figure 3, and the graph of H is pictured in Figure 4.

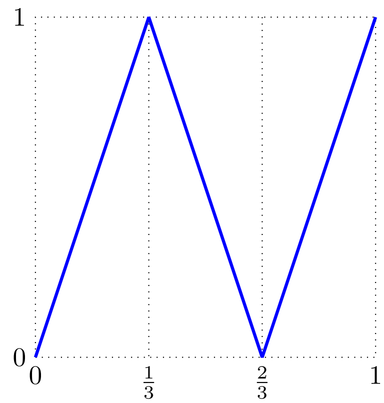

FIGURE 3. g

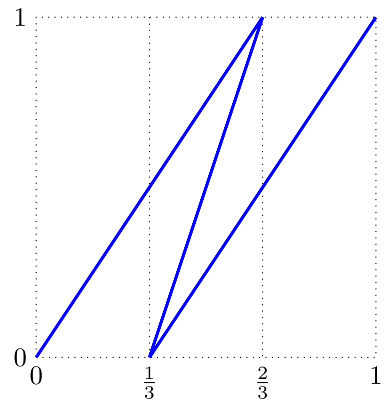

FIGURE 4. H

The functions F, g, and H are each irreducible functions whose inverses are comprised of 3 continuous functions. Hence, the following proposition follows from Theorem 2.3.

Proposition 4.2. *The inverse limits* $\varprojlim F$, $\varprojlim g$, *and* $\varprojlim H$ *are all homeomorphic.*

While these three functions have homeomorphic inverse limits, the inverse limits of their iterates are not homeomorphic to one another. We show this in steps beginning with Proposition 4.4.

The following notation will be useful in the proof of Proposition 4.4.

Notation 4.3. Given two points (x_1, y_1) and (x_2, y_2), we use the notation $\overline{(x_1, y_1)(x_2, y_2)}$ to represent the straight line segment whose endpoints are (x_1, y_1) and (x_2, y_2).

Proposition 4.4. *If* $n \geq 2$, *then* $\varprojlim H^n$ *contains a simple closed curve. In particular,* $\varprojlim H^n$ *is not chainable.*

Proof. The graph of H^2 is pictured in Figure 5. Note that this graph contains the simple closed curve $S = A \cup B \cup C \cup D$, where

$$\begin{aligned} A &= \overline{\left(\frac{1}{3}, 0\right)\left(\frac{4}{9}, \frac{1}{2}\right)}, \\ B &= \overline{\left(\frac{4}{9}, \frac{1}{2}\right)\left(\frac{2}{3}, 1\right)}, \\ C &= \overline{\left(\frac{2}{3}, 1\right)\left(\frac{5}{9}, \frac{1}{2}\right)}, \\ D &= \overline{\left(\frac{5}{9}, \frac{1}{2}\right)\left(\frac{1}{3}, 0\right)}. \end{aligned}$$

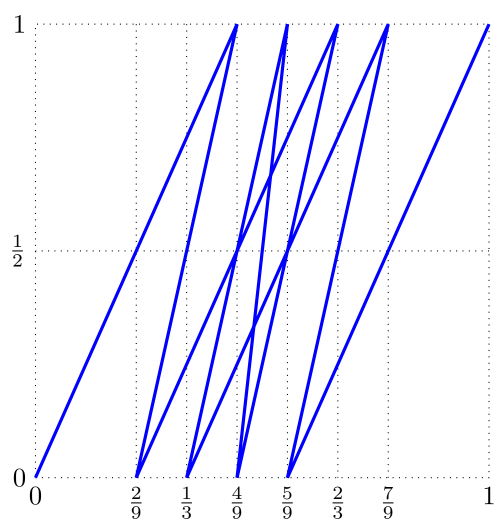

FIGURE 5. H^2

Now observe that the graph of H includes the graph of $y = 3x/2$ over the interval $[0, 2/3]$. Fix $n \in \mathbb{N}$, then the graph of H^n includes the graph of $y = 3^n x/2^n$ over the interval $[0, 2^n/3^n]$. Since this function is strictly increasing, it follows that the graph of F^{n+2} contains the simple closed curve $S_n = A_n \cup B_n \cup C_n \cup D_n$ where

$$\begin{aligned} A_n &= \overline{\left(\frac{2^n}{3^n}\cdot\frac{1}{3},0\right)\left(\frac{2^n}{3^n}\cdot\frac{4}{9},\frac{1}{2}\right)}, \\ B_n &= \overline{\left(\frac{2^n}{3^n}\cdot\frac{4}{9},\frac{1}{2}\right)\left(\frac{2^n}{3^n}\cdot\frac{2}{3},1\right)}, \\ C_n &= \overline{\left(\frac{2^n}{3^n}\cdot\frac{2}{3},1\right)\left(\frac{2^n}{3^n}\cdot\frac{5}{9},\frac{1}{2}\right)}, \\ D_n &= \overline{\left(\frac{2^n}{3^n}\cdot\frac{5}{9},\frac{1}{2}\right)\left(\frac{2^n}{3^n}\cdot\frac{1}{3},0\right)}. \end{aligned}$$

Therefore, we have that, for each $k \geq 2$, $\Gamma(H^k)$ contains a simple closed curve. It then follows from a result due to M. M. Marsh [8, Corollary 3.2] that $\varprojlim H^k$ contains a simple closed curve for each $k \geq 2$. As Marsh's results are more general, we include a proof for our specific case.

Fix $k \geq 2$. To show that $\varprojlim H^k$ contains a simple closed curve, we define a function $\phi\colon \Gamma(H^k) \to \varprojlim H^k$ by

$$\phi(x,y) = \left(y, x, \frac{2^k}{3^k}x, \frac{2^{2k}}{3^{2k}}x, \frac{2^{3k}}{3^{3k}}x, \dots\right).$$

Since ϕ is clearly continuous and injective, $\varprojlim H^k$ contains a subcontinuum homeomorphic to $\Gamma(H^k)$. It follows that $\varprojlim H^k$ contains a simple closed curve. $\square$

Proposition 4.5. *If $n \geq 2$, then the following hold:*

(1) $\varprojlim F^n$ *is not homeomorphic to* $\varprojlim g^m$ *or* $\varprojlim H^m$ *for any* $m \in \mathbb{N}$.
(2) $\varprojlim H^n$ *is not homeomorphic to* $\varprojlim F^m$ *or* $\varprojlim g^m$ *for any* $m \in \mathbb{N}$.

Proof. From Theorem 3.4, we have that for each $n \geq 2$, $\varprojlim F^n$ is not homeomorphic to $\varprojlim F$, and from Proposition 4.2, we have that $\varprojlim F$, $\varprojlim g$, and $\varprojlim H$ are all homeomorphic. It follows that, for each $n \geq 2$, $\varprojlim F^n$ is not homeomorphic to either $\varprojlim g$ or $\varprojlim H$.

Moreover, since g is a continuous, single-valued function, for every $m \in \mathbb{N}$, $\varprojlim g^m$ is homeomorphic to $\varprojlim g$, so it follows that, for any $n \geq 2$ and any $m \in \mathbb{N}$, $\varprojlim F^n$ is not homeomorphic to $\varprojlim g^m$.

Hence, it suffices to show that, for any $n \geq 2$ and any $m \in \mathbb{N}$, the inverse limit $\varprojlim H^n$ is homeomorphic to neither $\varprojlim F^m$ nor $\varprojlim g^m$. From Proposition 4.4, $\varprojlim H^n$ is not chainable for any $n \geq 2$, and for every $m \in \mathbb{N}$, both $\varprojlim F^m$ and $\varprojlim g^m$ are chainable. Therefore, the result follows. $\square$

Various functions satisfying Definition 2.1 have been studied by this author, as well as Ingram [2], James P. Kelly and Jonathan Meddaugh [6], Varagona [11], and Michel Smith and Varagona [10], primarily because their inverse limits are indecomposable continua. We show in Proposition 4.6 that this is not preserved under iteration. First, we make the following observation about $\varprojlim H^2$.

Proposition 4.6. *The continuum $\varprojlim H^2$ contains a subcontinuum which is homeomorphic to $\varprojlim F^2$.*

Proof. Let $\widetilde{H}\colon [0,1] \to 2^{[0,1]}$ be the function whose graph consists of the following line segments:

$$
\begin{aligned}
A_1 &= \overline{(0,0)\left(\frac{4}{9},1\right)}, & A_6 &= \overline{\left(\frac{4}{9},0\right)\left(\frac{5}{9},\frac{1}{2}\right)},\\
A_2 &= \overline{\left(\frac{4}{9},1\right)\left(\frac{2}{9},0\right)}, & A_7 &= \overline{\left(\frac{5}{9},\frac{1}{2}\right)\left(\frac{7}{9},1\right)},\\
A_3 &= \overline{\left(\frac{2}{9},0\right)\left(\frac{4}{9},\frac{1}{2}\right)}, & A_8 &= \overline{\left(\frac{7}{9},1\right)\left(\frac{5}{9},0\right)},\\
A_4 &= \overline{\left(\frac{4}{9},\frac{1}{2}\right)\left(\frac{5}{9},1\right)}, & A_9 &= \overline{\left(\frac{5}{9},0\right)(1,1)}.\\
A_5 &= \overline{\left(\frac{5}{9},1\right)\left(\frac{4}{9},0\right)},
\end{aligned}
$$

The graph of $\widetilde{H}$ is a subset of the graph of H^2 and is pictured in Figure 6.

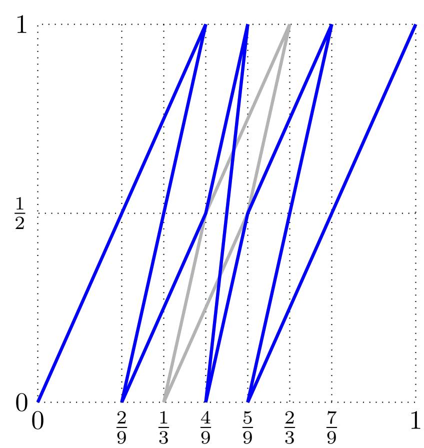

FIGURE 6. The graph of $\widetilde{H}$ as a subset of the graph of H^2

Since the graph of $\widetilde{H}$ is a subset of the graph of H^2, it follows that $\varprojlim \widetilde{H} \subseteq \varprojlim H^2$. Moreover, $\widetilde{H}$ is an irreducible function whose inverse is the union of seven continuous functions. Therefore, by Theorem 2.3, $\varprojlim \widetilde{H}$ is homeomorphic to $\varprojlim F^2$. □

Proposition 4.7. *The continuum* $\varprojlim H^2$ *is decomposable.*

Proof. Let $X = \varprojlim H^2$, and let $\widetilde{H}$ be defined as in the proof of Proposition 4.6. We define a sequence of irreducible functions $(H_i)_{i=1}^{\infty}$ by $H_1 = \widetilde{H}$ and $H_i = H^2$ for all $i \geq 2$. Since each H_i is surjective and has its inverse equal to a union of continuous, single-valued functions, $Y = \varprojlim H_i$ is a continuum. Moreover, since for each $i \in \mathbb{N}$, $\Gamma(H_i) \subseteq \Gamma(H^2)$, we have that $Y \subseteq X$.

Additionally, if I_1 is the open interval $(0, 1/2)$ and I_2 is the open interval $(0, 2/9)$, then $\varprojlim H_i$ contains the open set $\pi_1^{-1}(I_1) \cap \pi_2^{-1}(I_2)$. Therefore, X is decomposable. □

References

[1] Matevž Črepnjak, *All powers of a function can have non-homeomorphic inverse limits*, Topology Appl. **191** (2015), 65–75.

[2] W. T. Ingram, *Inverse limits with upper semi-continuous bonding functions: Problems and some partial solutions*, Topology Proc. **36** (2010), 353–373.

[3] ———, *An Introduction to Inverse Limits with Set-valued Functions*. Springer Briefs in Mathematics. New York: Springer, 2012.

[4] W. T. Ingram and William S. Mahavier, *Inverse limits of upper semi-continuous set valued functions*, Houston J. Math. **32** (2006), no. 1, 119–130.

[5] James P. Kelly, *Inverse limits with irreducible set-valued functions*, Topology Appl. **166** (2014), 15–31.

[6] James P. Kelly and Jonathan Meddaugh, *Indecomposability in inverse limits with set-valued functions*, Topology Appl. **160** (2013), no. 13, 1720–1731.

[7] William S. Mahavier, *Inverse limits with subsets of* $[0, 1] \times [0, 1]$, Topology Appl. **141** (2004), no. 1-3, 225–231.

[8] M. M. Marsh, *Some structure theorems for inverse limits with set-valued functions*, Topology Proc. **42** (2013), 237–258.

[9] Moshe Roitman, *On Zsigmondy primes*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 1913–1919.

[10] Michel Smith and Scott Varagona, *Generalized inverse limits with N-type bonding functions*. To appear Houston J. Math. **42** (2016), no. 2, 637–657.

[11] Scott Varagona, *Homeomorphisms between inverse limits with N-shaped set-valued functions*, Topology Proc. **44** (2014), 233–248.

Department of Mathematics; Christopher Newport University; Newport News, VA 23606

E-mail address: james.kelly@cnu.edu